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A

COLLEGE ALGEBRA.

BY

G. A. WENTWORTH,

PROFESSOR OF MATHEMATICS IN PHILLIPS EXETER ACADEMY.

TEACHERS' EDITION.

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PREFACE.

THIS edition is intended for teachers, and *for them only*. The publishers will take the utmost pains to keep the book from pupils; and teachers are expected to exercise due care not to lose their copies, and not to leave them where pupils can have access to them.

Every teacher who obtains a copy must understand that he comes under an implied obligation not to let the book go out of his possession unless he sells it to the publishers.

It is hoped that many teachers who are pressed for time will find this edition of great advantage.

G. A. WENTWORTH.

PHILLIPS EXETER ACADEMY,
October, 1889.

COLLEGE ALGEBRA.

Exercise 1.

1. Add

$$\begin{array}{rrr} 9a^2 & +3a & +4b \\ 2a^2 & -4a & +5b \\ -6a^2 & +5a & -2b \\ \hline 5a^2 & +4a & +7b \end{array}$$

2. Add

$$\begin{array}{rrr} 7x^2 & -2xy & +y^2 \\ & 4xy & -2y^2 \\ 8x^2 & -9xy & +12y^2 \\ \hline 15x^2 & -7xy & +11y^2 \end{array}$$

3. Add

$$\begin{array}{rrrr} & 7a^2b & +9ab^2 & -13b^3 \\ 3a^3 & & +2ab^2 & -7b^3 \\ -6a^3 & -a^2b & +ab^2 & \\ -7a^3 & & -ab^2 & +5b^3 \\ -2a^3 & +a^2b & & +4b^3 \\ \hline -12a^3 & +7a^2b & +11ab^2 & -11b^3 \end{array}$$

4. Add

$$\begin{array}{rrrrrr} & 5x^4 & & +2x^2 & & -7 \\ & & 4x^3 & & +x & -9 \\ & & & -x^2 & +x & +1 \\ x^5 & +x^4 & -x^3 & -x^2 & & -7 \\ & -4x^4 & +9x^3 & +9x^2 & -12x & +10 \\ \hline x^5 & +2x^4 & +12x^3 & +9x^2 & -10x & -12 \end{array}$$

5. Add

$$\begin{array}{rrrrr} 3m^4 & +2m^3n & +5m^2n^2 & & -9n^4 \\ & & -8m^2n^2 & -3mn^3 & +7n^4 \\ & 6m^3n & -4m^2n^2 & +11mn^3 & \\ 5m^4 & +2m^3n & & -15mn^3 & -7n^4 \\ \hline 8m^4 & +10m^3n & -7m^2n^2 & -7mn^3 & -9n^4 \end{array}$$

6. Add

$$\begin{array}{rrrrrrr} 2x^6 & +3x^5y & -4x^4y^2 & & & & \\ & & & -10x^3y^3 & +4x^2y^4 & -3xy^5 & +2y^6 \\ & & & 5x^3y^3 & +4x^2y^4 & & -9y^6 \\ & 8x^5y & -7x^4y^2 & +6x^3y^3 & -8x^2y^4 & & \\ \hline 2x^6 & +11x^5y & -11x^4y^2 & +x^3y^3 & & -3xy^5 & -7y^6 \end{array}$$

Exercise 2.

1. From $4a + 5b - 3c$ take $2a + 9b - 8c$.

$$\begin{array}{l} 4a + 5b - 3c \\ 2a + 9b - 8c \\ \hline 2a - 4b + 5c \end{array}$$

2. From $7x^3 - x^2 + 4x - 2$ take $2x^3 + 8x^2 - 9x + 8$.

$$\begin{array}{rrrr} 7x^3 - & x^2 + & 4x - & 2 \\ 2x^3 + & 8x^2 - & 9x + & 8 \\ \hline 5x^3 - & 9x^2 + & 13x - & 10 \end{array}$$

3. From $3a^3 + 3a^2b - 9ab^2 + 3b^3$ take $2a^3 - 5a^2b + 7ab^2 - 9b^3$.

$$\begin{array}{rrrr} 3a^3 + & 3a^2b - & 9ab^2 + & 3b^3 \\ 2a^3 - & 5a^2b + & 7ab^2 - & 9b^3 \\ \hline a^3 + & 8a^2b - & 16ab^2 + & 12b^3 \end{array}$$

4. From $\frac{1}{3}ab + 4a^2 - \frac{2}{5}b^2 + \frac{1}{4}a$ take $a^2 - \frac{1}{10}b^2 + \frac{1}{8}a$.

$$\begin{array}{rrrr} 4a^2 + & \frac{1}{3}ab - & \frac{2}{5}b^2 + & \frac{1}{4}a \\ a^2 & - & \frac{1}{10}b^2 + & \frac{1}{8}a \\ \hline 3a^2 + & \frac{1}{3}ab - & \frac{3}{10}b^2 + & \frac{1}{8}a \end{array}$$

5. From $4x^3 - 6x^2 + 8x - 7$ take the sum of $8x^3 + 7 - 8x^2 + 7x$ and $-9x^3 - 8x^2 + 4x + 4$.

$$\begin{array}{rrrr} 8x^3 - & 8x^2 + & 7x + & 7 \\ -9x^3 - & 8x^2 + & 4x + & 4 \\ \hline -\ x^3 - & 16x^2 + & 11x + & 11 \end{array}$$

$$\begin{array}{rrrr} 4x^3 - & 6x^2 + & 8x - & 7 \\ -\ x^3 - & 16x^2 + & 11x + & 11 \\ \hline 5x^3 + & 10x^2 - & 3x - & 18 \end{array}$$

6. Simplify $2 - 3x - (4 - 6x) - \{7 - (9 - 2x)\}$

$$\begin{aligned} &= 2 - 3x - 4 + 6x - \{7 - 9 + 2x\} \\ &= 2 - 3x - 4 + 6x - 7 + 9 - 2x \\ &= x. \end{aligned}$$

7. Simplify $3a - (a - b - c) - 2\{a + c - 2(b - c)\}$

$$\begin{aligned} &= 3a - a + b + c - 2\{a + c - 2b + 2c\} \\ &= 3a - a + b + c - 2a - 2c + 4b - 4c \\ &= 5b - 5c. \end{aligned}$$

8. Simplify $4a - [3a - \{2a - (a - b)\} + 5b]$

$$\begin{aligned} &= 4a - [3a - \{2a - a + b\} + 5b] \\ &= 4a - [3a - 2a + a - b + 5b] \\ &= 4a - 3a + 2a - a + b - 5b \\ &= 2a - 4b. \end{aligned}$$

9. Simplify $[8a - 3\{a - (b - a)\}] - 4[a - 2\{a - 2(a - b)\} + b]$

$$\begin{aligned} &= 8a - 3\{a - b + a\} - 4[a - 2\{a - 2a + 2b\} + b] \\ &= 8a - 3\{a - b + a\} - 4[a - 2a + 4a - 4b + b] \\ &= 8a - 3a + 3b - 3a - 4a + 8a - 16a + 16b - 4b \\ &= -10a + 15b. \end{aligned}$$

10. Simplify $x(y+z)+y[x-(y+z)]-z[y-x(z-x)]$

$$
\begin{aligned}
&= xy+xz+y[x-y-z]-z[y-xz+x^2]\\
&= xy+xz+xy-yz-y^2-yz+xz^2-zx^2\\
&= 2xy+xz-2yz-y^2+xz^2-zx^2.
\end{aligned}
$$

11. Simplify $2x^3(x-3a)-2[2x^4-a^2(x^2-a^2)]$

$$
\begin{aligned}
&\quad -3a[x^3-2x\{a^2+x(a-x)\}+a^3]\\
&= 2x^4-6ax^3-2[2x^4-a^2x^2+a^4]\\
&\quad -3a[x^3-2x\{a^2+ax-x^2\}+a^3]\\
&= 2x^4-6ax^3-2[2x^4-a^2x^2+a^4]\\
&\quad -3a[x^3-2a^2x-2ax^2+2x^3+a^3]\\
&= 2x^4-6ax^3-4x^4+2a^2x^2-2a^4-3ax^3+6a^3x\\
&\quad +6a^2x^2-6ax^3-3a^4\\
&= -2x^4-15ax^3+8a^2x^2+6a^3x-5a^4.
\end{aligned}
$$

Exercise 3.

1. Find the product of $3x+2y$ and $4x-5y$.

$$
\begin{array}{rrr}
3x+2y & & \\
4x-5y & & \\
\hline
12x^2 & +\ 8xy & \\
 & -15xy & -10y^2\\
\hline
12x^2 & -\ 7xy & -10y^2
\end{array}
$$

2. Find the product of $2x^2-5$ and $4x+3$.

$$
\begin{array}{rrrr}
2x^2-5 & & & \\
4x+3 & & & \\
\hline
8x^3 & & -20x & \\
 & 6x^2 & & -15\\
\hline
8x^3 & +6x^2 & -20x & -15
\end{array}
$$

3. Find the product of $2x^2+4x-3$ and $2x^2+3x-4$.

$$
\begin{array}{rrrrr}
2x^2 & +4x & -3 & & \\
2x^2 & +3x & -4 & & \\
\hline
4x^4 & +\ 8x^3 & -\ 6x^2 & & \\
 & +\ 6x^3 & +12x^2 & -\ 9x & \\
 & & -\ 8x^2 & -16x & +12\\
\hline
4x^4 & +14x^3 & -\ 2x^2 & -25x & +12
\end{array}
$$

4. Find the product of x^4+2x^2+4 and x^4-2x^2+4.

$$
\begin{array}{rrrrr}
x^4 & +2x^2 & +4 & & \\
x^4 & -2x^2 & +4 & & \\
\hline
x^8 & +2x^6 & +4x^4 & & \\
 & -2x^6 & -4x^4 & -8x^2 & \\
 & & +4x^4 & +8x^2 & +16\\
\hline
x^8 & & +4x^4 & & +16
\end{array}
$$

5. Find the product of $x^2+2xy-3y^2$ and $x^2-5xy+4y^2$.

$$\begin{array}{rrrrr}
x^2 & +2xy & -3y^2 & & \\
x^2 & -5xy & +4y^2 & & \\ \hline
x^4 & +2x^3y & -3x^2y^2 & & \\
 & -5x^3y & -10x^2y^2 & +15xy^3 & \\
 & & +4x^2y^2 & +8xy^3 & -12y^4 \\ \hline
x^4 & -3x^3y & -9x^2y^2 & +23xy^3 & -12y^4
\end{array}$$

6. Find the product of $9x^2+3xy+y^2-6x+2y+4$ and $3x-y+2$.

$$\begin{array}{rrrrrrrrrr}
9x^2 & +3xy & +y^2 & -6x & +2y & +4 & & & & \\
 & & & & 3x & -y & +2 & & & \\ \hline
27x^3 & +9x^2y & +3xy^2 & & -18x^2 & +6xy & & +12x & & \\
 & -9x^2y & -3xy^2 & -y^3 & & +6xy & -2y^2 & & -4y & \\
 & & & & +18x^2 & +6xy & +2y^2 & -12x & +4y & +8 \\ \hline
27x^3 & & & -y^3 & & +18xy & & & & +8
\end{array}$$

7. Find the product of

$$11a^3+4b^3-4ab(a-4b) \text{ and } a^2(b+3a)-4b^2(a+b).$$

$$\begin{aligned}
&11a^3+4b^3-4ab(a-4b)\\
&\quad=11a^3+4b^3-4a^2b+16ab^2\\
&\quad=11a^3-4a^2b+16ab^2+4b^3.
\end{aligned}
\qquad
\begin{aligned}
&a^2(b+3a)-4b^2(a+b)\\
&\quad=a^2b+3a^3-4ab^2-4b^3\\
&\quad=3a^3+a^2b-4ab^2-4b^3.
\end{aligned}$$

$$\begin{array}{rrrrrrr}
11 & -4 & +16 & +4 & & & \\
3 & +1 & -4 & -4 & & & \\ \hline
33 & -12 & +48 & +12 & & & \\
 & +11 & -4 & +16 & +4 & & \\
 & & -44 & +16 & -64 & -16 & \\
 & & & -44 & +16 & -64 & -16 \\ \hline
33 & -1 & +0 & +0 & -44 & -80 & -16
\end{array}$$

$$33a^6-a^5b-44a^2b^4-80ab^5-16b^6.$$

8. Find the product of $(a+b)^2+(a-b)^2$ and $(a+b)^2-(a-b)^2$.

$$\begin{aligned}
&(a+b)^2+(a-b)^2\\
&\quad=a^2+2ab+b^2+a^2-2ab+b^2\\
&\quad=2a^2+2b^2.
\end{aligned}$$

$$\begin{aligned}
&(a+b)^2-(a-b)^2\\
&\quad=a^2+2ab+b^2-a^2+2ab-b^2\\
&\quad=4ab.
\end{aligned}$$

$$4ab\times(2a^2+2b^2)=8a^3b+8ab^3.$$

9. Find the product of $x-2y+3z$ and $x-2y+3z$.

$$\begin{aligned}&(x-2y+3z)(x-2y+3z)\\ &\quad=(x-2y+3z)^2\\ &\quad=x^2+4y^2+9z^2-4xy+6xz-12yz.\end{aligned}$$

10. Find the product of x^3+2x^2-4x-1 and x^3+2x^2-4x-1.

$$\begin{aligned}&(x^3+2x^2-4x-1)(x^3+2x^2-4x-1)\\ &\quad=(x^3+2x^2-4x-1)^2\\ &\quad=x^6+4x^4+16x^2+1+4x^5-8x^4-2x^3-16x^3-4x^2+8x\\ &\quad=x^6+4x^5-4x^4-18x^3+12x^2+8x+1.\end{aligned}$$

11. Find the product of $39d^{x+y-1}-54d^{x-2y+1}+60d^{x+3y}$ and $30d^{2-x+2y}$.

$$\begin{array}{r}39\,d^{x+y-1}-54\,d^{x-2y+1}+60\,d^{x+3y}\\ 30\,d^{2-x+2y}\\ \hline 1170\,d^{1+3y}-1620\,d^{3}+1800\,d^{2+5y}.\end{array}$$

12. Find the product of $24x^{m+2n-1}-42x^{2m-3n+2}+25x^{2n+3m-2}$ and $25x^{2-m-2n}$.

$$\begin{array}{r}24\,x^{m+2n-1}-42\,x^{2m-3n+2}+25\,x^{2n+3m-2}\\ 25\,x^{2-m-2n}\\ \hline 600\,x-1050\,x^{m-5n+4}+625\,x^{2m}\end{array}$$

13. Find the product of $a^p-3a^{p-1}+4a^{p-2}-6a^{p-3}+5a^{p-4}$ and $2a^3-a^2+a$.

$$\begin{array}{l}a^p-3a^{p-1}+4a^{p-2}-6a^{p-3}+5a^{p-4}\\ \qquad\qquad\qquad\qquad 2a^3-a^2+a\end{array}$$

$$\begin{array}{rrrrrrr}2&-6&+8&-12&+10&&\\ &-1&+3&-4&+6&-5&\\ &&+1&-3&+4&-6&+5\\ \hline 2a^{p+3}&-7a^{p+2}&+12a^{p+1}&-19a^{p}&+20a^{p-1}&-11a^{p-2}&+5a^{p-3}\end{array}$$

14. Find the product of $a^{2n+1}-a^{n+1}-a^n+a^{n-1}$ and $a^{n+2}-a^2-a+1$.

$$\begin{array}{lllllllll}a^{2n+1}&-a^{n+1}&-a^{n}&+a^{n-1}&&&&&\\ &a^{n+2}&-a^{2}&-a&+1&&&&\\ \hline a^{3n+3}&-a^{2n+3}&-a^{2n+2}&+a^{2n+1}&&&&&\\ &-a^{2n+3}&&&+a^{n+3}&+a^{n+2}&-a^{n+1}&&\\ &&-a^{2n+2}&&&+a^{n+2}&+a^{n+1}&-a^{n}&\\ &&&+a^{2n+1}&&&-a^{n+1}&-a^{n}&+a^{n-1}\\ \hline a^{3n+3}&-2a^{2n+3}&-2a^{2n+2}&+2a^{2n+1}&+a^{n+3}&+2a^{n+2}&-a^{n+1}&-2a^{n}&+a^{n-1}\end{array}$$

15. Find the product of $a^p + 3a^{p-2} - 2a^{p-1}$ and $2a^{p+1} + a^{p+2} - 3a^p$.

$$\begin{array}{l}
a^p - 2a^{p-1} + 3a^{p-2} \\
\underline{a^{p+2} + 2a^{p+1} - 3a^p} \\
1 - 2 + 3 \\
\quad + 2 - 4 + 6 \\
\qquad\quad - 3 + 6 - 9 \\
\hline
a^{2p+2} - 4a^{2p} + 12a^{2p-1} - 9a^{2p-2}
\end{array}$$

Exercise 4.

1. Divide $(6a^2b^3c \times 35a^2b^5c^4)$ by $(21a^3b^9c^6 \times 2a^2c^3)$.

$$\begin{aligned}
&(6a^2b^3c \times 35a^2b^5c^4) \div (21a^3b^9c^6 \times 2a^2c^3) \\
&= 210a^4b^8c^5 \div 42a^5b^9c^9 \\
&= \frac{5}{abc^4}.
\end{aligned}$$

2. Divide $39a^3x^2 + 24a^4x^3 + 42a^2x^3 + 27a^4x^2$ by $6a^2x^2$.

$$\begin{aligned}
&39a^3x^2 + 24a^4x^3 + 42a^2x^3 + 27a^4x^2 \div 6a^2x^2 \\
&= \tfrac{13}{2}a + 4a^2x + 7x + \tfrac{9}{2}a^2.
\end{aligned}$$

3. Divide $35x^3 + 94ax^2 + 52a^2x + 8a^3$ by $5x + 2a$.

$$\begin{array}{l|l}
35 + 94 + 52 + 8 & 5 + 2 \\ \cline{2-2}
\underline{35 + 14} & 7 + 16 + 4 \\
\qquad 80 + 52 & 7x^2 + 16ax + 4a^2.\ \textit{Ans.} \\
\qquad \underline{80 + 32} & \\
\qquad\qquad 20 + 8 & \\
\qquad\qquad \underline{20 + 8} &
\end{array}$$

4. Divide $x^3 - 5ax - a^2x + 14a^3$ by $x^2 - 3ax - 7a^2$.

$$\begin{array}{l|l}
1 - 5 - 1 + 14 & 1 - 3 - 7 \\ \cline{2-2}
\underline{1 - 3 - 7} & 1 - 2 \\
\quad - 2 + 6 + 14 & x - 2a.\ \textit{Ans.} \\
\quad \underline{- 2 + 6 + 14} &
\end{array}$$

5. Divide $81x^4+36x^2y^2+16y^4$ by $9x^2-6xy+4y^2$.

$$
\begin{array}{l|l}
81x^4+36x^2y^2+16y^4 & 9x^2-6xy+4y^2 \\ \cline{2-2}
81x^4-54x^3y+36x^2y^2 & 9x^2+6xy+4y^2.\ \textit{Ans.} \\ \cline{1-1}
\qquad 54x^3y+16y^4 & \\
\qquad 54x^3y-36x^2y^2+24xy^3 & \\ \cline{1-1}
\qquad\qquad 36x^2y^2-24xy^3+16y^4 & \\
\qquad\qquad 36x^2y^2-24xy^3+16y^4 & \\ \cline{1-1}
\end{array}
$$

6. Divide $x^4+b^4-a^2x^2+2b^2x^2$ by x^2+b^2+ax.

$$
\begin{aligned}
&x^4+b^4-a^2x^2+2b^2x^2\\
&\quad=x^4+2b^2x^2+b^4-a^2x^2\\
&\quad=(x^2+b^2)^2-(ax)^2.\\
&[(x^2+b^2)^2-(ax)^2]\div(x^2+b^2+ax)\\
&\quad=x^2+b^2-ax.\ \textit{Ans.}
\end{aligned}
$$

7. Divide $a^2-2b^2-3c^2+ab+2ac+7bc$ by $a-b+3c$.

$$
\begin{array}{l|l}
a^2+ab+2ac-2b^2+7bc-3c^2 & a-b+3c \\ \cline{2-2}
a^2-ab+3ac & a+2b-c.\ \textit{Ans.} \\ \cline{1-1}
\qquad 2ab-ac-2b^2+7bc-3c^2 & \\
\qquad 2ab\qquad\ -2b^2+6bc & \\ \cline{1-1}
\qquad\qquad -ac\qquad +bc-3c^2 & \\
\qquad\qquad -ac\qquad +bc-3c^2 & \\ \cline{1-1}
\end{array}
$$

Divide $4x^4-5x^2y^2-8x^2-4y^2+4+y^4$ by $y^2+2x^2-2-3xy$.

$$
\begin{array}{l|l}
4x^4-5x^2y^2-8x^2-4y^2+y^4+4 & 2x^2-3xy+y^2-2 \\ \cline{2-2}
4x^4-6x^3y+2x^2y^2-4x^2 & 2x^2+3xy+y^2-2.\ \textit{Ans.} \\ \cline{1-1}
\qquad 6x^3y-7x^2y^2-4x^2-4y^2+y^4+4 & \\
\qquad 6x^3y-9x^2y^2+3xy^3-6xy & \\ \cline{1-1}
\qquad\qquad 2x^2y^2-3xy^3-4x^2+6xy-4y^2+y^4+4 & \\
\qquad\qquad 2x^2y^2-3xy^3+y^4\qquad -2y^2 & \\ \cline{1-1}
\qquad\qquad\qquad -4x^2+6xy-2y^2+4 & \\
\qquad\qquad\qquad -4x^2+6xy-2y^2+4 & \\ \cline{1-1}
\end{array}
$$

9. Divide $2a^{m+1} - 2a^{n+1} - a^{m+n} + a^{2n}$ by $a^n - 2a$.

$$\begin{array}{l|l}
a^{2n} - a^{m+n} - 2a^{n+1} + 2a^{m+1} & a^n - 2a \\ \cline{2-2}
\underline{a^{2n} \qquad\qquad - 2a^{n+1}\quad} & a^n - a^m. \ \textit{Ans.} \\
\multicolumn{1}{l}{\quad - a^{m+n} \qquad\qquad + 2a^{m+1}} & \\
\multicolumn{1}{l}{\quad \underline{- a^{m+n} \qquad\qquad + 2a^{m+1}}} &
\end{array}$$

10. Divide $625x^4 - 81y^4$ by $5x - 3y$.

$$\begin{array}{l|l}
625x^4 - 81y^4 & 5x - 3y \\ \cline{2-2}
\underline{625x^4 - 375x^3y} & 125x^3 + 75x^2y + 45xy^2 + 27y^3. \ \textit{Ans.} \\
\multicolumn{1}{l}{\qquad\qquad 375x^3y - 81y^4} & \\
\multicolumn{1}{l}{\qquad\qquad \underline{375x^3y - 225x^2y^2}} & \\
\multicolumn{1}{l}{\qquad\qquad\qquad\qquad 225x^2y^2 - 81y^4} & \\
\multicolumn{1}{l}{\qquad\qquad\qquad\qquad \underline{225x^2y^2 - 135xy^3}} & \\
\multicolumn{1}{l}{\qquad\qquad\qquad\qquad\qquad\qquad 135xy^3 - 81y^4} & \\
\multicolumn{1}{l}{\qquad\qquad\qquad\qquad\qquad\qquad \underline{135xy^3 - 81y^4}} &
\end{array}$$

11. Divide $x^{3n} + y^{3n}$ by $x^n + y^n$.

$$\begin{array}{l|l}
x^{3n} + y^{3n} & x^n + y^n \\ \cline{2-2}
\underline{x^{3n} + x^{2n}y^n} & x^{2n} - x^ny^n + y^{2n}. \ \textit{Ans.} \\
\multicolumn{1}{l}{\quad - x^{2n}y^n + y^{3n}} & \\
\multicolumn{1}{l}{\quad \underline{- x^{2n}y^n - x^ny^{2n}}} & \\
\multicolumn{1}{l}{\qquad\qquad\quad x^ny^{2n} + y^{3n}} & \\
\multicolumn{1}{l}{\qquad\qquad\quad \underline{x^ny^{2n} + y^{3n}}} &
\end{array}$$

12. Divide $\dfrac{27a^3}{125} - \dfrac{b^3}{64}$ by $\dfrac{3a}{5} - \dfrac{b}{4}$.

$$\begin{array}{l|l}
\dfrac{27a^3}{125} - \dfrac{b^3}{64} & \dfrac{3a}{5} - \dfrac{b}{4} \\ \cline{2-2}
\dfrac{27a^3}{125} - \dfrac{9a^2b}{100} & \dfrac{9a^2}{25} + \dfrac{3ab}{20} + \dfrac{b^2}{16}. \ \textit{Ans.} \\ \cline{1-1}
\multicolumn{1}{l}{\qquad\quad \dfrac{9a^2b}{100} - \dfrac{b^3}{64}} & \\
\multicolumn{1}{l}{\qquad\quad \dfrac{9a^2b}{100} - \dfrac{3ab^2}{80}} & \\ \cline{1-1}
\multicolumn{1}{l}{\qquad\qquad\qquad \dfrac{3ab^2}{80} - \dfrac{b^3}{64}} & \\
\multicolumn{1}{l}{\qquad\qquad\qquad \underline{\dfrac{3ab^2}{80} - \dfrac{b^3}{64}}} &
\end{array}$$

13. Divide $(a+2b)^3+(b-3c)^3$ by $a+3(b-c)$.

$$\begin{aligned}&(a+2b)^3+(b-3c)^3\\&\quad=a^3+6a^2b+12ab^2+8b^3+b^3-9b^2c+27bc^2-27c^3\\&\quad=a^3+6a^2b+12ab^2+9b^3-9b^2c+27bc^2-27c^3.\end{aligned}$$

$$\begin{array}{l|l}
a^3+6a^2b+12ab^2+9b^3-9b^2c+27bc^2-27c^3 & a+3b-3c \\ \cline{2-2}
\underline{a^3+3a^2b-3a^2c} & a^2+3ab+3ac+3b^2+9c^2. \\
\quad 3a^2b+3a^2c+12ab^2+9b^3-9b^2c+27bc^2-27c^3 & \qquad\qquad\qquad\qquad\text{Ans.} \\
\quad \underline{3a^2b\qquad\;\; +9ab^2-9abc} & \\
\qquad 3a^2c+\;3ab^2+9abc+9b^3-9b^2c+27bc^2-27c^3 & \\
\qquad \underline{3a^2c\qquad\quad +9abc-9ac^2} & \\
\qquad\qquad 3ab^2+9ac^2+9b^3-9b^2c+27bc^2-27c^3 & \\
\qquad\qquad \underline{3ab^2\qquad\quad +9b^3-9b^2c} & \\
\qquad\qquad\qquad 9ac^2\qquad\qquad +27bc^2-27c^3 & \\
\qquad\qquad\qquad \underline{9ac^2\qquad\qquad +27bc^2-27c^3} &
\end{array}$$

14. Divide $a^m-a^{m+1}+37a^{m+3}-55a^{m+4}+50a^{m+5}$ by $1-3a+10a^2$.

$$\begin{array}{l|l}
1-1+\;0+37-55+50 & 1-3+10 \\ \cline{2-2}
\underline{1-3+10} & 1+2-4+5 \\
\quad 2-10+37 & a^m+2a^{m+1}-4a^{m+2}+5a^{m+3}.\ \text{Ans.} \\
\quad \underline{2-\;\;6+20} & \\
\qquad -\;4+17-55 & \\
\qquad \underline{-\;4+12-40} & \\
\qquad\qquad 5-15+50 & \\
\qquad\qquad \underline{5-15+50} &
\end{array}$$

15. Divide $4h^{x+1}-30h^x+19h^{x-1}+5h^{x-2}+9h^{x-4}$

by $h^{x-3}-7h^{x-4}+2h^{x-5}-3h^{x-6}$.

$$\begin{array}{l|l}
4-30+19+\;5+0+9 & 1-7+2-3 \\ \cline{2-2}
\underline{4-28+\;8-12} & 4-2-3 \\
\quad -\;2+11+17+0 & 4h^4-2h^3-3h^2.\ \text{Ans.} \\
\quad \underline{-\;2+14-\;4+6} & \\
\qquad -\;3+21-6+9 & \\
\qquad \underline{-\;3+21-6+9} &
\end{array}$$

16. Divide $6x^{m-n+2}+x^{m-n+1}-22x^{m-n}+19^{m-n-1}-4^{m-n-2}$ by $3x^{3-n}-4x^{2-n}+x^{1-n}$.

$$\begin{array}{l|l}
6+1-22+19-4 & 3-4+1 \\ \cline{2-2}
\underline{6-8+\ 2} & 2+3-4 \\
\quad 9-24+19 & 2x^{m-1}+3x^{m-2}-4x^{m-3}.\ \textit{Ans.} \\
\quad \underline{9-12+\ 3} & \\
\qquad -12+16-4 & \\
\qquad \underline{-12+16-4} &
\end{array}$$

Exercise 5.

1. $9x^4+6x^3+3x^2+2x$
$=3x^2(3x^2+2x)+3x^2+2x$
$=(3x^2+1)(3x^2+2x)$
$=(3x^2+1)(3x+2)x.$

2. $2a^4-3a^3b-14a^2+21ab$
$=a^2(2a^2-3ab)-7(2a^2-3ab)$
$=(a^2-7)(2a^2-3ab)$
$=(a^2-7)(2a-3b)a.$

3. $5x^3+15x^2y-4xy^2-12y^3$
$=5x^2(x+3y)-4y^2(x+3y)$
$=(5x^2-4y^2)(x+3y).$

4. $a^2x^3-b^2xy^2-a^2cx^2+b^2cy^2$
$=x(a^2x^2-b^2y^2)-c(a^2x^2-b^2y^2)$
$=(x-c)(a^2x^2-b^2y^2)$
$=(x-c)(ax+by)(ax-by).$

5. x^2+8x+7
$=(x+1)(x+7).$

6. $x^2-17x+60$
$=(x-5)(x-12).$

7. $x^2+7x-18$
$=(x-2)(x+9).$

8. $x^2-2x-24$
$=(x+4)(x-6).$

9. $9x^2+30x+25$
$=(3x+5)^2.$

10. $16x^2-56x+49$
$=(4x-7)^2.$

11. x^2+x-72
$=(x-8)(x+9).$

12. $x^2-14x-176$
$=(x+8)(x-22).$

13. $81x^4-196x^2y^2$
$=x^2(81x^2-196y^2)$
$=x^2(9x-14y)(9x+14y).$

14. $729a^6 - x^6$
$= (27a^3 - x^3)(27a^3 + x^3)$
$= (3a - x)(9a^2 + 3ax + x^2)(3a + x)(9a^2 - 3ax + x^2).$

15. $64x^7 + xy^6$
$= x(64x^6 + y^6)$
$= x(4x^2 + y^2)(16x^4 - 4x^2y^2 + y^4).$

16. $(x^2 - y^2)^2 - y^4$
$= \{(x^2 - y^2) + y^2\}\{(x^2 - y^2) - y^2\}$
$= x^2(x^2 - 2y^2).$

17. $(a^2 + 2b^2)^2 - a^2b^2$
$= \{(a^2 + 2b^2) - ab\}\{(a^2 + 2b^2) + ab\}$
$= (a^2 - ab + 2b^2)(a^2 + ab + 2b^2).$

18. $(2x - 3y)^2 - (x - 2y)^2$
$= \{(2x - 3y) - (x - 2y)\}\{(2x - 3y) + (x - 2y)\}$
$= (x - y)(3x - 5y).$

19. $(2x^2 - 4x + 7)^2 - x^2(x + 4)^2$
$= \{(2x^2 - 4x + 7) - x(x + 4)\}\{(2x^2 - 4x + 7) + x(x + 4)\}$
$= (x^2 - 8x + 7)(3x^2 + 7)$
$= (x - 1)(x - 7)(3x^2 + 7).$

20. $x^4 - 2(b^2 - c^2)x^2 + b^4 - 2b^2c^2 + c^4$
$= x^4 - 2(b^2 - c^2)x^2 + (b^2 - c^2)^2$
$= \{x^2 - (b^2 - c^2)\}^2$
$= (x^2 - b^2 + c^2)^2.$

21. $15x^2 - 7x - 2$
$= (5x + 1)(3x - 2).$

22. $11x^2 - 54x + 63$
$= (11x - 21)(x - 3).$

23. $21x^2 + 26x - 15$
$= (3x + 5)(7x - 3)$

24. $70x^2 - 27x - 9$
$= (14x + 3)(5x - 3).$

25. $x^4 - 2abx^2 - a^4 - a^2b^2 - b^4$
$= x^4 - 2abx^2 + a^2b^2 - a^4 - 2a^2b^2 - b^4$
$= (x^2 - ab)^2 - (a^2 + b^2)^2$
$= \{(x^2 - ab) + (a^2 + b^2)\}\{(x^2 - ab) - (a^2 + b^2)\}$
$= (x^2 + a^2 - ab + b^2)(x^2 - a^2 - ab - b^2).$

26. $5x^4 + 4x^3 - 20x - 125$

$$= 5(x^4 - 25) + 4x(x^2 - 5)$$
$$= 5(x^2 + 5)(x^2 - 5) + 4x(x^2 - 5)$$
$$= (x^2 - 5)(5x^2 + 4x + 25).$$

27. $2x^4 - 5x^3 - x^2 - 2.$

Try as factors, $2x^2 + kx + 1$ and $x^2 + k'x - 2.$

$$(2x^2 + kx + 1)(x^2 + k'x - 2)$$
$$= 4x^2 + (2k' + k)x^3 + (kk' - 3)x^2 + (k' - 2k)x - 2.$$

This is equal to $2x^4 - 5x^3 - x^2 - 2$ if $k = -1$ and $k' = -2.$

$\therefore 2x^4 - 5x^3 - x^2 - 2 = (2x^2 - x + 1)(x^2 - 2x - 2).$

28. $6x^4 - ax^3 - 2a^2x^2 + 3a^3x - 2a^4.$

Try as factors, $3x^2 + kax - 2a^2$ and $2x^2 + k'ax + a^2.$

$$(3x^2 + kax - 2a^2)(2x^2 + k'ax + a^2)$$
$$= 6x^4 + (3k' + 2k)ax^3 + (kk - 1)a^2x^2 + (k - 2k')a^3x - 4a^4.$$

This is equal to $6x^4 - ax^3 - 2a^2x^2 + 3a^3x - 2a^4$
if $k = 1$ and $k' = -1.$

$\therefore 6x^4 - ax^3 - 2a^2x^2 + 3a^3x - 2a^4$

$$= (3x^2 + ax - 2a^2)(2x^2 - ax + a^2)$$
$$= (3x - 2a)(x + a)(2x^2 - ax + a^2)$$

29. $12x^5 + 10x^4y - 12x^3y^2 - 6x^2y^3 - 4y^5$

$$= 12(x^5 - x^3y^2) + 10x^4y - 6x^2y^3 - 4y^5$$
$$= 12x^3(x^2 - y^2) + y(10x^4 - 6x^2y^2 - 4y^4)$$
$$= 12x^3(x^2 - y^2) + y(10x^2 + 4y^2)(x^2 - y^2)$$
$$= (12x^3 + 10x^2y + 4y^3)(x^2 - y^2)$$
$$= 2(6x^3 + 5x^2y + 2y^3)(x + y)(x - y).$$

Exercise 6.

1. Find the H.C.F. of $12x^2 - 17x + 6$, $9x^2 + 6x - 8.$

$$12x^2 - 17x + 6 = (4x - 3)(3x - 2);$$
$$9x^2 + 6x - 8 = (3x + 4)(3x - 2).$$

$\therefore$ H.C.F. is $3x - 2.$

2. Find the H.C.F. of $x^4 - a^4$, $x^2 + 3ax - 4a^2$, $x^2 - 5ax + 4a^2.$

$$x^4 - a^4 = (x^2 + a^2)(x - a)(x + a);$$
$$x^2 + 3ax - 4a^2 = (x - a)(x + 4a);$$
$$x^2 - 5ax + 4a^2 = (x - 4a)(x - a).$$

$\therefore$ H.C.F. is $x - a.$

3. Find the H.C.F. of

$$x^4-6x^3+13x^2-12x+4,\ x^4-4x^3+8x^2-16x+16.$$

$$\begin{array}{r|r|l}
1-\ 6+13-\ 12+\ 4 & 1-4+\ 8-16+16 & 1\\
\underline{2\qquad\qquad\qquad\qquad} & \underline{1-6+13-12+\ 4} & \\
2-12+26-\ 24+\ 8 & 2-\ 5-\ 4+12 & 1-7\\
\underline{2-\ 5-\ 4+\ 12\quad} & \underline{2-\ 8+\ 8\quad} & \\
-\ 7+30-\ 36+\ 8 & 3-12+12 & \\
\underline{2\qquad\qquad\qquad} & \underline{3-12+12} & \\
-14+60-\ 72+\ 16 & & \\
\underline{-14+35+\ 28-\ 84} & & \\
25)\underline{25-100+100} & & \\
1-\ 4+\ 4 & & 2+3
\end{array}$$

$\therefore$ H.C.F. is x^2-4x+4.

4. Find the H.C.F. of

$$3x^4-x^3-2x^2+2x-8,\ 6x^4+13x^3+3x^2+20x.$$

Remove x from second expression.

$$\begin{array}{r|r|l}
3-\ 1-\ 2+\ 2-\ 8 & 6+13+3+20 & 1-5\\
\underline{2\qquad\qquad\qquad\qquad} & \underline{6-\ 2+8\quad} & \\
6-\ 2-\ 4+\ 4-\ 16 & 15-5+20 & \\
\underline{6+13+\ 3+20\quad} & \underline{15-5+20} & \\
-15-\ 7-16-\ 16 & & \\
\underline{2\qquad\qquad\qquad} & & \\
-30-14-32-\ 32 & & \\
\underline{-30-65-15-100} & & \\
17)\underline{51-17+\ 68} & & \\
3-\ 1+\ 4 & & 2+5
\end{array}$$

$\therefore$ H.C.F. is $3x^2-x+4$.

5. Find the H. C. F. of

$$96x^4+8x^3-2x,\ 32x^3-24x^2-8x+3.$$

Remove factor $2x$ from 1st expression.

$$\begin{array}{r|r|l}
48+4+0-1 & 32-24-8+3 & 3\\
2 & 5 & \\
96+8+0-2 & 160-120-40+15 & \\
96-72-24+9 & 160+48-22 & \\
80+24-11 & 3)-168-18+15 & 2-7\\
80-20 & -56-6+5 & \\
 & 10 & \\
44-11 & -560-60+50 & \\
44-11 & -560-168+77 & \\
 & 27)108-27 & \\
 & 4-1 & 20+11
\end{array}$$

$\therefore$ H. C. F. is $4x-1$.

6. Find the H. C. F. of

$$x^4+5x^3-7x^2-9x-10,\ 2x^4-4x^3+4x-8.$$

$$\begin{array}{r|r|l}
2)2-4+0+4-8 & 1+5-7-9-10 & 1\\
1-2+0+2-4 & 1-2+0+2-4 & \\
7 & 7-7-11-6 & 1-1\\
7-14+0+14-28 & 4 & \\
7-7-11-6 & 28-28-44-24 & \\
-7+11+20-28 & 28+63-238 & \\
-7+7+11+6 & -91+194-24 & \\
4+9-34 & 4 & \\
4-8 & -364+776-96 & 7-91\\
17-34 & -364-819+3094 & \\
17-34 & 1595)1595-3190 & \\
 & 1-2 & 4+17
\end{array}$$

$\therefore$ H. C. F. is $x-2$.

7. Find the H. C. F. of

$$2x^3-16x+6,\ 5x^6+15x^5+5x+15.$$

$$\begin{aligned}5x^6+15x^5+5x+15&=5(x^5+1)(x+3)\\&=5(x+1)(x+3)(x^4-x^3+x^2-x+1);\\2x^3-16x+6&=2(x^2-3x+1)(x+3).\end{aligned}$$

$\therefore$ H. C. F. $= x+3$.

8. Find the H. C. F. of

$$2a^4+3a^3x-9a^2x^2,\ 6a^4x-3ax^4-17a^3x^2+14a^2x^3.$$

$$\begin{aligned}2a^4+3a^3x-9a^2x^2&=a^2(2a^2+3ax-9x^2)\\&=a^2(2a-3x)(a+3x);\\6a^4x-3ax^4-17a^3x^2+17a^2x^3&=ax(6a^3-3x^3-17a^2x+14ax^2)\\&=ax(3a^2-4ax-x^2)(2a-3x).\end{aligned}$$

$\therefore$ H. C. F. $= a(2a-3x)$.

9. Find the H. C. F. of

$$2a^5-4a^4+8a^3-12a^2+6a,\ 3a^6-3a^5-6a^4+9a^3-3a^2.$$

$$\begin{aligned}3a^6-3a^5-6a^4+9a^3-3a^2&=3a^2(a^4-a^3-2a^2+3a-1)\\&=3a^2\{a^3(a-1)-(2a-1)(a-1)\}\\&=3a^2(a-1)(a^3-2a+1)\\&=3a^2(a-1)^2(a^2-a-1);\\2a^5-4a^4+8a^3-12a^2+6a&=2a(a-1)(a^3-a^2+3a-3)\\&=2a(a-1)^2(a^2+3).\end{aligned}$$

$\therefore$ H. C. F. $= a(a-1)^2$.

10. Find the H. C. F. of

$$3x^3-7x^2y-y^3+5xy^2,\ x^2y+3xy^2-3x^3-y^3,\ 3x^3+5x^2y+xy^2-y^3.$$

$$\begin{aligned}x^2y+3xy^2-3x^3-y^3&=y(x^2-y^2)+3x(y^2-x^2)\\&=(y-3x)(x+y)(x-y);\\3x^3-7x^2y-y^3+5xy^2&=(x-y)(3x^2-4xy+y^2)\\&=(x-y)^2(3x-y);\\3x^3+5x^2y+xy^2-y^3&=(3x-y)(x^2+2xy+y^2)\\&=(3x-y)(x+y)^2.\end{aligned}$$

$\therefore$ H. C. F. $= 3x-y$.

11. Find the H. C. F. of

$$36x^7-28x^5+32x^4+8x^3-16x^2,\quad 12x^5-14x^4-20x^3+10x^2+4x.$$

$$12x^5-14x^4-20x^3+10x^2+4x=2x(6x^4-7x^3-10x^2+5x+2);$$

$$36x^7-28x^5+32x^4+8x^3-16x^2=4x^2(9x^5-7x^3+8x^2+2x-4).$$

$$\begin{array}{r|r|l}
6-7-10+5+2 & 9+0-7+8+2-4 & 3+7 \\
9 & 2 & \\
\cline{1-2}
54-63-90+45+18 & 18+0-14+16+4-8 & \\
54+48-26-20 & 18-21-30+15+6 & \\
\cline{1-2}
-111-64+65+18 & 21+16+1-2-8 & \\
-108-96+52+40 & 2 & \\
\cline{1-2}
-3+32+13-22 & 42+32+2-4-16 & \\
3-32-13+22 & 42-49-70+35+14 & \\
\cline{2-2}
3+1-2 & 3)\,87+72-39-30 & 9 \\
\cline{1-2}
-33-11+22 & 27+24-13-10 & 2-4 \\
-33-11+22 & 27-288-117+198 & \\
\cline{1-2}
 & 104)\,312+104+208 & \\
\cline{2-2}
 & 3+1-2 & 1-11
\end{array}$$

$\therefore$ H.C.F. is $(3x^2+x-2)2x=6x^3+2x^2-4x$.

12. Find the L. C. M. of

$$x^2-3x-4,\quad x^2-x-12,\quad x^2+5x+4.$$

$$x^2-3x-4=(x-4)(x+1);$$

$$x^2-x-12=(x-4)(x+3);$$

$$x^2+5x+4=(x+4)(x+1).$$

$$\therefore \text{L.C.M.}=(x-4)(x+4)(x+1)(x+3)$$

$$=x^4+4x^3-13x^2-64x-48.$$

13. Find the L. C. M. of

$$6x^2-13x+6,\quad 6x^2+5x-6,\quad 9x^2-4.$$

$$6x^2-13x+6=(2x-3)(3x-2);$$

$$6x^2+5x-6=(2x+3)(3x-2);$$

$$9x^2-4=(3x-2)(3x+2).$$

$$\therefore \text{L.C.M.}=(2x-3)(2x+3)(3x-2)(3x+2)$$

$$=36x^4-97x^2+36.$$

14. Find the L. C. M. of

$$3x^4 - x^3 - 2x^2 + 2x - 8,\ 6x^3 + 13x^2 + 3x + 20.$$

$$\begin{array}{r|r|l}
6+13+3+20 & 3-\ 1-\ 2+\ 2-\ \ 8 & 1-5\\
6-\ 2+8 & 2\qquad\qquad\qquad & \\
\cline{1-2}
15-5+20 & 6-\ 2-\ 4+\ 4-\ 16 & \\
15-5+20 & 6+13+\ 3+20\quad & \\
\cline{1-2}
 & -15-\ 7-16-\ 16 & \\
 & 2\qquad\qquad\quad & \\
\cline{2-2}
 & -30-14-32-\ 32 & \\
 & -30-65-15-100 & \\
\cline{2-2}
 & 17)\underline{51-17+\ 68} & \\
 & 3-\ 1+\ \ 4 & 2+5
\end{array}$$

$\therefore$ H. C. F. is $3x^2 - x + 4$.

$$6x^3 + 13x^2 + 3x + 20 = (3x^2 - x + 4)(2x + 5);$$
$$3x^4 - x^3 - 2x^2 + 2x - 8 = (3x^2 - x + 4)(x^2 - 2).$$
$$\therefore \text{ L. C. M.} = (2x + 5)(x^2 - 2)(3x^2 - x + 4);$$
$$= 6x^5 + 13x^4 - 9x^3 - 6x^2 - 6x - 40.$$

15. Find the L. C. M. of

$$15a^2x^4 + 10a^4x^3 + 4a^5x^2 + 6a^6x - 3a^7,\ 12x^4 + 38ax^3 + 16a^2x^2 - 10a^3x.$$

$$12x^4 + 38ax^3 + 16a^2x^2 - 10a^3x$$
$$= 2x(6x^3 + 19ax^2 + 8a^2x - 5a^3);$$
$$15a^3x^4 - 10a^4x^3 + 4a^5x^2 + 6a^6x - 3a^7$$
$$= a^3(15x^4 + 10ax^3 + 4a^2x^2 + 6a^3 - 3a^4).$$

$$\begin{array}{r|r|l}
6+19+\ 8-5 & 15+\ 10+\ \ 4+\ \ 6-\ \ 3 & \\
6+\ 4-\ 2\quad & 2\qquad\qquad\qquad\qquad & \\
\cline{1-2}
15+10-5 & 30+\ 20+\ \ 8+\ 12-\ \ 6 & 5-25\\
15+10-5 & 30+\ 95+\ 40-\ 25\quad & \\
\cline{1-2}
 & -\ 75-\ 32+\ 37-\ \ 6 & \\
 & 2\qquad\qquad\qquad & \\
\cline{2-2}
 & -150-\ 64+\ 74-\ 12 & \\
 & -150-475-200+125 & \\
\cline{2-2}
 & 137)\underline{411+274-137} & \\
 & 3+\ \ 2-\ \ 1 & 2+5
\end{array}$$

$\therefore$ H. C. F. is $3x^2 + 2ax - a^2$.

$$\begin{aligned}
\text{1st expression} &= 2x(3x^2+2ax-a^2)(2x+5a).\\
\text{2d expression} &= a^3(3x^2+2ax-a^2)(5x^2+3a^2).\\
\therefore \text{ L. C. M.} &= 2a^3x(3x^2+2ax-a^2)(2x+5a)(5x^2+3a^2)\\
&= 60a^3x^6+190a^4x^5+116a^5x^4+64a^6x^3+48a^7x^2-30a^8x.
\end{aligned}$$

16. Find the L.C.M. of

$$2x^4+x^3-8x^2-x+6,\ 4x^4+12x^3-x^2-27x-18,$$
$$4x^4+4x^3-17x^2-9x+18.$$

$$\begin{array}{r|r|l}
2+1-8-1-6 & 4+4-17-\ \ 9+18 & 2\\
2-1-7+6 & 4+2-16\qquad -12 & \\
\hline
2-1-7-6 & 2-\ \ 1-\ \ 7+\ \ 6 & 1+1\\
\underline{2-1-7-6} & &
\end{array}$$

$$\therefore \text{ H.C.F.} = x+1.$$

$$\begin{aligned}
2x^4+x^3-8x^2-x+6 &= (2x^3-x^2-7x+6)(x+1)\\
&= \{(2x^3-2x^2)+(x^2-7x+6)\}(x+1)\\
&= (2x^2+x-6)(x-1)(x+1)\\
&= (2x-3)(x+2)(x-1)(x+1);\\
4x^4+4x^3-17x^2-9x+18 &= (2x-3)(x+2)(x-1)(2x+3);\\
4x^4+12x^3-x^2-27x-18 &= (x+2)(x+1)(2x+3)(2x-3).\\
\therefore \text{ L.C.M.} &= (2x-3)(x+2)(x-1)(x+1)(2x+3)\\
&= 4x^5+8x^4-13x^3-26x^2+9x+18.
\end{aligned}$$

Exercise 7.

1. Reduce to lowest terms $\dfrac{42a^3-30a^2x}{35ax^2-25x^3}$.

$$\frac{42a^3-30a^2x}{35ax^2-25x^3}=\frac{6a^2(7a-5x)}{5x^2(7a-5x)}=\frac{6a^2}{5x^2}.$$

2. Reduce to lowest terms $\dfrac{2x^3+5x^2-12x}{7x^3+25x^2-12x}$.

$$\frac{2x^3+5x^2-12x}{7x^3+25x^2-12x}=\frac{x(2x-3)(x+4)}{x(7x-3)(x+4)}=\frac{2x-3}{7x-3}.$$

3. Reduce to lowest terms $\dfrac{6a^2c^2 - 2a^4 + 18c^2 - 6a^2}{4a^4 + 2a^2c^2 + 12a^2 + 6c^2}$.

$$\begin{aligned}\frac{6a^2c^2 - 2a^4 + 18c^2 - 6a^2}{4a^4 + 2a^2c^2 + 12a^2 + 6c^2} &= \frac{2a^2(3c^2 - a^2) + 6(3c^2 - a^2)}{2a^2(2a^2 + c^2) + 6(2a^2 + c^2)}\\ &= \frac{(2a^2 + 6)(3c^2 - a^2)}{(2a^2 + 6)(2a^2 + c^2)}\\ &= \frac{3c^2 - a^2}{2a^2 + c^2}.\end{aligned}$$

4. Reduce to lowest terms $\dfrac{x^4 + (2b^2 - a^2)x^2 + b^4}{x^4 + 2ax^3 + a^2x^2 - b^4}$.

$$\begin{aligned}\frac{x^4 + (2b^2 - a^2)x^2 + b^4}{x^4 + 2ax^3 + a^2x^2 - b^4} &= \frac{x^4 + 2b^2x^2 + b^4 - a^2x^2}{x^2(x^2 + 2ax + a^2) - b^4}\\ &= \frac{(x^2 + b^2)^2 - (ax)^2}{\{x(x + a)\}^2 - (b^2)^2}\\ &= \frac{(x^2 + b^2 + ax)(x^2 + b^2 - ax)}{(x^2 + ax + b^2)(x^2 + ax - b^2)}\\ &= \frac{x^2 + b^2 - ax}{x^2 + ax - b^2}.\end{aligned}$$

5. Reduce to lowest terms $\dfrac{6x^5 - 9x^4 + 11x^3 + 6x^2 - 10x}{4x^6 + 10x^5 + 10x^4 + 4x^3 + 60x^2}$.

$$2x^2\underline{)4x^6 + 10x^5 + 10x^4 + 4x^3 + 60x^2} \qquad\qquad x\underline{)6x^5 - 9x^4 + 11x^3 + 6x^2 - 10x}$$
$$\quad 2x^4 + 5x^3 + 5x^2 + 2x + 30 \qquad\qquad\qquad 6x^4 - 9x^3 + 11x^2 + 6x - 10$$

$$\begin{array}{r|r|l}
2 + 5 + 5 + 2 + 30 & 6 - 9 + 11 + 6 - 10 & 3\\
3 & 6 + 15 + 15 + 6 + 90 & \\ \cline{1-2}
6 + 15 + 15 + 6 + 90 & -4)\underline{-24 - 4 + 0 - 100} & \\
6 + 1 + 0 + 25 & 6 + 1 + 0 + 25 & 1 + 7\\ \cline{1-1}
14 + 15 - 19 + 90 & 6 - 9 + 15 & \\ \cline{2-2}
3 & 10 - 15 + 25 & \\ \cline{1-1}
42 + 45 - 57 + 270 & 10 - 15 + 25 & \\ \cline{2-2}
42 + 7 + 0 + 175 & & \\ \cline{1-1}
19)\underline{38 - 57 + 95} & & \\
2 - 3 + 5 & & 3 + 5
\end{array}$$

$\therefore$ H.C.F. is $(2x^2 - 3x + 5)x$.

Hence,

$$6x^5 - 9x^4 + 11x^3 + 6x^2 - 10x = x(2x^2 - 3x + 5)(3x^2 - 2);$$

$$4x^6 + 10x^5 + 10x^4 + 4x^3 + 60x^2 = 2x^2(2x^2 - 3x + 5)(x^2 + 4x + 6).$$

$$\therefore \frac{6x^5 - 9x^4 + 11x^3 + 6x^2 - 10x}{4x^6 + 10x^5 + 10x^4 + 4x^3 + 60x^2} = \frac{3x^2 - 2}{2x(x^2 + 4x + 6)}.$$

6. Simplify $\frac{3x - 2y}{3} - \frac{4y + 2x}{5} + \frac{22y - 9x}{15}.$

L. C. D. $= 15.$

The multipliers are 5, 3, and 1.

$$\begin{aligned} 15x - 10 &= \text{first numerator.} \\ -6x - 12y &= \text{second numerator.} \\ \underline{-9x + 22y} &= \text{third numerator.} \\ 0 &= \text{sum of numerators.} \end{aligned}$$

$\therefore$ Sum of fractions $= 0.$

7. Simplify $\frac{2}{3a} - \frac{1}{2b} - \frac{2a + 3}{6a^2} + \frac{1}{2x^2} + \frac{3a - 2b}{6ab}.$

L. C. D. $= 6a^2bx^2.$

The multipliers are $2abx^2$, $3a^2x^2$, bx^2, $3a^2b$, and ax^2.

$$\begin{aligned} 4abx^2 &= \text{first numerator.} \\ -3a^2x^2 &= \text{second numerator.} \\ -(2a + 3)bx^2 &= \text{third numerator.} \\ 3a^2b &= \text{fourth numerator.} \\ \underline{(3a - 2b)ax^2} &= \text{fifth numerator.} \\ -3bx^2 + 3a^2b &= \text{sum of numerators} \\ &= 3b(a^2 - x^2). \end{aligned}$$

$$\therefore \text{Sum of fractions} = \frac{3b(a^2 - x^2)}{6a^2bx^2} = \frac{a^2 - x^2}{2a^2x^2}.$$

8. Simplify $\frac{3}{x - a} + \frac{4a}{(x - a)^2} - \frac{5a^2}{(x - a)^3}.$

L. C. D. $= (x - a)^3.$

The multipliers are $(x - a)^2$, $x - a$, and 1.

$$\begin{aligned}
3x^2-6ax+3a^2 &= \text{first numerator.}\\
4ax-4a^2 &= \text{second numerator.}\\
-5a^2 &= \text{third numerator.}\\
\hline
3x^2-2ax-6a^2 &= \text{sum of numerators.}
\end{aligned}$$

$$\therefore \text{ Sum of fractions} = \frac{3x^2-2ax-6a^2}{(x-a)^3}.$$

9. Simplify $\dfrac{a+b}{(b-c)(c-a)}+\dfrac{b+c}{(c-a)(a-b)}-\dfrac{a-c}{(a-b)(b-c)}$.

$$\text{L. C. D.} = (a-b)(b-c)(c-a).$$

The multipliers are $a-b$, $b-c$, and $c-a$.

$$\begin{aligned}
a^2-b^2 \qquad\qquad &= \text{first numerator.}\\
b^2-c^2 \qquad &= \text{second numerator.}\\
a^2 \qquad +c^2-2ac &= \text{third numerator.}\\
\hline
2a^2 \qquad\qquad -2ac &= \text{sum of numerators}\\
&= 2a(a-c).
\end{aligned}$$

$$\begin{aligned}
\therefore \text{ Sum of fractions} &= \frac{2a(a-c)}{(a-b)(b-c)(c-a)}.\\
&= \frac{-2a}{(a-b)(b-c)}.
\end{aligned}$$

10. Simplify $\dfrac{1}{a(a-b)(a-c)}+\dfrac{1}{b(b-c)(b-a)}+\dfrac{1}{c(c-a)(c-b)}$.

$$\text{L. C. D.} = abc(a-b)(a-c)(b-c).$$

The multipliers are $bc(b-c)$, $ac(c-a)$, and $ab(a-b)$.

$$\begin{aligned}
b^2c-bc^2 \qquad\qquad\qquad\qquad &= \text{first numerator.}\\
ac^2-a^2c \qquad\qquad &= \text{second numerator.}\\
a^2b-ab^2 &= \text{third numerator.}\\
\hline
b^2c-bc^2+ac^2-a^2c+a^2b-ab^2 &= \text{sum of numerators}\\
&= a^2b-ab^2+b^2c-a^2c+ac^2-bc^2\\
&= ab(a-b)+c(b^2-a^2)+c^2(a-b)\\
&= (ab-cb-ca+c^2)(a-b)\\
&= (a-c)(b-c)(a-b).
\end{aligned}$$

$$\begin{aligned}
\therefore \text{ Sum of fractions} &= \frac{(a-c)(b-c)(a-b)}{abc(a-b)(a-c)(b-c)}\\
&= \frac{1}{abc}.
\end{aligned}$$

11. Simplify

$$\begin{aligned}
&\frac{6x^2-17x+12}{12x^2-25x+12}+\frac{27x^2+18x-24}{12x^2+7x-12}+\frac{25x^2-25x+6}{20x^2-23x+6}\\
&\qquad=\frac{(3x-4)(2x-3)}{(3x-4)(4x-3)}+\frac{3(3x+4)(3x-2)}{(3x+4)(4x-3)}+\frac{(5x-2)(5x-3)}{(4x-3)(5x-2)}\\
&\qquad=\frac{2x-3}{4x-3}+\frac{3(3x-2)}{4x-3}+\frac{5x-3}{4x-3}\\
&\qquad=\frac{16x-12}{4x-3}\\
&\qquad=4.
\end{aligned}$$

12. Simplify $\frac{2a^8x^7}{3b^8}\times\frac{5a^4b^5}{4c^4x^6}\times\frac{15b^2c^2}{4a^9x}\div\frac{25a^4x}{18ab^2c^3}$

$$\begin{aligned}
&=\frac{150a^{12}b^7c^2x^7}{48a^9b^8c^4x^7}\div\frac{25a^4x}{18ab^2c^3}\\
&=\frac{25a^3}{8bc^2}\div\frac{25a^4x}{18ab^2c^3}\\
&=\frac{25a^3}{8bc^2}\times\frac{18ab^2c^3}{25a^4x}\\
&=\frac{9bc}{4x}.
\end{aligned}$$

13. Simplify $\left(\frac{x^4-y^4}{x^2-y^2}\div\frac{x+y}{x^2-xy}\right)\div\left(\frac{x^2+y^2}{x-y}\div\frac{x+y}{xy-y^2}\right)$

$$\begin{aligned}
&=\left((x^2+y^2)\times\frac{x^2-xy}{x+y}\right)\div\left(\frac{x^2+y^2}{x-y}\times\frac{xy-y^2}{x+y}\right)\\
&=\frac{(x^2+y^2)(x-y)x}{x+y}\div\frac{y(x^2+y^2)}{x+y}\\
&=\frac{(x^2+y^2)(x-y)x}{x+y}\times\frac{x+y}{y(x^2+y^2)}\\
&=\frac{x(x-y)}{y}.
\end{aligned}$$

14. Simplify $\left(\frac{a^2+b^2}{b}-a\right)\left(\frac{a^2-b^2}{a^3+b^3}\right)\div\left(\frac{1}{b}-\frac{1}{a}\right)$

$$\begin{aligned}
&=\frac{a^2+b^2-ab}{b}\times\frac{(a+b)(a-b)}{(a+b)(a^2-ab+b^2)}\div\frac{a-b}{ab}\\
&=\frac{a-b}{b}\div\frac{a-b}{ab}\\
&=\frac{a-b}{b}\times\frac{ab}{a-b}\\
&=a.
\end{aligned}$$

15. Simplify $\dfrac{x^2-7x+12}{x^2+5x+6}\times\dfrac{x^2+x-2}{x^2-5x+4}\times\dfrac{2x^2+5x-3}{3x^2-7x-6}$

$$=\frac{(x-3)(x-4)}{(x+3)(x+2)}\times\frac{(x-1)(x+2)}{(x-1)(x-4)}\times\frac{(2x-1)(x+3)}{(3x+2)(x-3)}$$

$$=\frac{2x-1}{3x+2}.$$

16. Simplify $\dfrac{6a^2-a-2}{8a^2-2a-3}\times\dfrac{8a^2-10a+3}{12a^2+a-6}\times\dfrac{12a^2+17a+6}{6a^2+a-2}$

$$=\frac{(3a-2)(2a+1)}{(4a-3)(2a+1)}\times\frac{(2a-1)(4a-3)}{(4a+3)(3a-2)}\times\frac{(4a+3)(3a+2)}{(3a+2)(2a-1)}$$

$$=1.$$

17. Simplify $\dfrac{\dfrac{2x+y}{y}-\dfrac{y}{2x+y}}{\dfrac{x}{x+y}-\dfrac{x+y}{x}}=\dfrac{\dfrac{(2x+y)^2-y^2}{y(2x+y)}}{\dfrac{x^2-(x+y)^2}{x(x+y)}}$

$$=\frac{4x^2+4xy}{y(2x+y)}\times\frac{x(x+y)}{-2xy-y^2}$$

$$=-\frac{4x(x+y)x(x+y)}{y(2x+y)y(2x+y)}$$

$$=-\frac{4x^2(x+y)^2}{y^2(2x+y)^2}.$$

18. Simplify $\dfrac{\dfrac{1+x}{1+x^2}-\dfrac{1+x^2}{1+x^3}}{\dfrac{1+x^2}{1+x^3}-\dfrac{1+x^3}{1+x^4}}=\dfrac{\dfrac{(1+x)(1+x^3)-(1+x^2)^2}{(1+x^2)(1+x^3)}}{\dfrac{(1+x^2)(1+x^4)-(1+x^3)^2}{(1+x^3)(1+x^4)}}$

$$=\frac{\dfrac{1+x+x^3+x^4-1-2x^2-x^4}{1+x^2}}{\dfrac{1+x^2+x^4+x^6-1-2x^3-x^6}{1+x^4}}$$

$$=\frac{x-2x^2+x^3}{1+x^2}\times\frac{1+x^4}{x^2-2x^3+x^4}$$

$$=\frac{1+x^4}{(1+x^2)x}.$$

19. Simplify $\dfrac{\left(a^2+\dfrac{b^4}{a^2-b^2}\right)(a^2+b^2)}{\dfrac{a}{a+b}+\dfrac{b}{a-b}} = \dfrac{\dfrac{(a^4-a^2b^2+b^4)(a^2+b^2)}{a^2-b^2}}{\dfrac{a^2-ab+ab+b^2}{a^2-b^2}}$

$$= \frac{(a^4-a^2b^2+b^4)(a^2+b^2)}{a^2-b^2} \times \frac{a^2-b^2}{a^2+b^2}$$

$$= a^4-a^2b^2+b^4$$

20. Simplify $\dfrac{\dfrac{64a^3-96a^2x+36ax^2}{36a^2-729x^2}}{\dfrac{48a^2-27x^2}{8a^2-72ax+162x^2}}$

$$= \frac{4a(16a^2-24ax+9x^2)}{9(4a^2-81x^2)} \times \frac{2(4a^2-36ax+81x^2)}{3(16a^2-9x^2)}$$

$$= \frac{4a(4a-3x)^2}{9(2a-9x)(2a+9x)} \times \frac{2(2a-9x)^2}{3(4a-3x)(4a+3x)}$$

$$= \frac{8a(4a-3x)(2a-9x)}{27(4a+3x)(2a+9x)}.$$

Exercise 8.

1. Solve $8(10-x) = 5(x+3).$

$$80-8x = 5x+15.$$
$$-13x = -65.$$
$$\therefore\ x = 5.$$

2. Solve $2x-3(2x-3) = 1-4(x-2).$

$$2x-6x+9 = 1-4x+8.$$
$$-4x+9 = -4x+9.$$

True for all values of x; the given equation is therefore an identical equation, and x may have any value (§ 79).

3. Solve $(x-5)(x+6) = (x-1)(x-2).$

$$x^2-5x+6x-30 = x^2-x-2x+2.$$
$$4x = 32.$$
$$\therefore\ x = 8.$$

4. Solve $(2x+3)(3x-2)=x^2+x(5x+3).$

$$6x^2+9x-4x-6=x^2+5x^2+3x.$$
$$2x=6.$$
$$\therefore\ x=3.$$

5. Solve $(x-3)(x+5)=(x+1)(2x-3)-x^2.$

$$x^2+2x-15=2x^2-x-3-x^2.$$
$$3x=12.$$
$$\therefore\ x=4.$$

6. Solve $(x+4)(x-2)=(x+3)(3x+4)-(2x+1)(x-6).$

$$x^2+2x-8=3x^2+13x+12-2x^2+11x+6.$$
$$-22x=26.$$
$$\therefore\ x=-\tfrac{13}{11}=-1\tfrac{2}{11}.$$

7. Solve $(x-3)(2x+5)=x(x+4)+(x+1)(x+3).$

$$2x^2-x-15=x^2+4x+x^2+4x+3.$$
$$-9x=18.$$
$$\therefore\ x=-2.$$

8. Solve $(x+2)^2+3x=(x-2)^2+5(16-x).$

$$x^2+4x+4+3x=x^2-4x+4+80-5x.$$
$$16x=80.$$
$$\therefore\ x=5.$$

9. Solve $(x-3)^2+(x-4)^2=(x-2)^2+(x+3)^2.$

$$x^2-6x+9+x^2-8x+16=x^2-4x+4+x^2+6x+9.$$
$$-16x=-12.$$
$$x=\tfrac{3}{4}.$$

10. Solve $\frac{3x}{5}-\frac{x}{6}=\frac{26}{15}.$

Multiply by 30,

$$18x-5x=52.$$
$$13x=52.$$
$$\therefore\ x=4.$$

11. Solve $\frac{x-2}{3x-5}=\frac{6}{19}.$

Multiply by $19(3x-5)$,

$$19x-38=18x-30.$$
$$\therefore\ x=8.$$

12. Solve $\frac{3x-5}{2x+10}=\frac{2}{3}$.

Multiply by $3(2x+10)$,

$$9x-15=4x+20.$$

$$5x=35.$$

$$\therefore\ x=7.$$

13. Solve $\frac{3(5x-3)}{2(4x+3)}=\frac{6}{5}$.

Multiply by $10(4x+3)$,

$$15(5x-3)=12(4x+3).$$

$$75x-45=48x+36.$$

$$27x=81.$$

$$\therefore\ x=3.$$

14. Solve $\frac{5x-6}{5}-\frac{3x}{4}=\frac{x-9}{10}$.

Multiply by 20,

$$20x-24-15x=2x-18.$$

$$3x=6.$$

$$\therefore\ x=2.$$

15. Solve $\frac{12-3x}{4}-\frac{3x-11}{3}=1$.

Multiply by 12,

$$36-9x-12x+44=12.$$

$$-21x=-68.$$

$$\therefore\ x=\tfrac{68}{21}=3\tfrac{5}{21}.$$

16. Solve $\frac{4x+17}{x+3}+\frac{3x-10}{x-4}=7$.

Multiply by $(x+3)(x-4)$,

$$(x-4)(4x+17)+(x+3)(3x-10)=7(x+3)(x-4).$$

$$4x^2+x-68+3x^2-x-30=7x^2-7x-84.$$

$$7x=14.$$

$$\therefore\ x=2.$$

17. Solve $\frac{x-3}{2x+1}+\frac{(2x-1)}{(4x-3)}=1$.

Multiply by $(4x-3)(2x+1)$,

$$(4x-3)(x-3)+(2x-1)(2x+1)=(4x-3)(2x+1).$$

$$4x^2-15x+9+4x^2-1=8x^2-2x-3.$$

$$-13x=-11.$$

$$\therefore\ x=\tfrac{11}{13}.$$

18. Solve $\frac{4x+3}{3x+4}-\frac{3x-4}{4x-3}=\frac{7}{12}$.

Multiply by $12(3x+4)(4x-3)$,

$$12(4x-3)(4x+3)-12(3x+4)(3x-4)=7(3x+4)(4x-3).$$

$$192x^2-108-108x^2+192=84x^2+49x-84.$$

$$-49x=-168.$$

$$\therefore\ x=\tfrac{24}{7}=3\tfrac{3}{7}.$$

19. Solve $\frac{6x+7}{3} - \frac{3}{x+2} = 2x + \frac{1}{2}$.

Multiply by 3, $6x + 7 - \frac{9}{x+2} = 6x + \frac{3}{2}$.

$$-\frac{9}{x+2} = -\frac{11}{2}.$$

$$\frac{9}{x+2} = \frac{11}{2}.$$

Multiply by $2(x+2)$, $18 = 11x + 22$.

$$11x = 4.$$

$$\therefore x = -\tfrac{4}{11}.$$

20. Solve $\frac{2x+1}{a+1} + \frac{2x}{a} = 5$.

Multiply by $(a+1)a$,

$$2ax + a + 2ax + 2x = 5a^2 + 5a.$$

$$4ax + 2x = 5a^2 + 4a.$$

$$(4a+2)x = 5a^2 + 4a.$$

$$\therefore x = \frac{5a^2 + 4a}{4a+2}.$$

21. Solve $\frac{ax-b}{c} - \frac{bx+c}{a} = abc$.

Multiply by ac,

$$a^2x - ab - bcx - c^2 = a^2bc^2.$$

$$(a^2 - bc)x = a^2bc^2 + ab + c^2.$$

$$\therefore x = \frac{a^2bc^2 + ab + c^2}{a^2 - bc}.$$

22. Solve $\frac{x+a}{3(x+b)} + \frac{x+b}{2(x+a)} = \frac{5}{6}$.

Multiply by $6(x+a)(x+b)$,

$$2(x+a)^2 + 3(x+b)^2 = 5(x+a)(x+b).$$

$$2x^2 + 4ax + 2a^2 + 3x^2 + 6bx + 3b^2 = 5x^2 + 5ax + 5bx + 5ab.$$

$$(-a+b)x = 5ab - 2a^2 - 3b^2.$$

$$(a-b)x = 2a^2 - 5ab + 3b^2.$$

$$\therefore\ x = \frac{2a^2 - 5ab + 3b^2}{a-b}.$$

$$x = \frac{(2a-3b)(a-b)}{a-b}.$$

$$x = 2a - 3b.$$

23. Solve $\dfrac{x-2a}{x+3a} - \dfrac{13a^2-2x^2}{x^2-9a^2} = 3.$

Multiply by $x^2 - 9a^2$,

$$(x-3a)(x-2a) - 13a^2 + 2x^2 = 3x^2 - 27a^2.$$

$$x^2 - 5ax + 6a^2 - 13a^2 + 2x^2 = 3x^2 - 27a^2.$$

$$-5ax = -20a^2.$$

$$\therefore\ x = 4a.$$

24. Solve $\quad \dfrac{a}{x} + \dfrac{x}{a} + \dfrac{a(x-a)}{x(x+a)} - \dfrac{x(x+a)}{a(x-a)} = \dfrac{ax}{a^2-x^2} - 2.$

Rearrange, $\quad \dfrac{a}{x} + \dfrac{a(x-a)}{x(x+a)} + \dfrac{x}{a} - \dfrac{x(x+a)}{a(x-a)} = \dfrac{ax}{a^2-x^2} - 2.$

$$\frac{a}{x}\left(1 + \frac{x-a}{x+a}\right) + \frac{x}{a}\left(1 - \frac{x+a}{x-a}\right) = \frac{ax}{a^2-x^2} - 2.$$

$$\frac{a}{x}\left(\frac{2x}{x+a}\right) - \frac{x}{a}\left(\frac{2a}{x-a}\right) = \frac{ax}{a^2-x^2} - 2.$$

$$\frac{2a}{x+a} - \frac{2x}{x-a} = \frac{ax}{a^2-x^2} - 2.$$

Multiply by $x^2 - a^2$,

$$2ax - 2a^2 - 2x^2 - 2ax = -ax - 2x^2 + 2a^2.$$

$$ax = 4a^2.$$

$$\therefore\ x = 4a.$$

Exercise 9.

1. The difference of two numbers is 3; and three times the greater number exceeds twice the less by 18. Find the numbers.

Let $\qquad x =$ the smaller number.

Then $\qquad x + 3 =$ the greater.

$3x + 9 =$ three times the greater.

$2x =$ twice the less.

$$\therefore\ 3x + 9 - 2x = 18.$$

$$\therefore\ x = 9.$$

$$x + 3 = 12.$$

$\therefore$ The numbers are 12 and 9.

2. If a certain number be increased by 16, the result is seven times the third part of the number. Find the given number.

Let $x =$ the number.

Then $x + 16 =$ the number increased by 16.

$\frac{x}{3} =$ the third part of the number.

$\frac{7x}{3} =$ seven times the third part of the number.

$$\therefore x + 16 = \frac{7x}{3}.$$

Multiply by 3, $3x + 48 = 7x.$

$$48 = 4x.$$

$$\therefore x = 12.$$

∴ The number is 12.

3. A boy was asked how many marbles he had. He replied, "If you take away 8 from twice the number I have, and divide the remainder by 3, the result is just one-half the number." How many marbles had he?

Let $x =$ the number of marbles he had.

Then $\frac{2x-8}{3} =$ one-third of twice the number less 8.

$\frac{x}{2} =$ one-half the number.

$$\therefore \frac{2x-8}{3} = \frac{x}{2}.$$

Multiply by 6, $4x - 16 = 3x.$

$$\therefore x = 16.$$

∴ He had 16 marbles.

4. The sum of the denominator and twice the numerator of a certain fraction is 26. If 3 be added to both numerator and denominator, the resulting fraction is $\frac{2}{3}$. Find the given fraction.

Let $x =$ the numerator.

Then $2x =$ twice the numerator.

$26 - 2x =$ the denominator.

$\frac{x}{26-2x} =$ the fraction.

$\frac{x+3}{29-2x} =$ the fraction after 3 is added to both numerator and denominator.

$$\therefore \frac{x+3}{29-2x} = \frac{2}{3}.$$

Clear of fractions, $3x + 9 = 58 - 4x.$

$$7x = 49.$$

$$\therefore\ x = 7.$$

$$26 - 2x = 12.$$

$\therefore$ The fraction is $\frac{7}{12}$.

5. A courier sent away with a despatch travels uniformly at the rate of 12 miles per hour; 2 hours after his departure a second courier starts to overtake the first, travelling uniformly at the rate of $13\frac{1}{2}$ miles per hour. In how many hours will the second courier overtake the first?

Let $x =$ the number of hours in which the second courier will overtake the first.

Then $x + 2 =$ the number of hours the first courier will have travelled when overtaken.

$12(x + 2) =$ the number of miles the first courier will have travelled.

$13\frac{1}{2}x =$ the number of miles the second courier will have travelled.

But these distances are equal.

$$\therefore\ 12(x + 2) = 13\tfrac{1}{2}x.$$

$$12x + 24 = 13\tfrac{1}{2}x.$$

$$24 = \tfrac{3}{2}x.$$

$$\therefore\ x = 16.$$

$\therefore$ The second courier will overtake the first in 16 hours.

6. Solve the above problem when the respective rates of the first and second couriers are a and b miles per hour, and the interval between their departures is c hours.

Let $x =$ the number of hours in which the second courier will overtake the first.

Then $x + c =$ the number of hours the first courier will have travelled.

$a(x + c) =$ the number of miles the first courier will have travelled.

$bx =$ the number of miles the second courier will have travelled.

But these distances are equal,

$$\therefore\ a(x+c)=bx.$$
$$ax+ac=bx.$$
$$(a-b)x=-ac.$$
$$(b-a)x=ac.$$
$$\therefore\ x=\frac{ac}{b-a}.$$

∴ The second courier will overtake the first in $\frac{ac}{b-a}$ hours.

7. A certain railroad train travels at a uniform rate. If the rate were 6 miles per hour faster, the distance travelled in 8 hours would exceed by 50 miles the distance travelled in 11 hours at a rate 7 miles per hour less than the actual rate. Find the actual rate of the train.

Let $x=$ the rate of the train per hour.
Then $x+6=$ its rate if it went 6 miles an hour faster.
$8(x+6)=$ the number of miles it would pass over at the increased rate in 8 hours.
$x-7=$ the rate if it went 7 miles an hour slower.
$11(x-7)=$ the number of miles it would pass over at the diminished rate in 11 hours.

$$\therefore\ 8(x+6)=11(x-7)+50.$$
$$8x+48=11x-77+50.$$
$$-3x=-75.$$
$$\therefore\ x=25.$$

∴ The rate of the train is 25 miles an hour.

8. A can do a piece of work in 10 days; A and B together can do it in 7 days. In how many days can B do it alone?

Let $x=$ the number of days in which B can do it alone.
Then $\frac{1}{x}=$ the part B can do in one day alone.
But $\frac{1}{10}=$ the part A can do in one day alone.
$\therefore\ \frac{1}{10}+\frac{1}{x}=$ the part A and B can do in one day together.
$\frac{1}{\frac{1}{10}+\frac{1}{x}}=$ the number of days in which A and B can do it together.

$$\therefore \frac{1}{\frac{1}{10}+\frac{1}{x}} = 7.$$

Reduce the complex to a simple fraction,

$$\frac{10x}{x+10} = 7.$$

Clear of fractions,

$$10x = 7x + 70.$$
$$3x = 70.$$
$$x = \tfrac{70}{3} = 23\tfrac{1}{3}.$$

$\therefore$ B can do the work alone in $23\tfrac{1}{3}$ days.

9. A can do a piece of work in a days; A and B together can do it in b days. In how many days can B do it alone?

Let $x =$ the number of days in which B can do it alone.

Then $\frac{1}{x} =$ the part B can do in one day alone.

But $\frac{1}{a} =$ the part A can do in one day alone.

$\therefore \frac{1}{x}+\frac{1}{a} =$ the part A and B can do in one day together.

$\frac{1}{\frac{1}{x}+\frac{1}{a}} =$ the number of days in which A and B can do it together.

$$\therefore \frac{1}{\frac{1}{x}+\frac{1}{a}} = b$$

Reduce the complex to a simple fraction.

$$\frac{ax}{a+x} = b.$$

Clear of fractions,

$$ax = ab + bx.$$
$$(a-b)x = ab.$$
$$\therefore x = \frac{ab}{a-b}.$$

$\therefore$ B can do the work alone in $\frac{ab}{a-b}$ days.

10. If A can do a piece of work in $2m$ days, B and A together in n days, and A and C in $m+\frac{n}{2}$ days, how long will it take them to do the work together?

Let $x =$ the number of days in which A, B, and C can do it together.

Then $\frac{1}{x} =$ the part they can do in one day together.

But $\frac{1}{2m} =$ the part A can do alone in one day.

$\frac{1}{m+\frac{n}{2}} =$ the part A and C can do in one day together.

$\therefore \frac{1}{m+\frac{n}{2}} - \frac{1}{2m} =$ the part C can do alone in one day.

Also $\frac{1}{n} =$ the part A and B can do in one day together.

$\therefore \frac{1}{m+\frac{n}{2}} - \frac{1}{2m} + \frac{1}{n} =$ the part A, B, and C can do in one day together.

$$\therefore \frac{1}{x} = \frac{1}{m+\frac{n}{2}} - \frac{1}{2m} + \frac{1}{n}.$$

$$\frac{1}{x} = \frac{2}{2m+n} - \frac{1}{2m} + \frac{1}{n}.$$

$$\frac{1}{x} = \frac{4mn - 2mn - n^2 + 4m^2 + 2mn}{2mn(2m+n)}.$$

$$\frac{1}{x} = \frac{4m^2 + 4mn - n^2}{2mn(2m+n)}.$$

$$\therefore x = \frac{2mn(2m+n)}{4m^2 + 4mn - n^2}.$$

$\therefore$ They can do the work together in $\frac{2mn(2m+n)}{4m^2+4mn-n^2}$ days.

11. A boatman moves 5 miles in ¾ of an hour, rowing with the tide; to return it takes him 1½ hours, rowing against a tide one-half as strong. What is the velocity of the stronger tide?

Let $x =$ the velocity of the stronger tide in miles per hour.

Then $\frac{3x}{4} =$ the number of miles the boat is carried by the stronger tide in $\frac{3}{4}$ of an hour.

$5 - \frac{3x}{4} =$ the number of miles the man himself rows in $\frac{3}{4}$ of an hour.

$\frac{4}{3}\left(5 - \frac{3x}{4}\right) =$ the number of miles he rows in one hour.

$\frac{x}{2} =$ the velocity of the weaker tide in miles per hour.

$\frac{3}{2} \times \frac{x}{2} =$ the number of miles the boat is carried backward by this tide in $\frac{3}{2}$ hours.

$5 + \frac{3}{2} \times \frac{x}{2} =$ the number of miles the man rows in $1\frac{1}{2}$ hours.

$\frac{2}{3}\left(5 + \frac{3}{2} \times \frac{x}{2}\right) =$ the number of miles he rows in one hour.

$$\therefore \frac{2}{3}\left(5 + \frac{3}{2} \times \frac{x}{2}\right) = \frac{4}{3}\left(5 - \frac{3x}{4}\right).$$

$$\frac{10}{3} + \frac{x}{2} = \frac{20}{3} - x.$$

$$\frac{3x}{2} = \frac{10}{3}.$$

$$\therefore x = \tfrac{20}{9} = 2\tfrac{2}{9}.$$

$\therefore$ The velocity of the stronger tide is $2\frac{2}{9}$ miles per hour.

12. A boatman, rowing with the tide, moves a miles in b hours. Returning, it takes him c hours to accomplish the same distance, rowing against a tide m times as strong as the first. What is the velocity of the stronger tide?

Let $x =$ the velocity of the weaker tide in miles per hour.

Then $bx =$ the number of miles the boat is carried by the tide in b hours.

$a - bx =$ the number of miles the man himself rows in b hours.

$\frac{a - bx}{b} =$ the number of miles he rows an hour.

$mx =$ the velocity of the stronger tide in miles per hour.

$cmx =$ the number of miles the boat is carried backward by this tide in c hours.

$a + cmx =$ the number of miles the man rows in c hours.

$$\frac{a+cmx}{c} = \text{the number of miles he rows in one hour.}$$

$$\therefore \frac{a+cmx}{c} = \frac{a-bx}{b}.$$

Clear of fractions,

$$ab + bcmx = ac - bcx.$$

$$bc\,(m+1)\,x = a\,(c-b).$$

$$\therefore x = \frac{a(c-b)}{bc(m+1)}.$$

$$mx = \frac{ma\,(c-b)}{bc\,(m+1)}.$$

$\therefore$ The velocity of the stronger tide is $\frac{ma(c-b)}{bc(m+1)}$ miles an hour.

18. If A, who is travelling, makes $\frac{1}{2}$ of a mile more per hour, he will be on the road only $\frac{4}{5}$ of the time; but if he makes $\frac{1}{2}$ of a mile less per hour, he will be on the road $2\frac{1}{2}$ hours more. Find the distance and the rate.

Let $x =$ his rate in miles per hour.

Then, $x + \frac{1}{2} =$ his rate if he travelled $\frac{1}{2}$ a mile per hour faster.

But, since he will be on the road only $\frac{4}{5}$ as long at the second rate as at the first, he must travel $\frac{5}{4}$ as fast.

$$\therefore x + \tfrac{1}{2} = \tfrac{5}{4}x.$$

$$\frac{1}{2} = \frac{x}{4}.$$

$$x = 2.$$

$\therefore$ His rate is 2 miles an hour.

Now let $y =$ the distance in miles.

Then $\frac{y}{2} =$ the number of hours it will take at 2 miles an hour,

$\frac{y}{\frac{3}{2}} =$ the number of hours it will take at $\frac{3}{2}$ miles an hour.

$$\therefore \frac{y}{\frac{3}{2}} = \frac{y}{2} + 2\tfrac{1}{2}.$$

$$\frac{2y}{3} = \frac{y}{2} + \frac{5}{2}.$$

$$\frac{y}{6} = \frac{5}{2}.$$

$$\therefore y = 15.$$

$\therefore$ The distance is 15 miles.

14. The circumference of a fore wheel of a carriage is a feet; that of a hind wheel, b feet. What distance will the carriage have passed over, when a fore wheel has made n more revolutions than a hind wheel?

Let $x =$ the number of feet in the required distance.

Then $\frac{x}{a} =$ the number of revolutions the fore wheel will have made.

$\frac{x}{b} =$ the number of revolutions the hind wheel will have made.

$$\therefore \frac{x}{a} - \frac{x}{b} = n.$$

$$x\frac{b-a}{ab} = n.$$

$$\therefore x = \frac{nab}{b-a}.$$

$\therefore$ The carriage will have gone $\frac{nab}{b-a}$ feet.

15. A wine merchant has two kinds of wine which he sells, one at a dollars, and the other at b dollars per gallon. He wishes to make a mixture of l gallons, which shall cost him on the average m dollars a gallon. How many gallons must he take of each?

Discuss the question (i.) when $a = b$; (ii.) when a or $b = m$; (iii.) when $a = b = m$; (iv.) when $a = b$ and $< m$; (v.) when $a > b$ and $b > m$.

Let $x =$ the number of gallons of the first kind.

Then $l - x =$ the number of gallons of the second kind.

$ax =$ the price of the x gallons of the first kind.

$b(l - x) =$ the price of the $l - x$ gallons of the second kind.

$ax + b(l - x) =$ the price of the whole mixture.

$\frac{ax + b(l-x)}{l} =$ the price of the mixture per gallon

$$\therefore \frac{ax + b(l-x)}{l} = m.$$

$$ax + bl - bx = ml.$$

$$(a-b)x = l(m-b).$$

$$\therefore x = \frac{l(m-b)}{a-b}.$$

$$l - x = \frac{la - lb - lm + lb}{a-b} = \frac{l(a-m)}{a-b}.$$

∴ He must take $\frac{l(m-b)}{a-b}$ gallons of the first kind, and $\frac{l(a-m)}{a-b}$ gallons of the second kind.

(i.) If $a=b$, and a and b are not equal to m, the two kinds of wine cost the same price per gallon, namely, a dollars, and *any* mixture of the two kinds will cost a dollars a gallon, and no mixture can be made to cost m dollars a gallon, since m is not equal to a. The values of x and $l-x$ in this case are each $=\frac{\text{a finite quantity}}{0}$, which is simply an algebraic statement that the problem is impossible.

(ii.) If $a=m$, $x=l$, *i.e.* all the wine must be of the first kind, since any admixture with the second kind would change the price. Similarly, if $b=m$, the wine must all be of the second kind.

(iii.) If $a=b=m$, the equation

$$\frac{ax+b(l-x)}{l}=m \text{ becomes } \frac{a(x+l-x)}{l}=m,$$

or $a=m$, and x is not determined. In this case he can evidently take *any* part from the one kind, and the rest from the other, since the prices of both, and that of *any* mixture, are the same per gallon.

(iv.) $a>b$ and $<m$. The price per gallon of any mixture must evidently lie between a and b. But in this case m is greater than either a or b. Hence no solution is possible. Algebraically, $l-x$ or $\frac{l(a-m)}{a-b}$ is negative, since $a-m$ is negative. x is therefore greater than l, which cannot be, since the part must be less than the whole.

(v.) $a>b$ and $b>m$. Here m is less than a or b, which cannot be. Algebraically, x or $\frac{l(m-b)}{a-b}$ is negative, since $m-b$ is negative, so that $l-x$ is greater than l, which is impossible.

Exercise 10.

1. Solve $6x+5y=46.$ (1)

$10x+3y=66.$ (2)

Multiply (1) by 5 and (2) by 3,

$$\begin{aligned}30x+25y&=230\\ 30x+\ \ 9y&=198\end{aligned}$$

Subtract, $16y=\ 32$

$\therefore\ y=2.$

Substitute value of y in (1),

$6x+10=46.$

$\therefore\ x=6.$

2. Solve $2x+7y=52.$ (1)

$3x-5y=16.$ (2)

Multiply (1) by 3 and (2) by 2,

$$\begin{aligned}6x+21y&=156\\ 6x-10y&=\ \ 32\end{aligned}$$

Subtract, $31y=124$

$\therefore\ y=4.$

Substitute value of y in (1),

$2x+28=52.$

$\therefore\ x=12.$

3. Solve $4x+\ 9y=79.$ (1)

$7x-17y=40.$ (2)

Multiply (1) by 7 and (2) by 4.

$$\begin{aligned}28x+63y&=553\\ 28x-68y&=160\end{aligned}$$

Subtract, $131y=393$

$\therefore\ y=3.$

Substitute value of y in (1).

$4x+27=79.$

$\therefore\ x=13.$

4. Solve $2x-7y=\ 8.$ (1)

$4y-9x=19.$ (2)

Multiply (1) by 4 and (2) by 7,

$$\begin{aligned}8x-28y&=\ \ 32\\ -63x+28y&=133\end{aligned}$$

Add, $-55x\ \ =165$

$\therefore\ x=-3.$

Substitute value of x in (1),

$-6-7y=8.$

$\therefore\ y=-2.$

5. Solve $x=16-4y.$ (1)

$y=34-4x.$ (2)

Substitute value of y from (2) in (1), $x=16-4(34-4x).$

$x=16-136+16x.$

$-15x=-120.$

$\therefore\ x=8.$

Substitute value of x in (2),

$y=34-32.$

$\therefore\ y=2.$

6. $5x=2y+\ 78.$ (1)

$3y=\ x+104.$ (2)

Transpose 104 in (2),

$3y-104=x.$

Substitute value of x in (1),

$5(3y-104)=2y+78.$

$15y-520=2y+78.$

$13y=598.$

$\therefore\ y=46.$

Substitute value of y in (1),

$5x=92+78.$

$\therefore\ x=34.$

7. Solve

$$\frac{2x}{3}+\frac{y}{2}=10. \quad (1)$$

$$\frac{y}{4}=\frac{5x-7}{19}. \quad (2)$$

Multiply (2) by 2.

$$\frac{y}{2}=\frac{10x-14}{19}.$$

Substitute value of $\frac{y}{2}$ in (1),

$$\frac{2x}{3}+\frac{10x-14}{19}=10.$$

Simplify,

$$38x+30x-42=570.$$

$$\therefore\ x=9.$$

Substitute value of x in (2),

$$\frac{y}{4}=\frac{45-7}{19}.$$

$$\therefore\ y=8.$$

8. Solve $4+y=\frac{3x}{4}.\quad(1)$

$$x-8=\frac{4y}{5}.\quad(2)$$

Transpose 8 in (2),

$$x=8+\frac{4y}{5}.$$

Substitute value of x in (1),

$$4+y=6+\frac{3y}{5}.$$

Simplify,

$$20+5y=30+3y.$$

$$\therefore\ y=5.$$

Substitute value of y in (2),

$$x=12.$$

9. Solve $\frac{x+y}{3}+x=15.\quad(1)$

$$\frac{x-y}{5}+y=6.\quad(2)$$

Simplify (1) and (2),

$$x+y+3x=45.$$

$$4x+y=45.\quad(3)$$

$$x-y+5y=30.$$

$$x+4y=30\quad(4)$$

4 × (3) is $16x+4y=180$

Subtract, $-15x=-150$

$$\therefore\ x=10.$$

Substitute value of x in (3),

$$y=5.$$

10. Solve

$$\frac{x-1}{8}+\frac{y-2}{5}=2.\quad(1)$$

$$\frac{2x}{7}+\frac{2y-5}{21}=3.\quad(2)$$

Simplify (1) and (2),

$$5x-5+8y-16=80.$$

$$5x+8y=101.\quad(3)$$

$$6x+2y-5=63.$$

$$6x+2y=68.\quad(4)$$

(3) is $5x+8y=101$

4 × (4) is $24x+8y=272$

Subtract, $-19x=-171$

$$\therefore\ x=9.$$

Substitute value of x in (4),

$$y=7.$$

11. Solve $\frac{3}{x}+\frac{8}{y}=3.\quad(1)$

$$\frac{15}{x}-\frac{4}{y}=4.\quad(2)$$

2 × (2) is $\frac{30}{x}-\frac{8}{y}=8$

(1) is $\frac{3}{x}+\frac{8}{y}=3$

Add, $\frac{33}{x}=11$

$$\therefore\ x=3.$$

Substitute value of x in (1),

$$y=4.$$

12. Solve
$$\frac{4}{5x}+\frac{5}{6y}=\frac{86}{15}. \qquad (1)$$
$$\frac{5}{4x}-\frac{4}{5y}=\frac{11}{20}. \qquad (2)$$

Multiply (1) by $\frac{5}{4}$ and (2) by $\frac{4}{5}$,
$$\frac{1}{x}+\frac{25}{24y}=\frac{86}{12}$$
$$\frac{1}{x}-\frac{16}{25y}=\frac{11}{25}$$

Subtract,
$$\frac{1009}{600y}=\frac{2018}{300}$$
$$\therefore y=\tfrac{1}{4}.$$

Substitute value of y in (2),
$$x=\tfrac{1}{3}.$$

13. Solve
$$\frac{x-2}{5}-\frac{10-x}{3}=\frac{y-10}{4}. \qquad (1)$$
$$\frac{2y+4}{3}=\frac{4x+y+13}{8}. \qquad (2)$$

Simplify (1), $32x-15y=74. \qquad (3)$

Simplify (2), $12x-13y=-7. \qquad (4)$

$3\times(3)$ is $96x-45y=222$

$8\times(4)$ is $96x-104y=-56$

Subtract,
$$59y=278$$
$$\therefore y=4\tfrac{42}{59}.$$

Substitute value of y in (4),
$$12x-\frac{13\times 278}{59}=-7.$$
$$12x=\tfrac{3201}{59}.$$
$$\therefore x=4\tfrac{123}{236}.$$

14. Solve
$$x-\frac{2y-x}{23-x}=20+\frac{2x-59}{2}. \qquad (1)$$
$$y-\frac{y-3}{x-18}=30-\frac{73-3y}{3}. \qquad (2)$$

(1) may be written,
$$x-\frac{2y-x}{23-x}=20+x-\frac{59}{2}.$$

Transpose and combine,
$$-\frac{2y-x}{23-x}=-\frac{19}{2}.$$

Simplify,
$$4y-2x=437-19x.$$
$$17x+4y=437. \qquad (3)$$

(2) may be written, $y - \frac{y-3}{x-18} = 30 - \frac{73}{3} + y.$

Transpose and combine, $-\frac{y-3}{x-18} = \frac{17}{3}.$

Simplify, $-3y + 9 = 17x - 306.$

$17x + 3y = 315. \qquad (4)$

Subtract (4) from (3), $y = 122.$

Substitute value of y in (4), $17x + 366 = 315.$

$\therefore x = -3.$

15. Solve $2x - 3y = 5a - b. \qquad (1)$

$3x - 2y = 5a + b. \qquad (2)$

$3 \times (1)$ is $6x - 9y = 15a - 3b$

$2 \times (2)$ is $6x - 4y = 10a + 2b$

Subtract, $-5y = 5a - 5b$

$\therefore y = -a + b.$

Substitute value of y in (2),

$$3x - 2a - 2b = 5a + b.$$
$$3x = 3a + 3b.$$
$$\therefore x = a + b.$$

16. Solve $\frac{x}{a} + \frac{y}{b} = 1 - \frac{x}{c}. \qquad (1)$

$\frac{x}{b} + \frac{y}{a} = 1 + \frac{y}{c}. \qquad (2)$

Simplify (1), $bcx + acy = abc - abx.$

Transpose and combine, $(bc + ab)x + acy = abc. \qquad (3)$

Simplify (2), $acx + bcy = abc + aby.$

Transpose and combine, $acx + (bc - ab)y = abc. \qquad (4)$

$ac \times (3)$ is $ac(bc + ab)x + a^2c^2y = a^2bc^2$

$(bc + ab) \times (4)$ is $ac(bc + ab)x + (b^2c^2 - a^2b^2)y = ab^2c^2 + a^2b^2c$

Subtract, $(a^2c^2 - b^2c^2 + a^2b^2)y = a^2bc^2 - ab^2c^2 - a^2b^2c$

$$\therefore y = \frac{a^2bc^2 - ab^2c^2 - a^2b^2c}{a^2c^2 - b^2c^2 + a^2b^2} = \frac{abc(bc + ab - ac)}{b^2c^2 - a^2b^2 - a^2c^2}.$$

Substitute value of y in (3),

$$(bc + ab)x + \frac{a^3bc^3 - a^2b^2c^3 - a^3b^2c^2}{a^2c^2 - b^2c^2 + a^2b^2} = abc.$$

$$(bc + ab)x = \frac{a^3bc^3 - ab^3c^3 + a^3b^3c - a^3bc^3 + a^2b^2c^3 + a^3b^2c^2}{a^2c^2 - b^2c^2 + a^2b^2}.$$

$$(bc + ab)x = \frac{a^2b^2c^3 + a^3b^2c^2 - ab^3c^3 + a^3b^3c}{a^2c^2 - b^2c^2 + a^2b^2}.$$

$$(bc + ab)x = \frac{a^2bc^2(bc + ab) - abc(b^2c^2 - a^2b^2)}{a^2c^2 - b^2c^2 + a^2b^2}.$$

$$\therefore\ x = \frac{a^2bc^2 - abc(bc - ab)}{a^2c^2 - b^2c^2 + a^2b^2}.$$

$$x = \frac{a^2bc^2 - ab^2c^2 + a^2b^2c}{a^2c^2 - b^2c^2 + a^2b^2} = \frac{abc(bc - ab - ac)}{b^2c^2 - a^2b^2 - a^2c^2}.$$

17. Solve
$$\frac{x + y}{x - y} = \frac{a}{b - c}. \quad (1)$$
$$\frac{x + c}{y + b} = \frac{a + b}{a + c}. \quad (2)$$

Simplify (1),
$$(b - c)\,x + (b - c)\,y = ax - ay.$$
$$(b - c - a)\,x + (b - c + a)\,y = 0. \quad (3)$$

Simplify (2), $(a + c)\,x + ac + c^2 = (a + b)\,y + b^2 + ab.$
$$(a + c)\,x - (a + b)\,y = b^2 + ab - ac - c^2. \quad (4)$$

Add (3) and (4), $bx - cy = b^2 + ab - ac - c^2. \quad (5)$

From (3), $(b - c - a)\,x = (c - b - a)\,y.$
$$\therefore\ x = \frac{c - b - a}{b - c - a}\,y.$$

Substitute value of x in (5),
$$\frac{b(c - b - a)}{b - c - a}\,y - cy = b^2 + ab - ac - c^2.$$
$$b(c - b - a)y - c(b - c - a)y = (b - c - a)(b^2 + ab - ac - c^2).$$
$$(bc - b^2 - ab - bc + c^2 + ac)y = (b - c - a)(b^2 + ab - ac - c^2).$$
$$(-b^2 - ab + ac + c^2)y = (b - c - a)(b^2 + ab - ac - c^2).$$
$$\therefore\ y = -(b - c - a) = a - b + c.$$

Substitute value of y in (3),
$$(b - c - a)\,x + (b - c - a)(a + c - b) = 0.$$
$$\therefore\ x = b - c + a = a + b - c.$$

18. Solve
$$\frac{x - a}{y - a} = \frac{a - b}{a + b}. \quad (1)$$
$$\frac{x}{y} = \frac{a^3 - b^3}{a^3 + b^3}.$$

Simplify (1), $(a + b)\,x - a^2 - ab = (a - b)\,y - a^2 + ab.$
$$(a + b)\,x - (a - b)\,y = 2\,ab. \quad (3)$$

Simplify (2),

$$(a^3 + b^3)\,x - (a^3 - b^3)\,y = 0 \qquad (4)$$

$(a^2 - ab + b^2) \times (3)$ is

$$\underline{(a^3 + b^3)\,x - (a^3 - 2a^2b + 2ab^2 - b^3)\,y = 2ab(a^2 - ab + b^2)}$$

Subtract,

$$(2a^2b - 2ab^2)\,y = 2ab(a^2 - ab + b^2)$$

$$2ab\,(a - b)\,y = 2ab(a^2 - ab + b^2).$$

$$\therefore\ y = \frac{a^2 - ab + b^2}{a - b}.$$

Substitute value of y in (3),

$$(a + b)\,x - a^2 + ab - b^2 = 2ab.$$

$$(a + b)\,x = a^2 + ab + b^2.$$

$$\therefore\ x = \frac{a^2 + ab + b^2}{a + b}.$$

19. Solve

$$8x + 4y - 3z = 6. \qquad (1)$$

$$x + 3y - z = 7. \qquad (2)$$

$$4x - 5y + 4z = 8. \qquad (3)$$

(1) is $8x + 4y - 3z = 6$

$3 \times (2)$ is $3x + 9y - 3z = 21$

Subtract, $5x - 5y = -15$

Divide by 5, $x - y = -3. \qquad (4)$

(3) is $4x - 5y + 4z = 8$

$4 \times (2)$ is $4x + 12y - 4z = 28$

Add, $8x + 7y = 36 \qquad (5)$

$7 \times (4)$ is $7x - 7y = -21$

Add, $15x = 15$

$$\therefore\ x = 1.$$

Substitute value of x in (4), $1 - y = -3.$

$$\therefore\ y = 4.$$

Substitute values of x and y in (2),

$$1 + 12 - z = 7.$$

$$\therefore\ z = 6.$$

20. Solve

$$3x - \frac{y}{4} + z = 7\tfrac{1}{2}. \qquad (1)$$

$$2x - \frac{y - 3z}{3} = 5\tfrac{1}{3}. \qquad (2)$$

$$2x - \frac{y}{2} + 4z = 11. \qquad (3)$$

(2) is $2x-\frac{y}{3}+z=5\frac{1}{3}$.

(1) is $3x-\frac{y}{4}+z=7\frac{1}{2}$

Subtract, $x+\frac{y}{12}=2\frac{1}{6}$ (4)

(3) is $2x-\frac{y}{2}+4z=11$

$4\times(1)$ is $12x-y+4z=30$

Subtract, $10x-\frac{y}{2}=19$ (5)

$6\times(4)$ is $6x+\frac{y}{2}=13$

Add, $16x=32$

$\therefore x=2.$

Substitute value of x in (5), $20-\frac{y}{2}=19.$

$\therefore y=2.$

Substitute values of x and y in (1),

$6-\frac{1}{2}+z=7\frac{1}{2}.$

$\therefore z=2.$

21. Solve $\frac{3y-2x}{3z-7}=\frac{1}{2}.$ (1)

$\frac{5z-x}{2y-3z}=1.$ (2)

$\frac{y-2z}{3y-2x}=1.$ (3)

Simplify (1),

$4x-6y+3z=7.$ (4)

Simplify (2),

$x+2y-8z=0$ (5)

Simplify (3),

$x-y-z=0.$ (6)

(4) is $4x-6y+3z=7$

$4\times(6)$ is $4x-4y-4z=0$

Subtract, $-2y+7z=7$ (7)

(5) is $x+2y-8z=0$

(6) is $x-y-z=0$

Subtract, $3y-7z=0$ (8)

(7) is $-2y+7z=7$

Add, $y=7$

Substitute value of y in (8),

$z=3.$

Substitute value of y and z in (6),

$x-3-7=0.$

$\therefore x=10.$

22. $$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=3. \quad (1)$$

$$\frac{a}{x}+\frac{b}{y}-\frac{c}{z}=1. \quad (2)$$

$$\frac{2a}{x}=\frac{b}{y}+\frac{c}{z}. (3)$$

Substitute value of $\frac{b}{y}+\frac{c}{z}$ from (3) in (1),

$$\frac{a}{x}+\frac{2a}{x}=3.$$

$$\frac{3a}{x}=3.$$

$$\therefore\ x=a.$$

Subtract (2) from (1),

$$\frac{2c}{z}=2.$$

$$\therefore\ z=c.$$

Substitute values of x and z in (1),

$$1+\frac{b}{y}+1=3.$$

$$\frac{b}{y}=1.$$

$$\therefore\ y=b.$$

23. $$\frac{xy}{x+y}=a. \quad (1)$$

$$\frac{xz}{x+z}=b. \quad (2)$$

$$\frac{yz}{y+z}=c. \quad (3)$$

The reciprocals of these equations are:

$$\frac{x+y}{xy}=\frac{1}{a},$$

$$\frac{x+z}{xz}=\frac{1}{b},$$

$$\frac{y+z}{yz}=\frac{1}{c},$$

or $$\frac{1}{x}+\frac{1}{y}=\frac{1}{a}, \quad (4)$$

$$\frac{1}{x}+\frac{1}{z}=\frac{1}{b}, \quad (5)$$

$$\frac{1}{y}+\frac{1}{z}=\frac{1}{c}. \quad (6)$$

Add (4), (5), and (6) together.

$$\frac{2}{x}+\frac{2}{y}+\frac{2}{z}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \quad (7)$$

$2\times(4)$ is $\frac{2}{x}+\frac{2}{y} \qquad =\frac{2}{a}$

Subtract, $$\frac{2}{z}=\frac{1}{b}+\frac{1}{c}-\frac{1}{a}$$

$$\frac{2}{z}=\frac{ac+ab-bc}{abc}.$$

$$\therefore\ z=\frac{2abc}{ac+ab-bc}.$$

Subtract $2\times(5)$ from (7),

$$\frac{2}{y}=\frac{1}{a}+\frac{1}{c}-\frac{1}{b}.$$

$$\frac{2}{y}=\frac{bc+ab-ac}{abc}.$$

$$\therefore\ y=\frac{2abc}{ab+bc-ac}.$$

Subtract $2\times(6)$ from (7),

$$\frac{2}{x}=\frac{1}{a}+\frac{1}{b}-\frac{1}{c}.$$

$$\frac{2}{x}=\frac{bc+ac-ab}{abc}.$$

$$\therefore\ x=\frac{2abc}{ac+bc-ab}.$$

Exercise 11.

1. Three times the greater of two numbers exceeds twice the less by 27; and the sum of twice the greater and five times the less is 94. Find the numbers.

Let $x =$ the greater number,
and $y =$ the less.
Then $3x - 2y = 27.$ (1)
$2x + 5y = 94.$ (2)

Multiply (1) by 2, and (2) by 3,

$$\begin{aligned} 6x - 4y &= 54 \\ \underline{6x + 15y} &\underline{= 282} \end{aligned}$$

Subtract, $19y = 228$
$\therefore\ y = 12.$

Substitute value of y in (1), $3x - 24 = 27.$
$\therefore\ x = 17.$

$\therefore$ The numbers are 17 and 12.

2. A fraction is such that if 3 be added to each of its terms, the resulting fraction is equal to $\frac{4}{5}$; and if 3 be subtracted from each of its terms, the result is equal to $\frac{1}{2}$. Find the fraction.

Let $x =$ the numerator,
and $y =$ the denominator.
Then $\frac{x}{y} =$ the fraction.

$$\therefore\ \frac{x+3}{y+3} = \frac{4}{5}. \quad (1)$$

$$\frac{x-3}{y-3} = \frac{1}{2}. \quad (2)$$

Simplify (1), $5x + 15 = 4y + 12.$
$5x - 4y = -3.$ (3)

Simplify (2), $2x - 6 = y - 3.$
$2x - y = 3.$ (4)

(3) is $5x - 4y = -3$
$4 \times (4)$ is $\underline{8x - 4y = 12}$
Subtract, $3x = 15$
$\therefore\ x = 5.$

Substitute value of x in (4), $10 - y = 3.$
$\therefore\ y = 7.$

$\therefore$ The fraction is $\frac{5}{7}$.

Substitute value of y in (3),

$$-5x + 35 = 0.$$
$$5x = 35.$$
$$\therefore x = 7.$$

$\therefore$ The number is 75.

7. A number is formed by two digits of which the sum is 8. If the digits be interchanged, 4 times the new number exceeds the original number by 2 more than 5 times the sum of the digits. Find the original number.

Let $x =$ the first digit,
and $y =$ the second.
Then $10x + y =$ the number,
and $10y + x =$ the number after the digits are interchanged.

$$\therefore x + y = 8. \quad (1)$$
$$4(10y + x) = 10x + y + 5(x + y) + 2. \quad (2)$$

Substitute value of $x + y$ from (1) in (2),

$$4(10y + x) = 10x + y + 40 + 2.$$

Transpose and combine,

$$-6x + 39y = 42$$
$$-2x + 13y = 14 \quad (3)$$

$2 \times (1)$ is $\underline{2x + \ 2y = 16}$

Add, $15y = 30$

$$\therefore y = 2.$$

Let $x =$ number of dollars the first has,
and $y =$ number of dollars the second has.
Then

$$x + \tfrac{3}{4}y = 1200. \quad (1)$$
$$\tfrac{8}{9}x + y = 1200. \quad (2)$$

Multiply (1) by 4, and (2) by 3,

$$4x + 3y = 4800$$
$$\underline{\tfrac{8}{3}x + 3y = 3600}$$

Subtract, $\tfrac{4}{3}x \quad = 1200$

$$\therefore x = 900.$$

Substitute value of x in (2), $800 + y = 1200.$

$$\therefore y = 400.$$

$\therefore$ The first has \$900, the second \$400.

$y =$ number of dollars B has,
and $z =$ number of dollars C has.

Exercise 11.

1. Three times the greater of two numbers exceeds twice the less by 27; and the sum of twice the greater and five times the less is 94. Find the numbers.

Let $x =$ the greater number,
and $y =$ the less.
Then $3x - 2y = 27.$ (1)
$2x + 5y = 94.$ (2)

Multiply (1) by 2, and (2) by 3,

$$\begin{aligned} 6x - 4y &= 54 \\ 6x + 15y &= 282 \end{aligned}$$

Subtract, $19y = 228$
$\therefore y = 12.$
Substitute value of y in (1), $3x - 24 = 27.$
$\therefore x = 17.$

$\therefore$ The numbers are 17 and 12.

2. A fraction is such that if 3 be added to each of its terms, the resulting fraction is equal to $\frac{4}{5}$; and if 3 be subtracted from each of its terms, the result is equal to $\frac{1}{2}$. Find the fraction.

Let $x =$ the numerator,
and $y =$ the denominator.
Then $\frac{x}{y} =$ the fraction.
$x + 3$ 4.

6. A number is formed by two digits. The sum of the digits is 6 times their difference. The number itself exceeds 6 times the sum of its digits by 3. Find the number.

Let $x =$ the first digit,
and $y =$ the second.
Then $10x + y =$ the number.
$\therefore x + y = 6(x - y).$ (1)
$10x + y - 6(x + y) = 3.$ (2)
From (1), $-5x + 7y = 0.$ (3)
From (2), $4x - 5y = 3.$ (4)
$4 \times (3)$ is $-20x + 28y = 0$
$5 \times (4)$ is $20x - 25y = 15$
$\therefore y = 7.$

$\therefore$ The fraction is $\frac{5}{7}$.

Substitute value of y in (3),

$$-5x + 35 = 0.$$
$$5x = 35.$$
$$\therefore\ x = 7.$$

$\therefore$ The number is 75.

7. A number is formed by two digits of which the sum is 8. If the digits be interchanged, 4 times the new number exceeds the original number by 2 more than 5 times the sum of the digits. Find the original number.

Let $x =$ the first digit,

and $y =$ the second.

Then $10x + y =$ the number,

and $10y + x =$ the number after the digits are interchanged.

$$\therefore\ x + y = 8. \qquad (1)$$
$$4(10y + x) = 10x + y + 5(x + y) + 2. \qquad (2)$$

Substitute value of $x + y$ from (1) in (2),

$$4(10y + x) = 10x + y + 40 + 2.$$

Transpose and combine,

$$\begin{aligned} -6x + 39y &= 42 \\ -2x + 13y &= 14 \qquad (3) \\ 2 \times (1) \text{ is } \quad 2x + \ \ 2y &= 16 \\ \hline \text{Add,} \qquad 15y &= 30 \\ \therefore\ y &= 2. \end{aligned}$$

Substitute value of y in (1),

$$x + 2 = 8.$$
$$\therefore\ x = 6.$$

$\therefore$ The original number is 62.

8. Three brothers, A, B, C, have together bought a house for \$32,000. A could pay the whole sum if B would give him $\frac{5}{8}$ of what he has; B could pay it if C would give him $\frac{8}{9}$ of what he has; and C could pay the whole sum if he had $\frac{1}{2}$ of what A has together with $\frac{3}{16}$ of what B has. How much does each have?

Let $x =$ number of dollars A has,

$y =$ number of dollars B has,

and $z =$ number of dollars C has.

Then $$x + \tfrac{5}{8}y = 32000. \tag{1}$$
$$y + \tfrac{8}{9}z = 32000. \tag{2}$$
$$z + \tfrac{1}{2}x + \tfrac{3}{16}y = 32000. \tag{3}$$
Simplify, $$8x + 5y = 256000. \tag{4}$$
$$9y + 8z = 288000. \tag{5}$$
$$8x + 3y + 16z = 512000. \tag{6}$$
Subtract (4) from (6),
$$-2y + 16z = 256000.$$
$$-\ y + \ 8z = 128000 \tag{7}$$
(5) is $$9y + \ 8z = 288000$$
Subtract, $$10y \qquad = 160000$$
$$\therefore\ y = 16000.$$
Substitute value of y in (7),
$$-16000 + 8z = 128000.$$
$$8z = 144000.$$
$$\therefore\ z = 18000.$$
Substitute value of y in (4),
$$8x + 80000 = 256000.$$
$$8x = 176000.$$
$$\therefore\ x = 22000.$$
$\therefore$ A has $22,000, B has $16,000, and C has $18,000.

9. A and B entered into partnership with a joint capital of $3400. A put in his money for 12 months; B put in his money for 16 months. In closing the business, B's share of the profits was greater than A's by $\frac{5}{59}$ of the total profit. Find the sum put in by each.

Let x = number of dollars A put in,
y = number of dollars B put in,
and z = number of dollars profit on one dollar in one month.
Then $12xz$ = number of dollars in A's share of profit.
$16yz$ = number of dollars in B's share of profit.
$$x + y = 3400. \tag{1}$$
$$16yz - 12xz = \tfrac{5}{59}(12xz + 16yz). \tag{2}$$
Simplify (2), $54 \times 16yz = 64 \times 12xz.$
Divide by $16z$, $9y = 8x.$
$$y = \frac{8x}{9}. \tag{3}$$

Substitute value of y in (1),

$$x + \frac{8x}{9} = 3400.$$

$$\frac{17x}{9} = 3400.$$

$$\therefore\ x = 1800.$$

Substitute value of x in (3),

$$y = 1600.$$

$\therefore$ A put in \$1800, B put in \$1600.

10. A capitalist makes two investments; the first at 3 per cent, the second at $3\frac{1}{2}$ per cent. His total income from the two investments is \$427. If \$1400 were taken from the second investment and added to the first, the incomes from the two investments would be equal. Find the amount of each investment.

Let $x =$ number of dollars in first investment,
and $y =$ number of dollars in second investment.

Then $\frac{3}{100}x + \frac{7}{200}y = 427.$ (1)
$\frac{3}{100}(x + 1400) = \frac{7}{200}(y - 1400).$ (2)
Simplify (1), $6x + 7y = 85400.$ (3)
Simplify (2), $6x + 8400 = 7y - 9800.$
$6x - 7y = -18200.$ (4)
Add (3) and (4), $12x = 67200.$
$\therefore\ x = 5600.$
Substitute value of x in (3),
$33600 + 7y = 85400.$
$7y = 51800.$
$\therefore\ y = 7400.$

$\therefore$ The investments were \$5600 and \$7400.

11. A cask contains 12 gallons of wine and 18 gallons of water; a second cask contains 9 gallons of wine and 3 gallons of water. How many gallons must be taken from each cask, so that, when mixed, there may be 14 gallons consisting half of water and half of wine?

Let $x =$ number of gallons to be taken from first cask,
and $y =$ number of gallons to be taken from second cask.

Then, $\frac{12x}{30} =$ number of gallons of wine taken from first cask.
$\frac{18x}{30} =$ number of gallons of water taken from first cask.

$\frac{9y}{12}$ = number of gallons of wine taken from second cask.

$\frac{3y}{12}$ = number of gallons of water taken from second cask.

Now
$$x + y = 14. \qquad (1)$$
$$\frac{12x}{30} + \frac{9y}{12} = \frac{18x}{30} + \frac{3y}{12}. \qquad (2)$$

Simplify (2),
$$24x + 45y = 36x + 15y.$$
$$-12x + 30y = 0.$$
$$-2x + 5y = 0 \qquad (4)$$

2 × (1) is
$$2x + 2y = 28 \qquad (5)$$

Add,
$$7y = 28$$
$$\therefore\ y = 4.$$

Substitute value of y in (1),
$$x = 10.$$

∴ 10 gallons must be taken from the first cask, and 4 gallons from the second.

12. A and B ran a race to a post and back. A returning meets B 30 yards from the post and beats him by 1 minute. If on arriving at the starting-place A had immediately returned to meet B, he would have run $\frac{1}{6}$ the distance to the post before meeting him. Find the distance run, and the time A and B each makes.

Let x = number of yards A runs in one minute,
y = number of yards B runs in one minute,
and z = number of yards to the post.
Then $z + 30$ = number of yards A runs before he meets B.
$z - 30$ = number of yards B runs before he meets A.
$\frac{z+30}{x}$ = number of minutes before A meets B.
$\frac{z-30}{y}$ = number of minutes before B meets A.

$$\therefore\ \frac{z+30}{x} = \frac{z-30}{y}. \qquad (1)$$

$\frac{2z}{x}$ = number of minutes it takes A to run to the post and back.

$\frac{2z}{y}$ = number of minutes it takes B to run to the post and back.

$$\therefore \frac{2z}{y} - \frac{2z}{x} = 1. \qquad (2)$$

$$\frac{2z + \frac{z}{6}}{x}$$ = number of minutes it would take A to run to the post and back and return $\frac{1}{6}$ of the distance.

$$\frac{2z - \frac{z}{6}}{y}$$ = number of minutes it would take B to reach the same point on his return.

$$\therefore \frac{2z + \frac{z}{6}}{x} = \frac{2z - \frac{z}{6}}{y}. \qquad (3)$$

Simplify (1), $yz + 30y = xz - 30x.$

$$(y - x)z + 30y + 30x = 0. \qquad (4)$$

Simplify (2), $2(x - y)z = xy.$ (5)

Simplify (3), $13zy = 11zx.$ (6)

Divide (6) by z, $13y = 11x.$

$$\therefore y = \tfrac{11}{13}x. \qquad (7)$$

Substitute value of y in (4),

$$-\tfrac{2}{13}xz + \tfrac{330}{13}x + 30x = 0.$$

$$-2xz + 330x + 390x = 0.$$

$$-2xz + 720x = 0.$$

Divide by x, $-2z + 720 = 0.$

$$\therefore z = 360.$$

Substitute values of y and z in (5),

$$\tfrac{4}{13} \times 360x = xy.$$

Divide by x, $\tfrac{1440}{13} = y.$

Substitute value of y in (7), $x = \tfrac{1440}{11}.$

$$2z = 720.$$

$$\frac{2z}{x} = \frac{720 \times 11}{1440} = \frac{11}{2} = 5\tfrac{1}{2}.$$

$$\frac{2z}{y} = \frac{720 \times 13}{1440} = \frac{13}{2} = 6\tfrac{1}{2}.$$

∴ The distance run is 720 yards. A runs it in $5\frac{1}{2}$ minutes, B in $6\frac{1}{2}$ minutes.

13. A and B together can do a piece of work in 15 days. After working together for 6 days, A leaves off, and B finishes the work in 30 days more. In how many days can each do the work?

Let $x =$ number of days in which A can do the work,

and $y =$ number of days in which B can do it.

Then, $\frac{1}{x} =$ part A can do in one day.

$\frac{1}{y} =$ part B can do in one day.

$\frac{1}{x} + \frac{1}{y} =$ part both can do together in one day.

$\frac{1}{\frac{1}{x} + \frac{1}{y}} =$ number of days in which both together can do it.

$$\therefore \frac{1}{\frac{1}{x} + \frac{1}{y}} = 15. \qquad (1)$$

$$\frac{6}{x} + \frac{6}{y} + \frac{30}{y} = 1. \qquad (2)$$

From (1), $$\frac{1}{x} + \frac{1}{y} = \frac{1}{15}. \qquad (3)$$

From (2), $$\frac{6}{x} + \frac{36}{y} = 1 \qquad (4)$$

6 × (3) is $$\frac{6}{x} + \frac{6}{y} = \frac{2}{5}$$

Subtract, $$\frac{30}{y} = \frac{3}{5}$$

$$\therefore y = 50. \qquad (5)$$

Substitute value of y in (3),

$$\frac{1}{x} + \frac{1}{50} = \frac{1}{15}.$$

$$\frac{1}{x} = \frac{7}{150}.$$

$$\therefore x = \tfrac{150}{7} = 21\tfrac{3}{7}.$$

∴ A can do the work in $21\frac{3}{7}$ days; B can do it in 50 days.

14. A and B together can do a piece of work in 12 days. After working together 9 days, however, they call in C to aid them, and the three finish the work in 2 days. C finds that he can do as much work in 5 days as A does in 6 days. In how many days can each do the work?

Let x = number of days in which A can do the work,
y = number of days in which B can do the work,
and z = number of days in which C can do the work.

Then, $\frac{1}{x}$ = part A can do in one day.
$\frac{1}{y}$ = part B can do in one day.
$\frac{1}{z}$ = part C can do in one day.

$$\therefore \frac{1}{x}+\frac{1}{y}=\frac{1}{12}. \tag{1}$$

$$\frac{9}{x}+\frac{9}{y}+\frac{2}{x}+\frac{2}{y}+\frac{2}{z}=1. \tag{2}$$

$$\frac{5}{z}=\frac{6}{x}. \tag{3}$$

From (2), $\frac{11}{x}+\frac{11}{y}+\frac{2}{z}=1$

$11 \times (1)$ is $\frac{11}{x}+\frac{11}{y}=\frac{11}{12}$

Subtract, $\frac{2}{z}=\frac{1}{12}$

$$\therefore z=24.$$

Substitute value of z in (3),

$$\frac{5}{24}=\frac{6}{x}.$$

$$\therefore x=\frac{144}{5}=28\frac{4}{5}.$$

Substitute value of x in (1),

$$\frac{5}{144}+\frac{1}{y}=\frac{1}{12}.$$

$$\frac{1}{y}=\frac{7}{144}.$$

$$y=\frac{144}{7}=20\frac{4}{7}.$$

$\therefore$ A can do the work in $28\frac{4}{5}$ days, B in $20\frac{4}{7}$ days, and C in 24 days.

15. A pedestrian has a certain distance to walk. After having passed over 20 miles, he increases his speed by 1 mile per hour. If he had walked the entire journey with this speed, he would have accomplished his walk in 40 minutes less time; but, by keeping his first pace, he would have arrived 20 minutes later than he did. What distance had he to walk?

Let $x =$ number of miles he has to walk,

and $y =$ number of miles he walks an hour.

Then $\frac{20}{y} =$ number of hours it takes him to walk the first 20 miles.

$\frac{x-20}{y+1} =$ number of hours it takes him to walk the remaining distance.

$\frac{x}{y+1} =$ number of hours it would take him to make the whole distance at the increased rate of speed.

40 minutes = $\frac{2}{3}$ hour, 20 minutes = $\frac{1}{3}$ hour.

$$\therefore \frac{20}{y} + \frac{x-20}{y+1} = \frac{x}{y+1} + \frac{2}{3} \qquad (1)$$

$$\frac{20}{y} + \frac{x-20}{y+1} = \frac{x}{y} - \frac{1}{3} \qquad (2)$$

Subtract,

$$0 = \frac{x}{y+1} - \frac{x}{y} + 1$$

or,

$$x\left(\frac{1}{y+1} - \frac{1}{y}\right) + 1 = 0. \qquad (3)$$

In (1) transpose $\frac{x}{y+1}$,

$$\frac{20}{y} + \frac{x-20-x}{y+1} = \frac{2}{3},$$

$$\frac{20}{y} - \frac{20}{y+1} = \frac{2}{3},$$

or,

$$\left(\frac{1}{y} - \frac{1}{y+1}\right) = \frac{1}{30}.$$

Substitute value of $\left(\frac{1}{y} - \frac{1}{y+1}\right)$ in (3),

$$-\frac{x}{30} + 1 = 0.$$

$$\therefore x = 30.$$

$\therefore$ He had 30 miles to walk.

Exercise 12.

Perform the indicated operation in:

1. $(2a^3)^4 = 2^4a^{3\times 4}$
$= 16a^{12}$.

2. $(3a^2x^3)^3 = 3^3a^{2\times 3}x^{3\times 3}$
$= 27a^6x^9$.

3. $$\left(\frac{2\,a^4b^2}{3\,c^3x^4}\right)^5=\frac{2^5\,a^{4\times5}b^{2\times5}}{3^5\,c^{3\times5}x^{4\times5}}$$
$$=\frac{32\,a^{20}b^{10}}{243\,c^{15}x^{20}}.$$

4. $$(-4\,b^2c)^3=-4^3\,b^{2\times3}\,c^3$$
$$=-64\,b^6c^3.$$

5. $$(-a^2b^3c)^4=a^{2\times4}b^{3\times4}c^4$$
$$=a^8b^{12}c^4.$$

6. $$\frac{(3\,a^4b^2)^4}{(9\,a^5b^3)^3}=\frac{3^4\,a^{4\times4}b^{2\times4}}{9^3\,a^{5\times3}b^{3\times3}}$$
$$=\frac{81\,a^{16}b^8}{729\,a^{15}b^9}$$
$$=\frac{a}{9\,b}.$$

7. $$(-5\,a^3b^2x^4)^3$$
$$=-5^3\,a^{3\times3}b^{2\times3}x^{4\times3}$$
$$=-125\,a^9b^6x^{12}.$$

8. $$(6\,a^2b^3c^4)^4$$
$$=6^4\,a^{2\times4}b^{3\times4}c^{4\times4}$$
$$=1296\,a^8b^{12}c^{16}.$$

9. $$\frac{(-3\,a^3x^2)^5}{(6\,a^5bx^2)^3}=\frac{-3^5\,a^{3\times5}x^{2\times5}}{6^3\,a^{5\times3}b^3x^{2\times3}}$$
$$=\frac{-243\,a^{15}x^{10}}{216\,a^{15}b^3x^6}$$
$$=-\frac{9\,x^4}{8\,b^3}.$$

10. $$\frac{(3\,a^2x^3)^2(4\,b^2x)^4}{(6\,b^3x^5)^2(a^2b)^2}$$
$$=\frac{3^2\,a^{2\times2}x^{3\times2}\times4^4\,b^{2\times4}x^4}{6^2\,b^{3\times2}x^{5\times2}\times a^{2\times2}b^2}$$
$$=\frac{9\,a^4x^6\times256\,b^8x^4}{36\,b^6x^{10}\times a^4b^2}$$
$$=64.$$

11. $$\frac{(4\,x^4y)^3}{(9\,x^2y^3)^4}\div\frac{(x^2y^5)^2}{(3\,y)^5}$$
$$=\frac{4^3\,x^{4\times3}y^3}{9^4\,x^{2\times4}y^{3\times4}}\div\frac{x^{2\times2}y^{5\times2}}{3^5y^5}$$
$$=\frac{64\,x^{12}y^3}{6561\,x^8y^{12}}\div\frac{x^4y^{10}}{243\,y^5}$$
$$=\frac{64\,x^4}{6561\,y^9}\div\frac{x^4y^5}{243}$$
$$=\frac{64\,x^4\times243}{6561\,x^4y^{14}}$$
$$=\frac{64}{27\,y^{14}}.$$

12. $$(x+3)^3=(x)^3+3\,(x)^2(3)+3\,(x)(3)^2+(3)^3$$
$$=x^3+9\,x^2+27\,x+27.$$

13. $$(1-2\,x)^4=1+4\,(-2\,x)+6\,(-2\,x)^2+4\,(-2\,x)^3+(-2\,x)^4$$
$$=1-8\,x+24\,x^2-32\,x^3+16\,x^4.$$

14. $$(3-x^2)^5=(3)^5+5\,(3)^4(-x^2)+10\,(3)^3(-x^2)^2+10\,(3)^2(-x^2)^3$$
$$+5\,(3)(-x^2)^4+(-x^2)^5$$
$$=243-405\,x^2+270\,x^4-90\,x^6+15\,x^8-x^{10}.$$

15. $$(1-4\,x)^6=1+6\,(-4\,x)+15\,(-4\,x)^2+20\,(-4\,x)^3+15\,(-4\,x)^4$$
$$+6\,(-4\,x)^5+(-4\,x)^6$$
$$=1-24\,x+240\,x^2-1280\,x^3+3840\,x^4-6144\,x^5+4096\,x^6.$$

16. $$\left(1-\frac{2x}{3}\right)^5=1+5\left(-\frac{2x}{3}\right)+10\left(-\frac{2x}{3}\right)^2+10\left(-\frac{2x}{3}\right)^3$$
$$+5\left(-\frac{2x}{3}\right)^4+\left(-\frac{2x}{3}\right)^5$$
$$=1-\frac{10}{3}x+\frac{40}{9}x^2-\frac{80}{27}x^3+\frac{80}{81}x^4-\frac{32}{243}x^5.$$

17. $$\left(1+\frac{3x^2}{4}\right)^4=1+4\left(\frac{3x^2}{4}\right)+6\left(\frac{3x^2}{4}\right)^2+4\left(\frac{3x^2}{4}\right)^3+\left(\frac{3x^2}{4}\right)^4$$
$$=1+3x^2+\frac{27}{8}x^4+\frac{27}{16}x^6+\frac{81}{256}x^8.$$

18. $$\left(\frac{3}{2x}-\frac{2x}{3}\right)^5=\left(\frac{3}{2x}\right)^5+5\left(\frac{3}{2x}\right)^4\left(-\frac{2x}{3}\right)+10\left(\frac{3}{2x}\right)^3\left(-\frac{2x}{3}\right)^2$$
$$+10\left(\frac{3}{2x}\right)^2\left(-\frac{2x}{3}\right)^3+5\left(\frac{3}{2x}\right)\left(-\frac{2x}{3}\right)^4+\left(-\frac{2x}{3}\right)^5$$
$$=\frac{243}{32x^5}-\frac{135}{8x^3}+\frac{15}{x}-\frac{20}{3}x+\frac{40}{27}x^3-\frac{32}{243}x^5.$$

19. $$(1+3x)^7=1+7(3x)+21(3x)^2+35(3x)^3+35(3x)^4+21(3x)^5$$
$$+7(3x)^6+(3x)^7$$
$$=1+21x+189x^2+945x^3+2835x^4+5103x^5+5103x^6+2187x^7.$$

20. $$\left(\frac{2}{x}-\frac{3x}{4}\right)^6=\left(\frac{2}{x}\right)^6+6\left(\frac{2}{x}\right)^5\left(-\frac{3x}{4}\right)+15\left(\frac{2}{x}\right)^4\left(-\frac{3x}{4}\right)^2$$
$$+20\left(\frac{2}{x}\right)^3\left(-\frac{3x}{4}\right)^3+15\left(\frac{2}{x}\right)^2\left(-\frac{3x}{4}\right)^4$$
$$+6\left(\frac{2}{x}\right)\left(-\frac{3x}{4}\right)^5+\left(-\frac{3x}{4}\right)^6$$
$$=\frac{64}{x^6}-\frac{144}{x^4}+\frac{135}{x^2}-\frac{135}{2}+\frac{1215}{64}x^2-\frac{729}{256}x^4+\frac{729}{4096}x^6.$$

21. $$\left(2a^2b^m-\frac{a^mb^2}{2}\right)^5$$
$$=(2a^2b^m)^5+5(2a^2b^m)^4\left(-\frac{a^mb^2}{2}\right)+10(2a^2b^m)^3\left(-\frac{a^mb^2}{2}\right)^2$$
$$+10(2a^2b^m)^2\left(-\frac{a^mb^2}{2}\right)^3+5(2a^2b^m)\left(-\frac{a^mb^2}{2}\right)^4$$
$$+\left(-\frac{a^mb^2}{2}\right)^5$$
$$=32a^{10}b^{5m}-40a^{m+8}b^{4m+2}+20a^{2m+6}b^{3m+4}-5a^{3m+4}b^{2m+6}$$
$$+\tfrac{5}{8}a^{4m+2}b^{m+8}-\tfrac{1}{32}a^{5m}b^{10}.$$

22. $\left(x^{n-2}-\frac{y^{n-4}}{4x^n}\right)^4$

$$= (x^{n-2})^4 + 4(x^{n-2})^3\left(-\frac{y^{n-4}}{4x^n}\right) + 6(x^{n-2})^2\left(-\frac{y^{n-4}}{4x^n}\right)^2$$

$$+ 4(x^{n-2})\left(-\frac{y^{n-4}}{4x^n}\right)^3 + \left(-\frac{y^{n-4}}{4x^n}\right)^4$$

$$= x^{4n-8} - x^{2n-6}y^{n-4} + \frac{3y^{2n-8}}{8x^4} - \frac{y^{3n-12}}{16x^{2n+2}} + \frac{y^{4n-16}}{256x^{4n}}.$$

23. $(1 + 3x - x^2)^4 = \{1 + (3x - x^2)\}^4$

$$= 1 + 4(3x - x^2) + 6(3x - x^2)^2 + 4(3x - x^2)^3 + (3x - x^2)^4$$

$$= 1 + 12x - 4x^2 + 54x^2 - 36x^3 + 6x^4 + 108x^3 - 108x^4 + 36x^5 - 4x^6$$

$$+ 81x^4 - 108x^5 + 54x^6 - 12x^7 + x^8.$$

$$= 1 + 12x + 50x^2 + 72x^3 - 21x^4 - 72x^5 + 50x^6 - 12x^7 + x^8.$$

24. $\left(1+\frac{x}{2}-\frac{x^2}{4}\right)^3 = \left\{1+\left(\frac{x}{2}-\frac{x^2}{4}\right)\right\}^3$

$$= 1 + 3\left(\frac{x}{2}-\frac{x^2}{4}\right) + 3\left(\frac{x}{2}-\frac{x^2}{4}\right)^2 + \left(\frac{x}{2}-\frac{x^2}{4}\right)^3$$

$$= 1 + \frac{3}{2}x - \frac{3}{4}x^2 + \frac{3}{4}x^2 - \frac{3}{4}x^3 + \frac{3}{16}x^4 + \frac{1}{8}x^3 - \frac{3}{16}x^4 + \frac{3}{32}x^5 - \frac{1}{64}x^6$$

$$= 1 + \frac{3}{2}x - \frac{5}{8}x^3 + \frac{3}{32}x^5 - \frac{x^6}{64}.$$

Exercise 13.

Simplify:

1. $\sqrt{16a^8b^4} = \pm 4a^4b^2.$

2. $\frac{\sqrt[3]{27a^9b^6}}{\sqrt{81a^4}} = \frac{3a^3b^2}{\pm 9a^2}$

$$= \pm\frac{ab^2}{3}.$$

3. $\sqrt[4]{81a^8b^{12}} = \pm 3a^2b^3.$

4. $\frac{\sqrt[4]{625x^8}}{\sqrt[3]{64x^6}} = \frac{\pm 5x^2}{4x^2}$

$$= \pm\frac{5}{4}.$$

5. $\sqrt[5]{1024a^{10}b^5} = 4a^2b.$

6. $\frac{\sqrt[3]{216a^6x^3}}{\sqrt[5]{32a^{15}x^{10}}} = \frac{6a^2x}{2a^3x^2}$

$$= \frac{3}{ax}.$$

Extract the square root of:

7. $1+4x+10x^2+12x^3+9x^4$.

$$\begin{array}{r|l}
\multicolumn{2}{l}{1+4x+10x^2+12x^3+9x^4\,\underline{|\,1+2x+3x^2}} \\
\multicolumn{2}{l}{1} \\
2+2x & 4x+10x^2 \\
 & \underline{4x+\;\;4x^2} \\
2+4x+3x^2 & 6x^2+12x^3+9x^4 \\
 & \underline{6x^2+12x^3+9x^4}
\end{array}$$

8. $9-24x-68x^2+112x^3+196x^4$.

$$\begin{array}{r|l}
\multicolumn{2}{l}{9-24x-68x^2+112x^3+196x^4\,\underline{|\,3-4x-14x^2}} \\
\multicolumn{2}{l}{9} \\
6-4x & -24x-68x^2 \\
 & \underline{-24x+16x^2} \\
6-8x-14x^2 & -84x^2+112x^3+196x^4 \\
 & \underline{-84x^2+112x^3+196x^4}
\end{array}$$

9. $4-12x+5x^2+26x^3-29x^4-10x^5+25x^6$.

$$\begin{array}{r|l}
\multicolumn{2}{l}{4-12x+5x^2+26x^3-29x^4-10x^5+25x^6\,\underline{|\,2-3x-x^2+5x^3}} \\
\multicolumn{2}{l}{4} \\
4-3x & -12x+5x^2 \\
 & \underline{-12x+9x^2} \\
4-6x-x^2 & -4x^2+26x^3-29x^4 \\
 & \underline{-4x^2+\;\;6x^3+\;\;\;\;x^4} \\
4-6x-2x^2+5x^3 & 20x^3-30x^4-10x^5+25x^6 \\
 & \underline{20x^3-30x^4-10x^5+25x^6}
\end{array}$$

10. $36x^2-120a^2x-12a^4x+100a^4+20a^6+a^8$.

$$\begin{array}{r|l}
\multicolumn{2}{l}{36x^2-120a^2x-12a^4x+100a^4+20a^6+a^8\,\underline{|\,6x-10a^2-a^4}} \\
\multicolumn{2}{l}{36x^2} \\
12x^2-10a^2 & -120a^2x-12a^4x+100a^4 \\
 & \underline{-120a^2x\qquad\qquad+100a^4} \\
12x-20a^2-a^4 & -12a^4x\qquad\qquad+20a^6+a^8 \\
 & \underline{-12a^4x\qquad\qquad+20a^6+a^8}
\end{array}$$

11. $4+9y^2-20x+25x^2+30xy-12y.$

$$\begin{array}{lll}
 & 25x^2+30xy+9y^2-20x-12y+4 & \underline{\mid 5x+3y-2} \\
 & 25x^2 & \\
10x+3y & \mid 30xy+9y^2 & \\
 & \mid \underline{30xy+9y^2} & \\
10x+6y-2 & \mid -20x-12y+4 & \\
 & \mid \underline{-20x-12y+4} &
\end{array}$$

12. $4x^4+9y^6-12x^2y^3+16x^2+16-24y^3.$

$$\begin{array}{lll}
 & 4x^4-12x^2y^3+9y^6+16x^2-24y^3+16 & \underline{\mid 2x^2-3y^3+4} \\
 & 4x^4 & \\
4x^2-3y^3 & \mid -12x^2y^3+9y^6 & \\
 & \mid \underline{-12x^2y^3+9y^6} & \\
4x^2-6y^3+4 & \mid 16x^2-24y^3+16 & \\
 & \mid \underline{16x^2-24y^3+16} &
\end{array}$$

13. $\frac{x^4}{4}+\frac{y^6}{9}+\frac{x^2}{4}-\frac{x^2y^3}{3}+\frac{1}{16}-\frac{y^3}{6}.$

$$\begin{array}{lll}
 & \frac{x^4}{4}-\frac{x^2y^3}{3}+\frac{y^6}{9}+\frac{x^2}{4}-\frac{y^3}{6}+\frac{1}{16} & \underline{\mid \frac{x^2}{2}-\frac{y^3}{3}+\frac{1}{4}} \\
 & \frac{x^4}{4} & \\
x^2-\frac{y^3}{3} & \mid -\frac{x^2y^3}{3}+\frac{y^6}{9} & \\
 & \mid \underline{-\frac{x^2y^3}{3}+\frac{y^6}{9}} & \\
x^2-\frac{2y^3}{3}+\frac{1}{4} & \mid \frac{x^2}{4}-\frac{y^3}{6}+\frac{1}{16} & \\
 & \mid \underline{\frac{x^2}{4}-\frac{y^3}{6}+\frac{1}{16}} &
\end{array}$$

Extract to four places of decimals the square root of:

14. 326.

$$\begin{array}{r l}
 & \dot{3}2\dot{6}(18.0554.....\\
 & 1\\
28) & \overline{226}\\
 & 224\\
3605) & \overline{20000}\\
 & 18025\\
36105) & \overline{197500}\\
 & 180525\\
361104) & \overline{1697500}\\
 & 1444416\\
 & \overline{253084}
\end{array}$$

To four places the root is 18.0554.

15. 1020.

$$\begin{array}{r l}
 & 1\dot{0}2\dot{0}(31.9374.....\\
 & 9\\
61) & \overline{120}\\
 & 61\\
629) & \overline{5900}\\
 & 5661\\
6383) & \overline{23900}\\
 & 19149\\
63867) & \overline{475100}\\
 & 447069\\
638744) & \overline{2803100}\\
 & 2554976\\
 & \overline{248124}
\end{array}$$

To four places the root is 31.9374.

16. 3.666.

$$\begin{array}{r l}
 & \dot{3}.6\dot{6}60(1.9146.....\\
 & 1\\
29) & \overline{266}\\
 & 261\\
381) & \overline{560}\\
 & 381\\
3824) & \overline{17900}\\
 & 15296\\
38286) & \overline{260400}\\
 & 229716\\
 & \overline{30694}
\end{array}$$

To four places the root is 1.9146.

17. 1.31213.

$$\begin{array}{r l}
 & \dot{1}.3\dot{1}2\dot{1}3\dot{0}(1.1454.....\\
 & 1\\
21) & \overline{31}\\
 & 21\\
224) & \overline{1021}\\
 & 896\\
2285) & \overline{12530}\\
 & 11425\\
22904) & \overline{110500}\\
 & 91616\\
 & \overline{18884}
\end{array}$$

To four places the root is 1.1454.

Exercise 14.

Extract the cube root of:

1. $27 - 108x + 144x^2 - 64x^3$.

$$\begin{array}{l r|l}
 & 27 - 108x + 144x^2 - 64x^3 & \underline{3 - 4x}\\
 & 27 & \\
\hline
3a^2 = 27 & & -108x + 144x^2 - 64x^3\\
3ab = \quad -36x & & \\
b^2 = \qquad\qquad + 4x^2 & & \\
\hline
 & 27 - 36x + 4x^2 & \underline{-108x + 144x^2 - 64x^3}
\end{array}$$

2. $x^6-3x^5+5x^3-3x-1.$

$$\begin{array}{rl|l}
 & x^6-3x^5+5x^3\qquad -3x-1 & \underline{x^2-x-1} \\
 & x^6 & \\
3(x^2)^2=3x^4 & -3x^5+5x^3 & \\
-x(3x^2-x)=\underline{\quad -3x^3+\ x^2} & & \\
3x^4-3x^3+\ x^2 & \underline{-3x^5+3x^4-x^3} & \\
3(x^2-x)^2=3x^4-6x^3+3x^2 & -3x^4+6x^3-3x-1 & \\
-1(3x^2-3x-1)=\underline{\qquad -3x^2+3x+1} & & \\
3x^4-6x^3\qquad +3x+1 & \underline{-3x^4+6x^3-3x-1} &
\end{array}$$

3. $a^3-a^2b+\frac{ab^2}{3}-\frac{b^3}{27}.$

$$\begin{array}{rl|l}
 & a^3-a^2b+\frac{ab^2}{3}-\frac{b^3}{27} & \underline{a-\frac{b}{3}} \\
 & a^3 & \\
3(a)^2=3a^2 & -a^2b+\frac{ab^2}{3}-\frac{b^3}{27} & \\
-\frac{b}{3}\left(3a-\frac{b}{3}\right)=\underline{\quad -ab+\frac{b^2}{9}} & & \\
3a^2-ab+\frac{b^2}{9} & \underline{-a^2b+\frac{ab^2}{3}-\frac{b^3}{27}} &
\end{array}$$

4. $1 - 6x + 21x^2 - 44x^3 + 63x^4 - 54x^5 + 27x^6$.

$$
\begin{array}{rl|l}
 & & 1 - 6x + 21x^2 - 44x^3 + 63x^4 - 54x^5 + 27x^6 \;\big|\; \underline{1 - 2x + 3x^2} \\
 & & 1 \\
 & 3 & -6x + 21x^2 - 44x^3 \\
3(-2x) = & -6x & \\
(-2x)^2 = & \underline{\qquad + 4x^2} & \\
 & 3 - 6x + 4x^2 & -6x + 12x^2 - 8x^3 \\
 & \underline{-6x + 4x^2} & 9x^2 - 36x^3 + 63x^4 - 54x^5 + 27x^6 \\
3(1 - 2x)^2 = & 3 - 12x + 12x^2 & \\
3(3x^2)(1 - 2x) = & 9x^2 - 18x^3 & \\
(3x^2)^2 = & \underline{\qquad + 9x^4} & \\
 & 3 - 12x + 21x^2 - 18x^3 + 9x^4 & \underline{9x^2 - 36x^3 + 63x^4 - 54x^5 + 27x^6}
\end{array}
$$

5. $27 + 296x^3 - 125x^6 - 108x + 9x^2 - 15x^4 - 300x^5$.

$$
\begin{array}{rl|l}
 & & -125x^6 - 300x^5 - 15x^4 + 296x^3 + 9x^2 - 108x + 27 \;\big|\; \underline{-5x^2 - 4x + 3} \\
 & & \underline{-125x^6} \\
 & 75x^4 & -300x^5 - 15x^4 + 296x^3 \\
-4x(-15x^2 - 4x) = & \underline{60x^3 + 16x^2} & \\
 & 75x^4 + 60x^3 + 16x^2 & -300x^5 - 240x^4 - 64x^3 \\
3(-5x^2 - 4x)^2 = & 75x^4 + 120x^3 + 48x^2 & 225x^4 + 360x^3 + 9x^2 - 108x + 27 \\
3(-15x^2 - 12x + 3) = & \underline{-45x^2 - 36x + 9} & \\
 & 75x^4 + 120x^3 + 3x^2 - 36x + 9 & \underline{225x^4 + 360x^3 + 9x^2 - 108x + 27}
\end{array}
$$

6. $12x^2-\frac{125}{x^3}-54x-59+\frac{135}{x}+8x^3+\frac{75}{x^2}.$

$$8x^3+12x^2-54x-59+\frac{135}{x}+\frac{75}{x^2}-\frac{125}{x^3}\,\Big|\,\underline{2x+1-\frac{5}{x}}$$

$$8x^3$$

$$12x^2-54x-59$$

$$\begin{aligned}3(2x)^2&=12x^2\\ 1(6x+1)&=\underline{\qquad\quad 6x+1}\\ &\ 12x^2+6x+1\end{aligned}$$

$$12x^2+6x+1$$

$$-60x-60+\frac{135}{x}+\frac{75}{x^2}-\frac{125}{x^3}$$

$$\begin{aligned}3(2x+1)^2&=12x^2+12x+3\\ -\frac{5}{x}\left(6x+3-\frac{5}{x}\right)&=\underline{\qquad\qquad -30-\frac{15}{x}+\frac{25}{x^2}}\\ &\ 12x^2+12x-27-\frac{15}{x}+\frac{25}{x^2}\end{aligned}$$

$$\underline{-60x-60+\frac{135}{x}+\frac{75}{x^2}-\frac{125}{x^3}}$$

7. $8x^3-36ax^2+\frac{a^6}{x^3}+\frac{33a^4}{x}+66a^2x-\frac{9a^5}{x^2}-63a^3.$

$$8x^3-36ax^2+66a^2x-63a^3+\frac{33a^4}{x}-\frac{9a^5}{x^2}+\frac{a^6}{x^3}\,\Big|\,\underline{2x-3a+\frac{a^2}{x}}$$

$$8x^3$$

$$-36ax^2+66a^2x-63a^3$$

$$\begin{aligned}3(2x)^2&=12x^2\\ -3a(6x-3a)&=\underline{\qquad -18ax+9a^2}\\ &\ 12x^2-18ax+9a^2\end{aligned}$$

$$-36ax+54a^2x-27a^3$$

$$12a^2x-36a^3+\frac{33a^4}{x}-\frac{9a^5}{x^2}+\frac{a^6}{x^3}$$

$$\begin{aligned}3(2x-3a)^2&=12x^2-36ax+27a^2\\ \frac{a^2}{x}\left(6x-9a+\frac{a^2}{x}\right)&=\underline{\qquad\qquad 6a^2-\frac{9a^3}{x}+\frac{a^4}{x^2}}\\ &\ 12x^2-36ax+33a^2-\frac{9a^3}{x}+\frac{a^4}{x^2}\end{aligned}$$

$$\underline{12a^2x-36a^3+\frac{33a^4}{x}-\frac{9a^5}{x^2}+\frac{a^6}{x^3}}$$

8. **Extract to three places of decimals the cube root of 517.**

$$\begin{array}{rcr|l}
 & & & 517\ \underline{|8.025} \\
 & & & 512 \\
3\times 80^2 & = & 19200 & 5000000 \\
3\times 800^2 & = & 1920000 & \\
3\times 800\times 2 & = & 4800 & \\
2^2 & = & \underline{4} & \\
 & & 1924804 & \underline{3849608} \\
 & & \underline{4804} & 1150392000 \\
3\times (8020)^2 & = & 192961200 & \\
3\times 8020\times 5 & = & 120300 & \\
5^2 & = & \underline{25} & \\
 & & 193081525 & \underline{965407625} \\
 & & & 184984375
\end{array}$$

To three places = 8.026.

9. **Extract to three places of decimals the cube root of 1637.**

$$\begin{array}{rcr|l}
 & & & \dot{1}63\dot{7}\ \underline{|11.785} \\
 & & & 1 \\
3\times 10^2 & = & 300 & 637 \\
3\times 10\times 1 & = & 30 & \\
1^2 & = & \underline{1} & \\
 & & 331 & \underline{331} \\
 & & \underline{31} & 306000 \\
3\times 110^2 & = & 36300 & \\
3\times 110\times 7 & = & 2310 & \\
7^2 & = & \underline{49} & \\
 & & 38659 & \underline{270613} \\
 & & \underline{2359} & 35387000 \\
3\times 1170^2 & = & 4106700 & \\
3\times 1170\times 8 & = & 28080 & \\
8^2 & = & \underline{64} & \\
 & & 4134844 & \underline{33078752} \\
 & & \underline{28144} & 2308248000 \\
3\times 11780^2 & = & 416305200 & \\
3\times 11780\times 5 & = & 176700 & \\
5^2 & = & \underline{25} & \\
 & & 416481925 & \underline{2082409625} \\
 & & & 25838375
\end{array}$$

To three places = 11.786.

10. Extract to three places of decimals the cube root of 3.25.

	$\dot{3}.25\dot{0}$ | 1.481
	1
$3 \times 10^2 = 300$	2.250
$3 \times 10 \times 4 = 120$	
$4^2 = 16$	
436	1744
136	506000
$3 \times 140^2 = 58800$	
$3 \times 140 \times 8 = 3360$	
$8^2 = 64$	
62224	497792
3424	8208000
$3 \times 1480^2 = 6571200$	
$3 \times 1480 \times 1 = 4440$	
$1^2 = 1$	
6575641	6575641
	163259

To three places = 1.481.

11. Extract to three places of decimals the cube root of 20.911.

	20.911 | 2.755
	8
$3 \times 20^3 = 1200$	12911
$3 \times 20 \times 7 = 420$	
$7^2 = 49$	
1669	11683
469	1228000
$3 \times 270^2 = 218700$	
$3 \times 270 \times 5 = 4050$	
$5^2 = 25$	
222775	1113875
4075	114125000
$3 \times 2750^2 = 22687500$	
$3 \times 2750 \times 5 = 41250$	
$5^2 = 25$	
22728775	113643875
	481125

To three places = 2.755.

Exercise 15.

1. Express with radical signs:

$a^{\frac{4}{5}};\ b^{\frac{2}{7}};\ c^{-\frac{1}{3}};\ x^{-\frac{3}{5}};\ (y^{\frac{4}{7}})^{-3}.$

$$a^{\frac{4}{5}}=\sqrt[5]{a^4}.$$

$$b^{\frac{2}{7}}=\sqrt[7]{b^2}.$$

$$c^{-\frac{1}{3}}=\frac{1}{c^{\frac{1}{3}}}=\frac{1}{\sqrt[3]{c}}.$$

$$x^{-\frac{3}{5}}=\frac{1}{x^{\frac{3}{5}}}=\frac{1}{\sqrt[5]{x^3}}.$$

$$(y^{\frac{4}{7}})^{-3}=\frac{1}{(y^{\frac{4}{7}})^3}=\frac{1}{y^{\frac{12}{7}}}=\frac{1}{\sqrt[7]{y^{12}}}.$$

2. Express with fractional exponents:

$\sqrt[3]{a^4};\ \sqrt[5]{b^{12}};\ \frac{1}{c\sqrt[5]{c^4}};\ \sqrt{\frac{1}{x^{\frac{5}{3}}}};\ \frac{1}{\sqrt[5]{y^3}}.$

$$\sqrt[3]{a^4}=a^{\frac{4}{3}}.$$

$$\sqrt[5]{b^{12}}=b^{\frac{12}{5}}.$$

$$\frac{1}{c\sqrt[5]{c^4}}=\frac{1}{c^{\frac{9}{5}}}=c^{-\frac{9}{5}}.$$

$$\sqrt{\frac{1}{x^{\frac{5}{3}}}}=\frac{1}{x^{\frac{5}{6}}}=x^{-\frac{5}{6}}.$$

$$\frac{1}{\sqrt[5]{y^3}}=\frac{1}{y^{\frac{3}{5}}}=y^{-\frac{3}{5}}.$$

3. Express with positive exponents.

$(a^{-3})^5;\ \sqrt[4]{b^{-3}};\ (\sqrt{c})^{-\frac{3}{5}};\ \left(\frac{1}{\sqrt[4]{x^{-5}}}\right)^{-2}.$

$$(a^{-3})^5=a^{-15}=\frac{1}{a^{15}}.$$

$$\sqrt[4]{b^{-3}}=b^{-\frac{3}{4}}=\frac{1}{b^{\frac{3}{4}}}.$$

$$(\sqrt{c})^{-\frac{3}{5}}=c^{-\frac{3}{10}}=\frac{1}{c^{\frac{3}{10}}}.$$

$$\left(\frac{1}{\sqrt[4]{x^{-5}}}\right)^{-2}=\left(\frac{1}{x^{-\frac{5}{4}}}\right)^{-2}=\frac{1}{x^{\frac{5}{2}}}.$$

4. Express with negative exponents and without denominators:

$\frac{a^2}{(4x)^3};\ \frac{a^{\frac{1}{2}}}{\sqrt{5x^3}};\ \frac{4x^{-3}}{3y^{-2}};\ \frac{2\sqrt{a^{-3}}}{3\sqrt[3]{x^5}}.$

$$\frac{a^2}{(4x)^3}=4^{-3}a^2.$$

$$\frac{a^{\frac{1}{2}}}{\sqrt{5x^3}}=\frac{a^{\frac{1}{2}}}{5^{\frac{1}{2}}x^{\frac{3}{2}}}=5^{-\frac{1}{2}}a^{\frac{1}{2}}x^{-\frac{3}{2}}.$$

$$\frac{4x^{-3}}{3y^{-2}}=3^{-1}\times 4x^{-3}y^2.$$

$$\frac{2\sqrt{a^{-3}}}{3\sqrt[3]{x^5}}=2\times 3^{-\frac{1}{2}}a^{-\frac{3}{2}}x^{-\frac{5}{3}}.$$

5. Simplify $a^{\frac{1}{2}}\times a^{-\frac{2}{3}}\times a^{-\frac{4}{9}};\ b^{\frac{2}{3}}\times b^{\frac{4}{9}}\sqrt[4]{b^{-3}};\ (\sqrt{c})^3\sqrt[3]{c^{-4}}.$

$$a^{\frac{1}{2}}\times a^{-\frac{2}{3}}\times a^{-\frac{4}{9}}=a^{\frac{1}{2}-\frac{2}{3}-\frac{4}{9}}=a^{-\frac{11}{18}}.$$

$$b^{\frac{2}{3}}\times b^{\frac{4}{9}}\sqrt[4]{b^{-3}}=b^{\frac{2}{3}}\times b^{\frac{4}{9}}\times b^{-\frac{3}{4}}=b^{\frac{2}{3}+\frac{4}{9}-\frac{3}{4}}=b^{\frac{13}{36}}.$$

$$(\sqrt{c})^3\sqrt[3]{c^{-4}}=c^{\frac{3}{2}}\times c^{-\frac{4}{3}}=c^{\frac{3}{2}-\frac{4}{3}}=c^{\frac{1}{6}}.$$

6. Simplify $a^{\frac{3}{4}} \times a^{\frac{1}{2}} \times \sqrt[3]{a^4}$; $b\sqrt[3]{c} \div (cx)^{\frac{1}{4}}$; $(a^{\frac{1}{4}}\sqrt[3]{ax})^{\frac{1}{6}}$.

$$a^{\frac{3}{4}} \times a^{\frac{1}{2}} \times \sqrt[3]{a^4} = a^{\frac{3}{4}} \times a^{\frac{1}{2}} \times a^{\frac{4}{3}} = a^{\frac{3}{4}+\frac{1}{2}+\frac{4}{3}} = a^{\frac{31}{12}}.$$

$$b\sqrt[3]{c} \div (cx)^{\frac{1}{4}} = bc^{\frac{1}{3}} \div c^{\frac{1}{4}}x^{\frac{1}{4}} = bc^{\frac{1}{3}-\frac{1}{4}}x^{-\frac{1}{4}} = \frac{bc^{\frac{1}{12}}}{x^{\frac{1}{4}}}.$$

$$(a^{\frac{1}{4}}\sqrt[3]{ax})^{\frac{1}{6}} = (a^{\frac{1}{4}}a^{\frac{1}{3}}x^{\frac{1}{3}})^{\frac{1}{6}} = (a^{\frac{7}{12}}x^{\frac{1}{3}})^{\frac{1}{6}} = a^{\frac{7}{72}}x^{\frac{1}{18}}.$$

7. Simplify $(3a)^{\frac{1}{4}}\sqrt{(16x)^3}$; $\left(\frac{16a^{-4}}{81x^3}\right)^{-\frac{3}{4}}$; $\left(\frac{9a^4}{16x^{-3}}\right)^{-\frac{3}{2}}$; $\left(\frac{27a^3}{\sqrt{9x^4}}\right)^{-\frac{4}{3}}$.

$$(3a)^{\frac{1}{4}}\sqrt{(16x)^3} = 3^{\frac{1}{4}}a^{\frac{1}{4}} \times 4^3x^{\frac{3}{2}} = 64 \times 3^{\frac{1}{4}}a^{\frac{1}{4}}x^{\frac{3}{2}}.$$

$$\left(\frac{16a^{-4}}{81x^3}\right)^{-\frac{3}{4}} = \frac{16^{-\frac{3}{4}}a^3}{81^{-\frac{3}{4}}x^{-\frac{9}{4}}} = \frac{81^{\frac{3}{4}}a^3x^{\frac{9}{4}}}{16^{\frac{3}{4}}} = \frac{27a^3x^{\frac{9}{4}}}{8}.$$

$$\left(\frac{9a^4}{16x^{-3}}\right)^{-\frac{3}{2}} = \frac{9^{-\frac{3}{2}}a^{-6}}{16^{-\frac{3}{2}}x^{\frac{9}{2}}} = \frac{16^{\frac{3}{2}}}{9^{\frac{3}{2}}a^6x^{\frac{9}{2}}} = \frac{64}{27a^6x^{\frac{9}{2}}}.$$

$$\left(\frac{27a^3}{\sqrt{9x^4}}\right)^{-\frac{4}{3}} = \frac{27^{-\frac{4}{3}}a^{-4}}{(3x^2)^{-\frac{4}{3}}} = \frac{(3x^2)^{\frac{4}{3}}}{27^{\frac{4}{3}}a^4} = \frac{3^{\frac{4}{3}}x^{\frac{8}{3}}}{81a^4} = \frac{3 \times 3^{\frac{1}{3}}x^{\frac{8}{3}}}{81a^4} = \frac{3^{\frac{1}{3}}x^{\frac{8}{3}}}{27a^4}.$$

8. Multiply $x^{\frac{4}{3}} - x^{\frac{2}{3}} + 1$ by $x^{\frac{2}{3}} + 1$.

$$\begin{array}{rrrr}
x^{\frac{4}{3}} & - x^{\frac{2}{3}} & + 1 & \\
 & x^{\frac{2}{3}} & + 1 & \\
\hline
x^2 - x^{\frac{4}{3}} & + x^{\frac{2}{3}} & & \\
+ x^{\frac{4}{3}} & - x^{\frac{2}{3}} & + 1 & \\
\hline
x^2 & & + 1 &
\end{array}$$

9. Multiply $x^{2p} + x^py^p + y^{2p}$ by $x^{2p} - x^py^p + y^{2p}$.

$$\begin{array}{lllll}
x^{2p} + x^py^p + y^{2p} & & & & \\
x^{2p} - x^py^p + y^{2p} & & & & \\
\hline
x^{4p} & + x^{3p}y^p & + x^{2p}y^{2p} & & \\
 & - x^{3p}y^p & - x^{2p}y^{2p} & - x^py^{3p} & \\
 & & + x^{2p}y^{2p} & + x^py^{3p} & + y^{4p} \\
\hline
x^{4p} & & + x^{2p}y^{2p} & & + y^{4p}
\end{array}$$

10. Multiply $8a^{\frac{3}{7}}+4a^{\frac{2}{7}}b^{-\frac{1}{7}}+5a^{\frac{1}{7}}b^{-\frac{2}{7}}+9b^{-\frac{3}{7}}$ by $2a^{\frac{4}{7}}-b^{-\frac{4}{7}}$.

$$\begin{array}{r}
8a^{\frac{3}{7}}+4a^{\frac{2}{7}}b^{-\frac{1}{7}}+5a^{\frac{1}{7}}b^{-\frac{2}{7}}+9b^{-\frac{3}{7}} \\
2a^{\frac{4}{7}}-b^{-\frac{4}{7}} \\
\hline
16a+8a^{\frac{6}{7}}b^{-\frac{1}{7}}+10a^{\frac{5}{7}}b^{-\frac{2}{7}}+18a^{\frac{4}{7}}b^{-\frac{3}{7}} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \\
-8a^{\frac{3}{7}}b^{-\frac{4}{7}}-4a^{\frac{2}{7}}b^{-\frac{5}{7}}-5a^{\frac{1}{7}}b^{-\frac{6}{7}}-9b^{-1} \\
\hline
16a+8a^{\frac{6}{7}}b^{-\frac{1}{7}}+10a^{\frac{5}{7}}b^{-\frac{2}{7}}+18a^{\frac{4}{7}}b^{-\frac{3}{7}}-8a^{\frac{3}{7}}b^{-\frac{4}{7}}-4a^{\frac{2}{7}}b^{-\frac{5}{7}}-5a^{\frac{1}{7}}b^{-\frac{6}{7}}-9b^{-1}
\end{array}$$

11. Divide $x^{5n}+y^{5n}$ by $x^{n}+y^{n}$.

$$\begin{array}{l|l}
x^{5n}+y^{5n} & x^{n}+y^{n} \\
\cline{2-2}
\underline{x^{5n}+x^{4n}y^{n}} & x^{4n}-x^{3n}y^{n}+x^{2n}y^{2n}-x^{n}y^{3n}+y^{4n} \\
\quad -x^{4n}y^{n}+y^{5n} & \\
\quad \underline{-x^{4n}y^{n}-x^{3n}y^{2n}} & \\
\qquad\qquad x^{3n}y^{2n}+y^{5n} & \\
\qquad\qquad \underline{x^{3n}y^{2n}+x^{2n}y^{3n}} & \\
\qquad\qquad\qquad -x^{2n}y^{3n}+y^{5n} & \\
\qquad\qquad\qquad \underline{-x^{2n}y^{3n}-x^{n}y^{4n}} & \\
\qquad\qquad\qquad\qquad\quad x^{n}y^{4n}+y^{5n} & \\
\qquad\qquad\qquad\qquad\quad \underline{x^{n}y^{4n}+y^{5n}} &
\end{array}$$

12. Divide $x-y^{-4}$ by $x^{\frac{3}{4}}-x^{\frac{1}{2}}y^{-1}+x^{\frac{1}{4}}y^{-2}-y^{-3}$.

$$\begin{array}{l|l}
x-y^{-4} & x^{\frac{3}{4}}-x^{\frac{1}{2}}y^{-1}+x^{\frac{1}{4}}y^{-2}-y^{-3} \\
\cline{2-2}
\underline{x-x^{\frac{3}{4}}y^{-1}+x^{\frac{1}{2}}y^{-2}-x^{\frac{1}{4}}y^{-3}} & x^{\frac{1}{4}}+y^{-1} \\
\quad x^{\frac{3}{4}}y^{-1}-x^{\frac{1}{2}}y^{-2}+x^{\frac{1}{4}}y^{-3}-y^{-4} & \\
\quad \underline{x^{\frac{3}{4}}y^{-1}-x^{\frac{1}{2}}y^{-2}+x^{\frac{1}{4}}y^{-3}+y^{-4}} &
\end{array}$$

13. Divide $a^{\frac{3}{7}} + b + c^{-\frac{1}{2}} - 3a^{\frac{1}{7}}b^{\frac{1}{3}}c^{-\frac{1}{6}}$ by $a^{\frac{1}{7}} + b^{\frac{1}{3}} + c^{-\frac{1}{6}}$.

$$a^{\frac{3}{7}} + b + c^{-\frac{1}{2}} - 3a^{\frac{1}{7}}b^{\frac{1}{3}}c^{-\frac{1}{6}} \;\underline{\big|\, a^{\frac{1}{7}} + b^{\frac{1}{3}} + c^{-\frac{1}{6}}}$$

$$\underline{a^{\frac{3}{7}} + a^{\frac{2}{7}}b^{\frac{1}{3}} + a^{\frac{2}{7}}c^{-\frac{1}{6}}} \qquad a^{\frac{2}{7}} - a^{\frac{1}{7}}b^{\frac{1}{3}} - a^{\frac{1}{7}}c^{-\frac{1}{6}} + b^{\frac{2}{3}} + c^{-\frac{1}{3}} - b^{\frac{1}{3}}c^{-\frac{1}{6}}$$

$$-a^{\frac{2}{7}}b^{\frac{1}{3}} - a^{\frac{2}{7}}c^{-\frac{1}{6}} + b + c^{-\frac{1}{2}} - 3a^{\frac{1}{7}}b^{\frac{1}{3}}c^{-\frac{1}{6}}$$

$$\underline{-a^{\frac{2}{7}}b^{\frac{1}{3}} - a^{\frac{1}{7}}b^{\frac{2}{3}} \qquad\qquad - a^{\frac{1}{7}}b^{\frac{1}{3}}c^{-\frac{1}{6}}}$$

$$-a^{\frac{2}{7}}c^{-\frac{1}{6}} + a^{\frac{1}{7}}b^{\frac{2}{3}} + b + c^{-\frac{1}{2}} - 2a^{\frac{1}{7}}b^{\frac{1}{3}}c^{-\frac{1}{6}}$$

$$\underline{-a^{\frac{2}{7}}c^{-\frac{1}{6}} - a^{\frac{1}{7}}c^{-\frac{1}{3}} \qquad\qquad - a^{\frac{1}{7}}b^{\frac{1}{3}}c^{-\frac{1}{6}}}$$

$$a^{\frac{1}{7}}b^{\frac{2}{3}} + a^{\frac{1}{7}}c^{-\frac{1}{3}} + b + c^{-\frac{1}{2}} - a^{\frac{1}{7}}b^{\frac{1}{3}}c^{-\frac{1}{6}}$$

$$\underline{a^{\frac{1}{7}}b^{\frac{2}{3}} \qquad\qquad + b + b^{\frac{2}{3}}c^{-\frac{1}{6}}}$$

$$a^{\frac{1}{7}}c^{-\frac{1}{3}} + c^{-\frac{1}{2}} - b^{\frac{2}{3}}c^{-\frac{1}{6}} - a^{\frac{1}{7}}b^{\frac{1}{3}}c^{-\frac{1}{6}}$$

$$\underline{a^{\frac{1}{7}}c^{-\frac{1}{3}} + c^{-\frac{1}{2}} + b^{\frac{1}{3}}c^{-\frac{1}{3}}}$$

$$-a^{\frac{1}{7}}b^{\frac{1}{3}}c^{-\frac{1}{6}} - b^{\frac{2}{3}}c^{-\frac{1}{6}} - b^{\frac{1}{3}}c^{-\frac{1}{3}}$$

$$\underline{-a^{\frac{1}{7}}b^{\frac{1}{3}}c^{-\frac{1}{6}} - b^{\frac{2}{3}}c^{-\frac{1}{6}} - b^{\frac{1}{3}}c^{-\frac{1}{3}}}$$

Exercise 16.

1. Express as entire surds $3\sqrt{5}$; $5\sqrt{32}$; $a^2b\sqrt{bc}$; $3y^2\sqrt[4]{x^3y}$; $a^3\sqrt[4]{a^3b^2}$.

$$3\sqrt{5} = \sqrt{5 \times (3)^2} = \sqrt{45}.$$

$$5\sqrt{32} = \sqrt{32 \times (5)^2} = \sqrt{800}.$$

$$a^2b\sqrt{bc} = \sqrt{bc \times (a^2b)^2} = \sqrt{a^4b^3c}.$$

$$3y^2\sqrt[4]{x^3y} = \sqrt[4]{x^3y \times (3y^2)^4} = \sqrt{81x^3y^9}.$$

$$a^3\sqrt[4]{a^3b^2} = \sqrt[4]{a^3b^2 \times (a^3)^4} = \sqrt[4]{a^{15}b^2}.$$

2. Express as entire surds

$$5abc\sqrt{abc^{-1}};\ \frac{2}{7}\sqrt{\frac{91}{8}};\ (x+y)\sqrt{\frac{xy}{x^2+2xy+y^2}}.$$

$$5abc\sqrt{abc^{-1}} = \sqrt{abc^{-1} \times (5abc)^2} = \sqrt{25a^3b^3c}.$$

$$\frac{2}{7}\sqrt{\frac{91}{8}}=\sqrt{\frac{91}{8}\times\left(\frac{2}{7}\right)^2}=\sqrt{\frac{91\times 4}{8\times 49}}=\sqrt{\frac{13}{14}}.$$

$$(x+y)\sqrt{\frac{xy}{x^2+2xy+y^2}}=\frac{(x+y)\sqrt{xy}}{x+y}=\sqrt{xy}.$$

3. Express as mixed surds

$$\sqrt[3]{160x^4y^7};\ \sqrt[3]{54x^2y^3};\ \sqrt[4]{64x^5y^6};\ \sqrt[3]{1372a^{15}b^{16}}.$$

$$\sqrt[3]{160x^4y^7}=\sqrt[3]{8x^3y^6\times 20xy}=2xy^2\sqrt[3]{20xy}.$$

$$\sqrt[3]{54x^2y^3}=\sqrt[3]{27y^3\times 2x^2}=3y\sqrt[3]{2x^2}.$$

$$\sqrt[4]{64x^5y^6}=\sqrt[4]{16x^4y^4\times 4xy^2}=2xy\sqrt[4]{4xy^2}.$$

$$\sqrt[3]{1372a^{15}b^{16}}=\sqrt[3]{343a^{15}b^{15}\times 4b}=7a^5b^5\sqrt[4]{4b}.$$

4. Simplify

$$2\sqrt[4]{80a^5b^2c^6};\ 7\sqrt{396x};\ \sqrt{1\tfrac{11}{16}};\ \sqrt{3\tfrac{1}{8}};\ \sqrt{\frac{3a^2bx}{4cy^3}}.$$

$$2\sqrt[4]{80a^5b^2c^6}=2\sqrt[4]{16a^4c^4\times 5ab^2c^2}=4ac\sqrt[4]{5ab^2c^2}.$$

$$7\sqrt{396x}=7\sqrt{36\times 11x}=42\sqrt{11x}.$$

$$\sqrt{1\tfrac{11}{16}}=\sqrt{\tfrac{27}{16}}=\sqrt{\tfrac{9}{16}\times 3}=\tfrac{3}{4}\sqrt{3}.$$

$$\sqrt{3\tfrac{1}{8}}=\sqrt{\tfrac{25}{8}}=\sqrt{\tfrac{25}{16}\times 2}=\tfrac{5}{4}\sqrt{2}.$$

$$\sqrt{\frac{3a^2bx}{4cy^3}}=\sqrt{\frac{a^2}{4y^2}\times\frac{3bx}{cy}}=\frac{a}{2y}\sqrt{\frac{3bx}{cy}}.$$

5. Simplify $\left(\frac{x^3y^2}{z^2}\right)\left(\frac{z^5}{x^5y^5}\right)^{\frac{1}{2}}$; $\left(\frac{a^3b^2}{c^4}\right)\left(\frac{c^9b^3}{a}\right)^{\frac{1}{3}}$; $(2a^2b^4)\times(b^2x^3)^{\frac{1}{3}}$.

$$\left(\frac{x^3y^2}{z^2}\right)\left(\frac{z^5}{x^5y^5}\right)^{\frac{1}{2}}=\frac{x^3y^2z^{\frac{5}{2}}}{z^2x^{\frac{5}{2}}y^{\frac{5}{2}}}=\frac{x^{\frac{1}{2}}z^{\frac{1}{2}}}{y^{\frac{1}{2}}}=\sqrt{\frac{xz}{y}};$$

$$\left(\frac{a^3b^2}{c^4}\right)\left(\frac{c^9b^3}{a}\right)^{\frac{1}{3}}=\frac{a^3b^2c^3b}{c^4a^{\frac{1}{3}}}=\frac{a^{\frac{8}{3}}b^3}{c}=\frac{a^2b^3}{c}\sqrt[3]{a^2};$$

$$(2a^2b^4)\times(b^2x^3)^{\frac{1}{3}}=2a^2b^4b^{\frac{2}{3}}x=2a^2b^4x\sqrt[3]{b^2}.$$

6. **Show that $\sqrt{20}$, $\sqrt{45}$, $\sqrt{\frac{4}{5}}$ are similar surds.**

$$\sqrt{20} = \sqrt{4 \times 5} = 2\sqrt{5}.$$
$$\sqrt{45} = \sqrt{9 \times 5} = 3\sqrt{5}.$$
$$\sqrt{\frac{4}{5}} = \sqrt{\frac{4}{25} \times 5} = \frac{2}{5}\sqrt{5}.$$

Since they all have the same surd factor they are similar surds.

7. **Show that $2\sqrt[3]{a^3b^2}$, $\sqrt[3]{8b^5}$, $\frac{1}{2}\sqrt[3]{\frac{a^6}{b}}$ are similar surds.**

$$2\sqrt[3]{a^3b^2} = 2a\sqrt[3]{b^2}.$$
$$\sqrt[3]{8b^5} = \sqrt[3]{8b^3 \times b^2} = 2b\sqrt[3]{b^2}.$$
$$\frac{1}{2}\sqrt[3]{\frac{a^6}{b}} = \frac{1}{2}\sqrt[3]{\frac{a^6}{b^3} \times b^2} = \frac{a^2}{2b}\sqrt[3]{b^2}.$$

Since they all have the same surd factor they are similar surds.

8. **Arrange in order of magnitude $9\sqrt{3}$, $6\sqrt{7}$, $5\sqrt{10}$.**

$$9\sqrt{3} = \sqrt{243}.$$
$$6\sqrt{7} = \sqrt{252}.$$
$$5\sqrt{10} = \sqrt{250}.$$

Therefore the descending order of magnitude is $6\sqrt{7}$, $5\sqrt{10}$, $9\sqrt{3}$.

9. **Arrange in order of magnitude $4\sqrt[3]{4}$, $3\sqrt[3]{5}$, $5\sqrt[3]{3}$.**

$$4\sqrt[3]{4} = \sqrt[3]{256}.$$
$$3\sqrt[3]{5} = \sqrt[3]{135}.$$
$$5\sqrt[3]{3} = \sqrt[3]{375}.$$

Therefore the descending order of magnitude is $5\sqrt[3]{3}$, $4\sqrt[3]{4}$, $3\sqrt[3]{5}$.

10. **Multiply $3\sqrt{2}$ by $4\sqrt{6}$; $\frac{1}{2}\sqrt[3]{4}$ by $2\sqrt[3]{2}$.**

$$3\sqrt{2} \times 4\sqrt{6} = 12\sqrt{12} = 12\sqrt{4 \times 3} = 24\sqrt{3};$$
$$\frac{1}{2}\sqrt[3]{4} \times 2\sqrt[3]{2} = \sqrt[3]{8} = 2.$$

11. Divide $2\sqrt{5}$ by $3\sqrt{15}$; $\frac{3}{8}\sqrt{21}$ by $\frac{9}{16}\sqrt{\frac{7}{20}}$.

$$2\sqrt{5} \div 3\sqrt{15} = \frac{2\sqrt{5}}{3\sqrt{3 \times 5}} = \frac{2}{3\sqrt{3}} = \frac{2\sqrt{3}}{9}.$$

$$\frac{3}{8}\sqrt{21} \div \frac{9}{16}\sqrt{\frac{7}{20}} = \frac{3 \times 16}{8 \times 9}\sqrt{\frac{21 \times 20}{7}} = \frac{2}{3}\sqrt{60} = \frac{4}{3}\sqrt{15}.$$

12. Simplify $\frac{2\sqrt{10}}{3\sqrt{27}} \times \frac{7\sqrt{48}}{5\sqrt{14}} \div \frac{4\sqrt{15}}{15\sqrt{21}}$.

$$\begin{aligned}
\frac{2\sqrt{10}}{3\sqrt{27}} \times \frac{7\sqrt{48}}{5\sqrt{14}} \div \frac{4\sqrt{15}}{15\sqrt{21}} &= \frac{2\sqrt{10} \times 7\sqrt{48} \times 15\sqrt{21}}{3\sqrt{27} \times 5\sqrt{14} \times 4\sqrt{15}} \\
&= \frac{2 \times 7 \times 15 \times \sqrt{10} \times \sqrt{48} \times \sqrt{21}}{3 \times 5 \times 4 \times \sqrt{27} \times \sqrt{14} \times \sqrt{15}} \\
&= \frac{7 \times \sqrt{2 \times 5} \times \sqrt{2^4 \times 3} \times \sqrt{3 \times 7}}{2 \times \sqrt{3^3} \times \sqrt{2 \times 7} \times \sqrt{3 \times 5}} \\
&= \frac{7 \times 2^2 \times 3 \times \sqrt{2 \times 5 \times 7}}{2 \times 3^2 \times \sqrt{2 \times 5 \times 7}} \\
&= \frac{7 \times 2}{3} \\
&= 4\tfrac{2}{3}.
\end{aligned}$$

13. Arrange in order of magnitude $2\sqrt[3]{3}$, $3\sqrt{2}$, $\frac{5}{2}\sqrt[4]{4}$.

$$2\sqrt[3]{3} = \sqrt[3]{24} = \sqrt[6]{(24)^2} = \sqrt[6]{576}.$$
$$3\sqrt{2} = \sqrt{18} = \sqrt[6]{(18)^3} = \sqrt[6]{5832}.$$
$$\tfrac{5}{2}\sqrt[4]{4} = \tfrac{5}{2}\sqrt{2} = \sqrt{\tfrac{25}{2}} = \sqrt[6]{(\tfrac{25}{2})^3} = \sqrt[6]{\tfrac{15625}{8}}.$$

Therefore the descending order of magnitude is $3\sqrt{2}$, $\frac{5}{2}\sqrt[4]{4}$, $2\sqrt[3]{3}$.

14. Arrange in order of magnitude $3\sqrt{19}$, $5\sqrt[3]{2}$, $3\sqrt[3]{3}$.

$$3\sqrt{19} = \sqrt{171} = \sqrt[6]{(171)^3} = \sqrt[6]{5000211}.$$
$$5\sqrt[3]{2} = \sqrt[3]{250} = \sqrt[6]{(250)^2} = \sqrt[6]{62500}.$$
$$3\sqrt[3]{3} = \sqrt[3]{81} = \sqrt[6]{(81)^2} = \sqrt[6]{6561}.$$

Therefore the descending order of magnitude is $3\sqrt{19}$, $5\sqrt[3]{2}$, $3\sqrt[3]{3}$.

15. Simplify $\sqrt[4]{a^2xy^3} \times \sqrt[5]{a^2xy}$; $3\sqrt[3]{4\,ab^2} \div \sqrt{2\,a^3b}$.

$$\sqrt[4]{a^2xy^3} \times \sqrt[5]{a^2xy} = \sqrt[20]{a^{10}x^5y^{15}} \times \sqrt[20]{a^8x^4y^4} = \sqrt[20]{a^{18}x^9y^{19}}.$$

$$\begin{aligned} 3\sqrt[3]{4\,ab^2} \div \sqrt{2\,a^3b} &= 3\sqrt[6]{16\,a^2b^4} \div \sqrt[6]{8\,a^9b^3} \\ &= 3\sqrt[6]{\frac{16\,a^2b^4}{8\,a^9b^3}} = 3\sqrt[6]{\frac{2\,b}{a^7}} = \frac{3}{a}\sqrt[6]{\frac{2\,b}{a}}. \end{aligned}$$

16. Simplify $\sqrt{\left(\frac{16}{25}\right)^7} \times \sqrt{\left(\frac{25}{64}\right)^6}$; $(\sqrt[7]{a^3b})^3 \times \sqrt[7]{(a^3b^{12})^4}$.

$$\sqrt{\left(\frac{16}{25}\right)^7} \times \sqrt{\left(\frac{25}{64}\right)^6} = \left(\frac{16}{25}\right)^3\sqrt{\frac{16}{25}} \times \left(\frac{25}{64}\right)^3 = \frac{16^3}{64^3} \times \frac{4}{5} = \frac{1}{4^2 \times 5} = \frac{1}{80}.$$

$$(\sqrt[7]{a^3b})^3 \times \sqrt[7]{(a^3b^{12})^4} = \sqrt[7]{a^9b^3} \times \sqrt[7]{a^{12}b^{48}} = \sqrt[7]{a^{21}b^{51}} = a^3b^7\sqrt[7]{b^2}.$$

Exercise 17.

1. Simplify $\sqrt{27} + 2\sqrt{48} + 3\sqrt{108}$; $7\sqrt[3]{54} + 3\sqrt[3]{16} + \sqrt[3]{432}$.

$$\sqrt{27} + 2\sqrt{48} + 3\sqrt{108} = 3\sqrt{3} + 8\sqrt{3} + 18\sqrt{3} = 29\sqrt{3}.$$

$$7\sqrt[3]{54} + 3\sqrt[3]{16} + \sqrt[3]{432} = 21\sqrt[3]{2} + 6\sqrt[3]{2} + 6\sqrt[3]{2} = 33\sqrt[3]{2}.$$

2. Simplify $2\sqrt{3} + 3\sqrt{1\frac{1}{3}} - \sqrt{5\frac{1}{3}}$; $2\sqrt{\frac{5}{3}} + \sqrt{60} - \sqrt{15} - \sqrt{\frac{3}{5}}$.

$$\begin{aligned} 2\sqrt{3} + 3\sqrt{1\tfrac{1}{3}} - \sqrt{5\tfrac{1}{3}} &= 2\sqrt{3} + \frac{6}{\sqrt{3}} - \frac{4}{\sqrt{3}} \\ &= 2\sqrt{3} + 2\sqrt{3} - \tfrac{4}{3}\sqrt{3} = \tfrac{8}{3}\sqrt{3}. \end{aligned}$$

$$2\sqrt{\frac{5}{3}} + \sqrt{60} - \sqrt{15} - \sqrt{\frac{3}{5}} = \tfrac{2}{3}\sqrt{15} + 2\sqrt{15} - \sqrt{15} - \tfrac{1}{5}\sqrt{15} = \tfrac{22}{15}\sqrt{15}.$$

3. Simplify $\sqrt{\frac{a^4c}{b^3}} - \sqrt{\frac{a^3c^3}{bd^2}} - \sqrt{\frac{a^2cd^2}{bm^2}}$; $3\sqrt{\frac{2}{5}} + 2\sqrt{\frac{1}{10}} - 4\sqrt{\frac{1}{40}}$.

$$\begin{aligned} \sqrt{\frac{a^4c}{b^3}} - \sqrt{\frac{a^2c^3}{bd^2}} - \sqrt{\frac{a^2cd^2}{bm^2}} &= \frac{a^2}{b}\sqrt{\frac{c}{b}} - \frac{ac}{d}\sqrt{\frac{c}{b}} - \frac{ad}{m}\sqrt{\frac{c}{b}} \\ &= \left(\frac{a^2}{b} - \frac{ac}{d} - \frac{ad}{m}\right)\sqrt{\frac{c}{b}} \\ &= \left(\frac{a^2}{b^2} - \frac{ac}{bd} - \frac{ad}{bm}\right)\sqrt{bc}. \end{aligned}$$

$$3\sqrt{\frac{2}{5}} + 2\sqrt{\frac{1}{10}} - 4\sqrt{\frac{1}{40}} = 6\sqrt{\frac{1}{10}} + 2\sqrt{\frac{1}{10}} - 2\sqrt{\frac{1}{10}}$$
$$= 6\sqrt{\frac{1}{10}} = \tfrac{3}{5}\sqrt{10}.$$

4. **Simplify** $2\sqrt[3]{40} + 3\sqrt[3]{108} + \sqrt[3]{500} - \sqrt[3]{320} - 2\sqrt[3]{1372}.$

$$2\sqrt[3]{40} + 3\sqrt[3]{108} + \sqrt[3]{500} - \sqrt[3]{320} - 2\sqrt[3]{1372}$$
$$= 4\sqrt[3]{5} + 9\sqrt[3]{4} + 5\sqrt[3]{4} - 4\sqrt[3]{5} - 14\sqrt[3]{4} = 0.$$

5. **Simplify** $(\sqrt[9]{8})^4;\ (\sqrt[9]{27})^4;\ (\sqrt[8]{64})^3;\ (\sqrt[3]{4})^2.$

$$(\sqrt[9]{8})^4 = (\sqrt[9]{2^3})^4 = (\sqrt[3]{2})^4 = 2\sqrt[3]{2};$$
$$(\sqrt[9]{27})^4 = (\sqrt[9]{3^3})^4 = (\sqrt[3]{3})^4 = 3\sqrt[3]{3};$$
$$(\sqrt[8]{64})^3 = (\sqrt[8]{2^6})^3 = (\sqrt[4]{2^3})^3 = 4\sqrt[4]{2};$$
$$(\sqrt[3]{4})^2 = (\sqrt[3]{2^2})^2 = \sqrt[3]{2^4} = 2\sqrt[3]{2}.$$

6. **Simplify** $(a\sqrt[3]{a})^{-3};\ (x\sqrt[3]{x})^{-\frac{1}{2}};\ (p^2\sqrt{p})^{\frac{1}{5}};\ (a^{-3}\sqrt[4]{a^{-3}})^{-\frac{1}{4}}.$

$$(a\sqrt[3]{a})^{-3} = (a^{\frac{4}{3}})^{-3} = a^{-4} = \frac{1}{a^4};$$
$$(x\sqrt[3]{x})^{-\frac{1}{2}} = (x^{\frac{4}{3}})^{-\frac{1}{2}} = x^{-\frac{2}{3}} = \frac{1}{x^{\frac{2}{3}}};$$
$$(p^2\sqrt{p})^{\frac{1}{5}} = (p^{\frac{5}{2}})^{\frac{1}{5}} = p^{\frac{1}{2}};$$
$$(a^{-3}\sqrt[4]{a^{-3}})^{-\frac{1}{4}} = (a^{-\frac{15}{4}})^{-\frac{1}{4}} = a^{\frac{15}{16}}.$$

7. **Find the square root of** $x^{4m} + 6x^{3m}y^n + 11x^{2m}y^{2n} + 6x^my^{3n} + y^{4n}.$

$$\begin{array}{r|l|l}
 & x^{4m} + 6x^{3m}y^n + 11x^{2m}y^{2n} + 6x^my^{3n} + y^{4n} & \underline{x^{2m} + 3x^my^n + y^{2n}} \\
 & x^{4m} & \\
2x^{2m} + 3x^my^n & 6x^{3m}y^n + 11x^{2m}y^{2n} & \\
 & \underline{6x^{3m}y^n + 9x^{2m}y^{2n}} & \\
2x^{2m} + 6x^my^n + y^{2n} & 2x^{2m}y^{2n} + 6x^my^{3n} + y^{4n} & \\
 & \underline{2x^{2m}y^{2n} + 6x^my^{3n} + y^{4n}} &
\end{array}$$

8. Find the square root of

$$1+4x^{-\frac{1}{3}}-2x^{-\frac{2}{3}}-4x^{-1}+25x^{-\frac{4}{3}}-24x^{-\frac{5}{3}}+16x^{-2}.$$

$$\begin{array}{r|l}
 & 1+4x^{-\frac{1}{3}}-2x^{-\frac{2}{3}}-4x^{-1}+25x^{-\frac{4}{3}}-24x^{-\frac{5}{3}}+16x^{-2} \;\big|\; 1+2x^{-\frac{1}{3}}-3x^{-\frac{2}{3}}+4x^{-1} \\
 & 1 \\ \hline
2+2x^{-\frac{1}{3}} & 4x^{-\frac{1}{3}}-2x^{-\frac{2}{3}} \\
 & 4x^{-\frac{1}{3}}+4x^{-\frac{2}{3}} \\ \hline
2+4x^{-\frac{1}{3}}-3x^{-\frac{2}{3}} & -6x^{-\frac{2}{3}}-4x^{-1}+25x^{-\frac{4}{3}} \\
 & -6x^{-\frac{2}{3}}-12x^{-1}+9x^{-\frac{4}{3}} \\ \hline
2+4x^{-\frac{1}{3}}-6x^{-\frac{2}{3}}+4x^{-1} & 8x^{-1}+16x^{-\frac{4}{3}}-24x^{-\frac{5}{3}}+16x^{-2} \\
 & 8x^{-1}+16x^{-\frac{4}{3}}-24x^{-\frac{5}{3}}+16x^{-2} \\ \hline
\end{array}$$

9. Simplify $\left(x^{\frac{2}{3}}\sqrt{\left(\frac{x^{\frac{1}{2}}}{\sqrt[3]{x}}\right)^5}\right)^{\frac{3}{26}}.$

$$\left(x^{\frac{2}{3}}\sqrt{\left(\frac{x^{\frac{1}{2}}}{\sqrt[3]{x}}\right)^5}\right)^{\frac{3}{26}}=\left(x^{\frac{2}{3}}\sqrt{\frac{x^{\frac{5}{2}}}{x^{\frac{5}{3}}}}\right)^{\frac{3}{26}}$$

$$=\left(x^{\frac{2}{3}}\sqrt{x^{\frac{5}{6}}}\right)^{\frac{3}{26}}=\left(x^{\frac{2}{3}}\times x^{\frac{5}{12}}\right)^{\frac{3}{26}}$$

$$=\left(x^{\frac{13}{12}}\right)^{\frac{3}{26}}=x^{\frac{3}{24}}=x^{\frac{1}{8}}.$$

10. Simplify $\left(\frac{\sqrt[5]{a^3}\sqrt{b}}{c\sqrt[4]{b^3}}\right)\times\left(a\sqrt[5]{ab^5}\right).$

$$\left(\frac{\sqrt[5]{a^3}\sqrt{b}}{c\sqrt[4]{b^3}}\right)\times\left(a\sqrt[5]{ab^5}\right)=\frac{a^{\frac{3}{5}}b^{\frac{1}{2}}}{cb^{\frac{3}{4}}}\times a^{\frac{6}{5}}b=\frac{a^{\frac{9}{5}}b^{\frac{3}{4}}}{c}.$$

11. Simplify $\left(\frac{\sqrt[4]{a^3}\sqrt[3]{b^2}}{5\sqrt[4]{c^5}}\right)\left(\frac{2b^{\frac{3}{2}}}{5a\sqrt[12]{b^2c^9}}\right).$

$$\left(\frac{\sqrt[4]{a^3}\sqrt[3]{b^2}}{5\sqrt[4]{c^5}}\right)\left(\frac{2b^{\frac{3}{2}}}{5a\sqrt[12]{b^2c^9}}\right)=\frac{a^{\frac{3}{4}}b^{\frac{2}{3}}}{5c^{\frac{5}{4}}}\times\frac{2b^{\frac{3}{2}}}{5ab^{\frac{1}{6}}c^{\frac{3}{4}}}=\frac{2b^2}{25a^{\frac{1}{4}}c^2}.$$

12. Simplify $\left(\frac{10\sqrt[3]{a^2}}{\sqrt[4]{5b^{11}}}\right)\left(\frac{5a\sqrt[3]{a^2}}{4b\sqrt[5]{a^2}}\right)$.

$$\left(\frac{10\sqrt[3]{a^2}}{\sqrt[4]{5b^{11}}}\right)\left(\frac{5a\sqrt[3]{a^2}}{4b\sqrt[5]{a^2}}\right)=\frac{10a^{\frac{2}{3}}}{5^{\frac{1}{4}}b^{\frac{11}{4}}}\times\frac{5a^{\frac{5}{3}}}{4ba^{\frac{2}{5}}}=\frac{5\times5^{\frac{3}{4}}a^{\frac{29}{15}}}{2b^{\frac{15}{4}}}.$$

13. Simplify $\left(\frac{x^{p+q}}{x^q}\right)^p\left(\frac{x^{q-p}}{x^q}\right)^{p-q}$.

$$\left(\frac{x^{p+q}}{x^q}\right)^p\left(\frac{x^{q-p}}{x^q}\right)^{p-q}=(x^p)^p(x^{-p})^{p-q}$$
$$=x^{p^2}\times x^{-p^2+pq}=x^{pq}.$$

14. Simplify $\frac{x^{2p(q+1)}-y^{2q(p-1)}}{x^{p(q+1)}+y^{q(p-1)}}$.

$$\frac{x^{2p(q+1)}-y^{2q(p-1)}}{x^{p(q+1)}+y^{q(p-1)}}=\frac{\left(x^{p(q+1)}+y^{q(p-1)}\right)\left(x^{p(q+1)}-y^{q(p-1)}\right)}{x^{p(q+1)}+y^{q(p-1)}}$$
$$=x^{p(q+1)}-y^{q(p-1)}.$$

15. Find equivalent fractions with rational denominators for the following, and find their approximate values.

$$\frac{3}{\sqrt{7}+\sqrt{5}};\quad\frac{7}{2\sqrt{5}-\sqrt{6}};\quad\frac{4-\sqrt{2}}{1+\sqrt{2}};\quad\frac{6}{5-2\sqrt{6}}.$$

$$(1)\quad\frac{3}{\sqrt{7}+\sqrt{5}}=\frac{3(\sqrt{7}-\sqrt{5})}{(\sqrt{7}+\sqrt{5})(\sqrt{7}-\sqrt{5})}$$
$$=\frac{3(\sqrt{7}-\sqrt{5})}{7-5}.$$

$$\sqrt{7}=2.6457.\qquad\sqrt{5}=2.2361.$$

$$\therefore\quad\frac{3}{\sqrt{7}+\sqrt{5}}=\frac{3(2.6457-2.2361)}{2}$$
$$=\frac{3\times.4096}{2}=.614.\ \textit{Ans.}$$

$$(2)\quad\frac{7}{2\sqrt{5}-\sqrt{6}}=\frac{7(2\sqrt{5}+\sqrt{6})}{(2\sqrt{5}-\sqrt{6})(2\sqrt{5}+\sqrt{6})}$$
$$=\frac{7(2\sqrt{5}+\sqrt{6})}{20-6}=\frac{2\sqrt{5}+\sqrt{6}}{2}.$$

$$\sqrt{5}=2.2361\qquad\sqrt{6}=2.4495.$$

$$\therefore\quad\frac{7}{2\sqrt{5}-\sqrt{6}}=\frac{4.4722+2.4495}{2}=3.461.\ \textit{Ans.}$$

(3) $$\frac{4-\sqrt{2}}{1+\sqrt{2}}=\frac{(4-\sqrt{2})(1-\sqrt{2})}{(1+\sqrt{2})(1-\sqrt{2})}$$
$$=\frac{6-5\sqrt{2}}{-1}=5\sqrt{2}-6.$$
$\sqrt{2}=1.4142.$ $5\sqrt{2}-6=1.071.$ *Ans.*

(4) $$\frac{6}{5-2\sqrt{6}}=\frac{6(5+2\sqrt{6})}{(5-2\sqrt{6})(5+2\sqrt{6})}=30+12\sqrt{6}.$$
$\sqrt{6}=2.4495.$ $30+12\sqrt{6}=59.394.$ *Ans.*

16. Find equivalent fractions with rational denominators for the following, and find their approximate values:

$$\frac{2}{\sqrt{3}};\ \frac{1}{\sqrt{5}-\sqrt{2}};\ \frac{7\sqrt{5}}{\sqrt{7}+\sqrt{3}};\ \frac{7-2\sqrt{3}+3\sqrt{2}}{3+3\sqrt{3}-2\sqrt{2}}.$$

(1) $$\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{3}.$$ $\sqrt{3}=1.7320.$
$$\therefore\ \frac{2}{\sqrt{3}}=\frac{3.4640}{3}=1.154.\ \textit{Ans.}$$

(2) $$\frac{1}{\sqrt{5}-\sqrt{2}}=\frac{\sqrt{5}+\sqrt{2}}{(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2})}$$
$$=\frac{\sqrt{5}+\sqrt{2}}{3}.$$
$\sqrt{5}=2.2361.$ $\sqrt{2}=1.4142.$
$$\therefore\ \frac{1}{\sqrt{5}-\sqrt{2}}=\frac{3.6503}{3}=1.216.\ \textit{Ans.}$$

(3) $$\frac{7\sqrt{5}}{\sqrt{7}+\sqrt{3}}=\frac{7\sqrt{5}(\sqrt{7}-\sqrt{3})}{(\sqrt{7}+\sqrt{3})(\sqrt{7}-\sqrt{3})}$$
$$=\frac{(7\sqrt{35}-\sqrt{15})}{7-3}.$$
$\sqrt{35}=5.9161.$ $\sqrt{15}=3.8729.$
$$\therefore\ \frac{7\sqrt{5}}{\sqrt{7}+\sqrt{3}}=\frac{7(5.9161-3.8729)}{4}=3.576.\ \textit{Ans.}$$

(4)

$$\frac{7-2\sqrt{3}+3\sqrt{2}}{3+3\sqrt{3}-2\sqrt{2}}$$

$$=\frac{(7-2\sqrt{3}+3\sqrt{2})(3+3\sqrt{3}+2\sqrt{2})}{(3+3\sqrt{3}-2\sqrt{2})(3+3\sqrt{3}+2\sqrt{2})}$$

$$=\frac{21+21\sqrt{3}+14\sqrt{2}-6\sqrt{3}-18-4\sqrt{6}+9\sqrt{2}+9\sqrt{6}+12}{9+18\sqrt{3}+27-8}$$

$$=\frac{15+15\sqrt{3}+23\sqrt{2}+5\sqrt{6}}{28+18\sqrt{3}}$$

$$=\frac{(15+15\sqrt{3}+23\sqrt{2}+5\sqrt{6})(28-18\sqrt{3})}{(28+18\sqrt{3})(28-18\sqrt{3})}$$

$$=\frac{420+420\sqrt{3}+644\sqrt{2}+140\sqrt{6}-270\sqrt{3}-870-414\sqrt{6}-270\sqrt{2}}{784-972}$$

$$=\frac{-390+150\sqrt{3}-274\sqrt{6}+374\sqrt{2}}{-188}$$

$$=\frac{195-75\sqrt{3}+137\sqrt{6}-187\sqrt{2}}{94}$$

$\sqrt{3}=1.732$ $\sqrt{6}=2.449$ $\sqrt{2}=1.414$.

$$\therefore \frac{7-2\sqrt{3}+3\sqrt{2}}{3+3\sqrt{3}-2\sqrt{2}}$$

$$=\frac{195-75\times 1.732+137\times 2.449-187\times 1.414}{94}$$

$$=\frac{136.195}{94}=1.449. \textit{ Ans.}$$

Exercise 18.

1. Solve

$$\frac{x^2-5}{3}+\frac{2x^2+1}{6}=\frac{1}{2}.$$

Simplify,

$$2x^2-10+2x^2+1=3.$$
$$4x^2=12.$$
$$\therefore x^2=3.$$
$$\therefore x=\pm\sqrt{3}.$$

2. Solve

$$\frac{3}{1+x}+\frac{3}{1-x}=8.$$

Simplify,

$$3-3x+3+3x=8-8x^2$$
$$8x^2=2$$
$$x^2=\tfrac{1}{4}.$$
$$\therefore x=\pm\tfrac{1}{2}.$$

3. Solve $\frac{3}{4x^2}-\frac{1}{6x^2}=\frac{7}{3}.$

Simplify,

$$9-2=28x^2.$$
$$x^2=\tfrac{1}{4}.$$
$$\therefore\ x=\pm\tfrac{1}{2}.$$

4. Solve $5x^2-9=2x^2+24.$

$$3x^2=33$$
$$x^2=11.$$
$$\therefore\ x=\pm\sqrt{11}.$$

5. Solve $\frac{x^2}{5}-\frac{x^2-10}{15}=7-\frac{50+x^2}{25}.$

Simplify, $15x^2-5x^2+50=525-150-3x^2.$

$$13x^2=325.$$
$$x^2=25.$$
$$\therefore\ x=\pm 5.$$

6. Solve $\frac{3x^2-27}{x^2+3}+\frac{90+4x^2}{x^2+9}=7.$

Simplify,

$$(3x^2-27)(x^2+9)+(x^2+3)(90+4x^2)=7(x^2+3)(x^2+9).$$
$$3x^4-243+4x^4+102x^2+270=7x^4+84x^2+189.$$
$$18x^2=162.$$
$$x^2=9.$$
$$\therefore\ x=\pm 3.$$

7. Solve $\frac{4x^2+5}{10}-\frac{2x^2-5}{15}=\frac{7x^2-25}{20}.$

Simplify, $24x^2+30-8x^2+20=21x^2-75.$

$$5x^2=125.$$
$$x^2=25.$$
$$\therefore\ x=\pm 5.$$

8. Solve $\frac{10x^2+17}{18}-\frac{12x^2+2}{11x^2-8}=\frac{5x^2-4}{9}.$

Transpose the 3d term and combine with 1st.

$$\frac{10x^2+17-10x^2+8}{18}-\frac{12x^2+2}{11x^2-8}=0.$$
$$\frac{25}{18}-\frac{12x^2+2}{11x^2-8}=0.$$

Simplify, $275x^2-200-216x^2-36=0.$

$$59x^2=236.$$
$$x^2=4.$$
$$\therefore\ x=\pm 2.$$

9. Solve

$$x^2 + bx + a = bx(1 - bx).$$

$$x^2 + bx + a = bx - b^2x^2.$$

Transpose and combine,

$$(1 + b^2)x^2 = -a.$$

$$x^2 = \frac{-a}{1 + b^2}.$$

$$\therefore\ x = \pm\sqrt{\frac{-a}{1 + b^2}}.$$

10. Solve $ax^2 + b = c$.

$$ax^2 = c - b.$$

$$x^2 = \frac{c - b}{a}.$$

$$\therefore\ x = \pm\sqrt{\frac{c - b}{a}}.$$

11. Solve

$$x^2 - ax + b = ax(x - 1).$$

$$x^2 - ax + b = ax^2 - ax.$$

Transpose and combine,

$$(1 - a)x^2 = -b.$$

$$x^2 = \frac{-b}{1 - a}.$$

$$x^2 = \frac{b}{a - 1}.$$

$$\therefore\ x = \pm\sqrt{\frac{b}{a - 1}}.$$

12. Solve

$$\frac{ab - x}{b - ax} = \frac{b - cx}{bc - x}.$$

Simplify,

$$ab^2c - abx - bcx + x^2$$
$$= b^2 - abx - bcx + acx^2.$$

$$(1 - ac)x^2 = b^2(1 - ac).$$

$$x^2 = b^2.$$

$$\therefore\ x = \pm b.$$

13. Solve $\qquad \dfrac{3(x + a)}{4x - a} - \dfrac{2x + a}{2a + x} = 1.$

Simplify,

$$3(x + a)(2a + x) - (4x - a)(2x + a) = (4x - a)(2a + x).$$

$$6a^2 + 9ax + 3x^2 + a^2 - 2ax - 8x^2 = -2a^2 + 7ax + 4x^2.$$

$$-9x^2 = -9a^2.$$

$$x^2 = a^2.$$

$$\therefore\ x = \pm a.$$

14. Solve $\qquad \dfrac{3a}{x - 5a} + \dfrac{x + 4a}{x + 3a} = \dfrac{7a^2 + 2ax - x^2}{(x - 5a)(x + 3a)}.$

Simplify,

$$3ax + 9a^2 + x^2 - ax - 20a^2 = 7a^2 + 2ax - x^2.$$

$$2x^2 = 18a^2.$$

$$x^2 = 9a^2.$$

$$\therefore\ x = \pm 3a.$$

15. Solve $\dfrac{2(a+2b)}{a+2x}+\dfrac{a-2x}{a+b}=\dfrac{b^2}{(a+b)(a+2x)}.$

Simplify,

$$2a^2+6ab+4b^2+a^2-4x^2=b^2.$$

$$4x^2=3(a^2+2ab+b^2).$$

$$x^2=\tfrac{3}{4}(a+b)^2.$$

$$\therefore\ x=\pm\frac{a+b}{2}\times\sqrt{3}.$$

Exercise 19.

1. Solve $x^2-2x=15$.

Complete the square,

$$x^2-2x+1=16.$$

Extract the root,

$$x-1=\pm 4.$$

$$\therefore\ x=5,\text{ or }-3.$$

2. Solve $x^2-14x=-48$.

Complete the square,

$$x^2-14x+49=1.$$

Extract the root,

$$x-7=\pm 1.$$

$$\therefore\ x=6,\text{ or }8.$$

3. Solve $x^2-x=12$.

Complete the square,

$$x^2-x+\tfrac{1}{4}=\tfrac{49}{4}.$$

Extract the root,

$$x-\tfrac{1}{2}=\pm\tfrac{7}{2}.$$

$$\therefore\ x=4,\text{ or }-3.$$

4. Solve $x^2-3x=28$.

Complete the square,

$$x^2-3x+\tfrac{9}{4}=30\tfrac{1}{4}.$$

Extract the root,

$$x-\tfrac{3}{2}=\pm 5\tfrac{1}{2}.$$

$$\therefore\ x=7,\text{ or }-4.$$

5. Solve

$$x^2-13x+42=0.$$

$$x^2-13x=-42.$$

Complete the square,

$$x^2-13x+42\tfrac{1}{4}=\tfrac{1}{4}.$$

Extract the root,

$$x-6\tfrac{1}{2}=\pm\tfrac{1}{2}.$$

$$\therefore\ x=7,\text{ or }6.$$

6. Solve

$$x^2-21x+108=0.$$

$$x^2-21x=-108.$$

Multiply by 4,

$$4x^2-84x=-432.$$

Complete the square,

$$4x^2-84x+441=9.$$

Extract the root,

$$2x-21=\pm 3.$$

$$2x=24,\text{ or }18.$$

$$\therefore\ x=12,\text{ or }9.$$

7. Solve $2x^2+x=6$.

Multiply by 8,

$$16x^2+8x=48.$$

Complete the square,

$$16x^2+8x+1=49.$$

Extract the root,

$$4x+1=\pm 7.$$

$$4x=6,\text{ or }-8.$$

$$\therefore\ x=1\tfrac{1}{2},\text{ or }-2.$$

8. Solve

$$4x^2 + 7x = 15.$$

Multiply by 16,

$$64x^2 + 112x = 240.$$

Complete the square,

$$64x^2 + 112x + 49 = 289.$$

Extract the root,

$$8x + 7 = \pm 17.$$

$$8x = 10, \text{ or } -24.$$

$$\therefore\ x = 1\tfrac{1}{4}, \text{ or } -3.$$

9. Solve

$$3x^2 - 19x + 28 = 0.$$

Multiply by 12, and transpose,

$$36x^2 - 228x = -336.$$

Complete the square,

$$36x^2 - 228x + 361 = 25.$$

Extract the root,

$$6x - 19 = \pm 5.$$

$$6x = 24, \text{ or } 14.$$

$$\therefore\ x = 4, \text{ or } 2\tfrac{1}{3}.$$

10. Solve

$$4x^2 + 17x - 15 = 0.$$

Multiply by 16, and transpose,

$$64x^2 + 272x = 240.$$

Complete the square,

$$64x^2 + 272x + 289 = 529.$$

Extract the root,

$$8x + 17 = \pm 23.$$

$$8x = 6, \text{ or } -40.$$

$$\therefore\ x = \tfrac{3}{4}, \text{ or } -5.$$

11. Solve

$$6x^2 - x = 12$$

Multiply by 6,

$$36x^2 - 6x = 72.$$

Complete the square,

$$36x^2 - 6x + \tfrac{1}{4} = 72\tfrac{1}{4}.$$

$$6x - \tfrac{1}{2} = \pm 8\tfrac{1}{2}.$$

$$6x = 9, \text{ or } -8.$$

$$\therefore\ x = 1\tfrac{1}{2}, \text{ or } -1\tfrac{1}{3}.$$

12. Solve

$$5x^2 - \tfrac{28}{3}x + 4 = 0.$$

Multiply by 5,

$$25x^2 - \tfrac{140}{3}x = -20.$$

Complete the square,

$$25x^2 - \tfrac{140}{3}x + \tfrac{196}{9} = \tfrac{16}{9}.$$

Extract the root,

$$5x - \tfrac{14}{3} = \pm \tfrac{4}{3}.$$

$$5x = 6, \text{ or } \tfrac{10}{3}.$$

$$\therefore\ x = 1\tfrac{1}{5}, \text{ or } \tfrac{2}{3}.$$

13. Solve

$$6x^2 - 7x + \tfrac{5}{3} = 0.$$

Multiply by 6,

$$36x^2 - 42x = -10.$$

Complete the square,

$$36x^2 - 42x + \tfrac{49}{4} = \tfrac{9}{4}.$$

Extract the root,

$$6x - \tfrac{7}{2} = \pm \tfrac{3}{2}.$$

$$6x = 5, \text{ or } 2.$$

$$\therefore\ x = \tfrac{5}{6}, \text{ or } \tfrac{1}{3}.$$

14. Solve

$$\frac{x^2 + 1}{17} + (x + 1)(x + 2) = 0.$$

Simplify,

$$x^2 + 1 + 17x^2 + 51x + 34 = 0.$$

$$18x^2 + 51x = -35.$$

Multiply by 8,

$$144x^2 + 408x = -280.$$

Complete the square,

$$144x^2 + 408x + 289 = 9.$$

Extract the root,

$$12x + 17 = \pm 3.$$

$$12x = -14, \text{ or } -20.$$

$$\therefore\ x = -1\tfrac{1}{6}, \text{ or } -1\tfrac{2}{3}.$$

15. Solve
$(x-5)^2+x^2-5=16(x+3).$
$x^2-10x+25+x^2-5=16x+48.$
$2x^2-26x=28.$
Multiply by 2,
$4x^2-52x=56.$
Complete the square,
$4x^2-52x+169=225.$
Extract the root,
$2x-13=\pm 15.$
$2x=28, \text{ or } -2.$
$\therefore x=14, \text{ or } -1.$

16. Solve
$\frac{x^2}{6}+\frac{3x-19}{3}=\frac{11+x}{3}.$
Simplify,
$x^2+6x-38=22+2x.$
$x^2+4x=60.$
Complete the square,
$x^2+4x+4=64.$
Extract the root,
$x+2=\pm 8.$
$\therefore x=6, \text{ or } -10.$

17. Solve
$\frac{2x^2-11}{2x+3}=\frac{x+1}{2}.$
Simplify,
$4x^2-22=2x^2+5x+3.$
$2x^2-5x=25.$
Multiply by 8,
$16x^2-40x=200.$
Complete the square,
$16x^2-40x+25=225.$
Extract the root,
$4x-5=\pm 15.$
$4x=20, \text{ or } -10.$
$\therefore x=5, \text{ or } -2\frac{1}{2}.$

18. Solve
$\frac{x+1}{x}+\frac{x}{6}=\frac{11}{2x}.$
Simplify,
$6x+6+x^2=33.$
$x^2+6x=27.$
Complete the square,
$x^2+6x+9=36.$
Extract the root,
$x+3=\pm 6.$
$\therefore x=3, \text{ or } -9.$

19. Solve $\frac{x^2-4}{3x}+\frac{2x}{5}=x+\frac{1-2x}{5}.$
Simplify, $5x^2-20+6x^2=15x^2+3x-6x^2.$
$2x^2-3x=20.$
Multiply by 8, $16x^2-24x=160.$
Complete the square, $16x^2-24x+9=169.$
Extract the root, $4x-3=\pm 13.$
$4x=16, \text{ or } -10.$
$\therefore x=4, \text{ or } -2\frac{1}{2}.$

20. Solve $x+\frac{x+6}{x-6}=2(x-2).$

Transpose and combine, $\frac{x+6}{x-6}=x-4.$

Simplify, $x + 6 = x^2 - 10x + 24.$

$x^2 - 11x = -18.$

Complete the square, $x^2 - 11x + \frac{121}{4} = \frac{49}{4}.$

Extract the root, $x - \frac{11}{2} = \pm \frac{7}{2}.$

$\therefore x = 9, \text{ or } 2.$

21. Solve $\frac{6}{2x-6} + \frac{x}{3-x} = \frac{8}{x}.$

$\frac{6}{2(x-3)} - \frac{x}{x-3} = \frac{8}{x}.$

Simplify, $6x - 2x^2 = 16x - 48.$

Transpose and combine, $2x^2 + 10x - 48 = 0.$

$x^2 + 5x = 24.$

Complete the square, $x^2 + 5x + \frac{25}{4} = \frac{121}{4}.$

Extract the root, $x + \frac{5}{2} = \pm \frac{11}{2}.$

$\therefore x = 3, \text{ or } -8.$

22. Solve $\frac{x+2}{x-1} - \frac{4-x}{2x} = \frac{7}{3}.$

Simplify, $6x(x+2) + 3(x-1)(x-4) = 7(x-1)2x.$

$6x^2 + 12x + 3x^2 - 15x + 12 = 14x^2 - 14x.$

$5x^2 - 11x = 12.$

Multiply by 5, $25x^2 - 55x = 60.$

Complete the square, $25x^2 - 55x + \frac{121}{4} = \frac{361}{4}.$

Extract the root, $5x - \frac{11}{2} = \pm \frac{19}{2}.$

$5x = 15, \text{ or } -4.$

$\therefore x = 3, \text{ or } -\frac{4}{5}.$

23. Solve $\frac{x-6}{x-2} + \frac{x+5}{2x+1} = 1.$

Simplify, $2x^2 - 11x - 6 + x^2 + 3x - 10 = 2x^2 - 3x - 2.$

$x^2 - 5x = 14.$

Complete the square, $x^2 - 5x + \frac{25}{4} = \frac{81}{4}.$

Extract the root, $x - \frac{5}{2} = \pm \frac{9}{2}.$

$\therefore x = 7, \text{ or } -2.$

24. Solve $\frac{x-3}{x+4} + \frac{x-4}{2(x-1)} = \frac{1}{2}.$

Simplify, $2x^2 - 8x + 6 + x^2 - 16 = x^2 + 3x - 4.$

$2x^2 - 11x = 6.$

Multiply by 8, $16x^2 - 88x = 48.$

Complete the square, $16x^2 - 88x + 121 = 169.$
Extract the root, $4x - 11 = \pm 13.$
$4x = 24, \text{ or } -2.$
$\therefore x = 6, \text{ or } -\frac{1}{2}.$

25. Solve $$\frac{x+1}{x^2-4} + \frac{1-x}{x+2} = \frac{2}{5(x-2)}.$$

Simplify, $5x + 5 + 5(x-2)(1-x) = 2(x+2).$
$5x + 5 + 5x - 5x^2 - 10 + 10x = 2x + 4.$
$5x^2 - 18x = -9.$
Multiply by 5, $25x^2 - 90x = -45.$
Complete the square, $25x^2 - 90x + 81 = 36.$
Extract the root, $5x - 9 = \pm 6.$
$5x = 15, \text{ or } 3.$
$\therefore x = 3, \text{ or } \frac{3}{5}.$

26. Solve $$\frac{x-5}{x+3} + \frac{x-8}{x-3} = \frac{80}{x^2-9} + \frac{1}{2}.$$

Simplify,
$2(x-5)(x-3) + 2(x-8)(x+3) = 160 + x^2 - 9.$
$2x^2 - 16x + 30 + 2x^2 - 10x - 48 = 160 + x^2 - 9.$
$3x^2 - 26x = 169.$
Multiply by 3, $9x^2 - 78x = 507.$
Complete the square, $9x^2 - 78x + 169 = 676.$
Extract the root, $3x - 13 = \pm 26.$
$3x = 39, \text{ or } -13.$
$\therefore x = 13, \text{ or } -4\frac{1}{3}.$

27. Solve $$\frac{1}{x-3} + \frac{7}{x+3} = \frac{14}{x^2-9} - \frac{x-4}{x+3}.$$

Transpose the 4th term, and combine with the 2d,

$$\frac{1}{x-3} + 1 = \frac{14}{x^2-9}.$$

Simplify, $x + 3 + x^2 - 9 = 14.$
$x^2 + x = 20.$
Complete the square, $x^2 + x + \frac{1}{4} = 20\frac{1}{4}.$
Extract the root, $x + \frac{1}{2} = \pm 4\frac{1}{2}.$
$\therefore x = 4, \text{ or } -5.$

28. Solve $\frac{3x+5}{x+3}+\frac{x+3}{x-3}=\frac{x-1}{x^2-9}.$

Simplify, $(3x+5)(x-3)+(x+3)^2=x-1.$

$$3x^2-4x-15+x^2+6x+9=x-1.$$

$$4x^2+x=5.$$

Multiply by 16, $64x^2+16x=80.$

Complete the square, $64x^2+16x+1=81.$

Extract the root, $8x+1=\pm 9.$

$$8x=8,\text{ or }-10.$$

$$\therefore\ x=1,\text{ or }-1\tfrac{1}{4}.$$

29. Solve $\frac{x+1}{x-1}+\frac{x+2}{x-2}=\frac{2x+13}{x+1}.$

This may be written, $1+\frac{2}{x-1}+1+\frac{4}{x-2}=2+\frac{11}{x+1}.$

$$\frac{2}{x-1}+\frac{4}{x-2}=\frac{11}{x+1}.$$

Simplify, $2(x-2)(x+1)+4(x-1)(x+1)=11(x-1)(x-2).$

$$2x^2-2x-4+4x^2-4=11x^2-33x+22.$$

$$5x^2-31x=-30.$$

Multiply by 20, $100x^2-620x=-600.$

Complete the square, $100x^2-620x+961=361.$

Extract the root, $10x-31=\pm 19.$

$$10x=50,\text{ or }12.$$

$$\therefore\ x=5,\text{ or }1\tfrac{1}{5}.$$

30. Solve $\frac{2x-1}{x+1}+\frac{3x-1}{x+2}+\frac{7-x}{x-1}=4.$

This may be written,

$$2-\frac{3}{x+1}+3-\frac{7}{x+2}-1+\frac{6}{x-1}=4.$$

$$\frac{3}{x+1}+\frac{7}{x+2}-\frac{6}{x-1}=0.$$

Simplify,

$$3(x+2)(x-1)+7(x+1)(x-1)-6(x+1)(x+2)=0.$$

$$3x^2+3x-6+7x^2-7-6x^2-18x-12=0.$$

$$4x^2-15x=25.$$

Multiply by 16, $64x^2-240x=400.$

Complete the square, $64x^2-240x+225=625.$

Extract the root, $8x-15=\pm 25.$

$$8x=40,\text{ or }-10.$$

$$\therefore\ x=5,\text{ or }-1\tfrac{1}{4}.$$

31. Solve

$$\frac{3x+2}{1-5x}+\frac{x-7}{1+5x}+\frac{6(x^2-x+1)}{25x^2-1}+5=0.$$

Simplify,

$$(1+5x)(3x+2)+(1-5x)(x-7)-6(x^2-x+1)+5(1-25x^2)=0.$$

$$15x^2+13x+2-5x^2+36x-7-6x^2+6x-6+5-125x^2=0.$$

$$121x^2-55x+6=0.$$

$$121x^2-55x=-6.$$

Complete the square, $121x^2-55x+\frac{25}{4}=\frac{1}{4}.$

Extract the root, $11x-\frac{5}{2}=\pm\frac{1}{2}.$

$$11x=3, \text{ or } 2.$$

$$\therefore\ x=\tfrac{3}{11}, \text{ or } \tfrac{2}{11}.$$

32. Solve $\frac{x+7}{9-4x^2}-\frac{1-x}{2x+3}=\frac{4}{2x-3}.$

Simplify, $x+7-(1-x)(3-2x)=-4(3+2x).$

$$x+7-3+5x-2x^2=-12-8x.$$

$$2x^2-14x=16.$$

Multiply by 2, $4x^2-28x=32.$

Complete the square, $4x^2-28x+49=81.$

Extract the root, $2x-7=\pm 9.$

$$2x=16, \text{ or } -2.$$

$$\therefore\ x=8, \text{ or } -1.$$

33. Solve $\frac{2x+1}{x+3}+\frac{2(x+1)}{x+2}=2\tfrac{1}{12}.$

Simplify,

$$12(x+2)(2x+1)+24(x+3)(x+1)=25(x+3)(x+2).$$

$$24x^2+60x+24+24x^2+96x+72=25x^2+125x+150.$$

$$23x^2+31x=54.$$

Multiply by 4×23,

$$4\times 23^2x^2+4\times 23\times 31x=4968.$$

Complete the square,

$$2^2\times 23^2x^2+4\times 23\times 31x+961=5929.$$

Extract the root, $2\times 23x+31=\pm 77.$

$$46x=46, \text{ or } -108.$$

$$\therefore\ x=1, \text{ or } -2\tfrac{8}{23}.$$

Exercise 20.

1. Solve $x^2-2ax=3a^2$.

Complete the square,

$$x^2-2ax+a^2=4a^2.$$

Extract the root,

$$x-a=\pm 2a.$$

$$\therefore\ x=3a, \text{ or } -a.$$

2. Solve $x^2+7a^2=8ax$.

$$x^2-8ax=-7a^2.$$

Complete the square,

$$x^2-8ax+16a^2=9a^2.$$

Extract the root,

$$x-4a=\pm 3a.$$

$$\therefore\ x=7a, \text{ or } a.$$

3. Solve

$$4x(x-a)+a^2=b^2.$$

$$4x^2-4ax+a^2=b^2.$$

Extract the root,

$$2x-a=\pm b.$$

$$2x=a\pm b.$$

$$\therefore\ x=\frac{a\pm b}{2}.$$

4. Solve

$$\frac{x^2}{2}-\frac{ax}{3}=2a(x+2a).$$

$$3x^2-2ax=12ax+24a^2.$$

$$3x^2-14ax=24a^2.$$

$$9x^2-42ax=72a^2.$$

Complete the square,

$$9x^2-42ax+49a^2=121a^2.$$

Extract the root,

$$3x-7a=\pm 11a.$$

$$3x=18a, \text{or} -4a.$$

$$\therefore\ x=6a, \text{ or } -\frac{4a}{3}.$$

5. Solve $x^2=ax+b$.

$$x^2-ax=b.$$

$$4x^2-4ax=4b.$$

Complete the square,

$$4x^2-4ax+a^2=a^2+4b.$$

Extract the root,

$$2x-a=\pm\sqrt{a^2+4b}.$$

$$2x=a\pm\sqrt{a^2+4b}.$$

$$x=\frac{a\pm\sqrt{a^2+4b}}{2}.$$

6. Solve

$$\frac{(x+a)^2}{a^2}=\frac{(x-a)^2}{b^2}.$$

$$\frac{x+a}{a}=\pm\frac{x-a}{b}.$$

(1)

$$\frac{x+a}{a}=+\frac{x-a}{b}.$$

$$bx+ab=ax-a^2.$$

$$(b-a)x=-a^2-ab.$$

$$\therefore\ x=\frac{a(a+b)}{a-b}.$$

(2)

$$\frac{x+a}{a}=-\frac{x-a}{b}.$$

$$bx+ab=-ax+a^2.$$

$$(a+b)x=a^2-ab.$$

$$\therefore\ x=\frac{a(a-b)}{a+b}.$$

7. Solve $x^2 - \frac{x}{a} = \frac{3}{4a^2}$.

$$4a^2x^2 - 4ax = 3.$$

Complete the square,

$$4a^2x^2 - 4ax + 1 = 4.$$

Extract the root,

$$2ax - 1 = \pm 2.$$

$$2ax = 3, \text{ or } -1.$$

$$\therefore x = \frac{3}{2a}, \text{ or } -\frac{1}{2a}.$$

8. Solve

$$x^2 - (a + b)x = -ab.$$

$$4x^2 - 4(a + b)x = -4ab.$$

Complete the square,

$$4x^2 - 4(a + b)x + (a + b)^2 = a^2 - 2ab + b^2.$$

Extract the root,

$$2x - (a + b) = \pm (a - b).$$

$$2x = 2a, \text{ or } 2b.$$

$$\therefore x = a, \text{ or } b.$$

9. Solve

$$x^2 - \frac{m^2 + n^2}{mn}x + 1 = 0.$$

$$x^2 - \frac{m^2 + n^2}{mn}x = -1.$$

$$x^2 - \frac{m^2 + n^2}{mn}x + \frac{(m^2 + n^2)^2}{4m^2n^2} = \frac{m^4 + 2m^2n^2 + n^4}{4m^2n^2} - 1 = \frac{(m^2 - n^2)^2}{4m^2n^2}.$$

$$x - \frac{m^2 + n^2}{2mn} = \pm \frac{m^2 - n^2}{2mn}.$$

$$\therefore x = \frac{m}{n}, \text{ or } \frac{n}{m}.$$

10. Solve

$$\frac{2x(a - x)}{3a - 2x} = \frac{a}{4}.$$

$$8ax - 8x^2 = 3a^2 - 2ax.$$

$$8x^2 - 10ax = -3a^2.$$

$$16x^2 - 20ax = -6a^2.$$

$$16x^2 - 20ax + \frac{25}{4}a^2 = \frac{a^2}{4}.$$

$$4x - \frac{5a}{2} = \pm \frac{a}{2}.$$

$$4x = 3a, \text{ or } 2a.$$

$$\therefore x = \frac{3a}{4}, \text{ or } \frac{a}{2}.$$

11. Solve

$$2x^2 + \frac{ab}{2} = (a + b)x.$$

$$2x^2 - (a + b)x = -\frac{ab}{2}.$$

$$16x^2 - 8(a + b)x = -4ab.$$

$$16x^2 - 8(a + b)x + (a + b)^2 = a^2 - 2ab + b^2.$$

$$4x - (a + b) = \pm (a - b).$$

$$4x = 2a, \text{ or } 2b.$$

$$\therefore x = \frac{a}{2}, \text{ or } \frac{b}{2}.$$

12. Solve

$$(x + m)^2 + (x - m)^2 = 5mx.$$

$$x^2 + 2mx + m^2 + x^2 - 2mx + m^2 = 5mx.$$

$$2x^2 - 5mx = -2m^2.$$

$$16x^2 - 40mx = -16m^2.$$

$$16x^2 - 40mx + 25m^2 = 9m^2.$$

$$4x - 5m = \pm 3m.$$

$$4x = 8m, \text{ or } 2m.$$

$$x = 2m, \text{ or } \frac{m}{2}.$$

13. Solve $$ax^2 + 5a^2x + \frac{9a^3}{4} = 0.$$
$$4a^2x^2 + 20a^3x = -9a^3.$$
$$4a^2x^2 + 20a^3x + 25a^4 = 25a^4 - 9a^3.$$
$$2ax + 5a^2 = \pm a\sqrt{25a^2 - 9a}.$$
$$2ax = -5a^2 \pm a\sqrt{25a^2 - 9a}.$$
$$\therefore\ x = \frac{-5a \pm \sqrt{25a^2 - 9a}}{2}.$$

14. Solve $$b(a - x)^2 = (b - 1)x^2.$$
Expand, $$ba^2 - 2abx + bx^2 = bx^2 - x^2.$$
$$x^2 - 2abx = -a^2b.$$
$$x^2 - 2abx + a^2b^2 = a^2b^2 - a^2b.$$
$$x - ab = \pm a\sqrt{b^2 - b}.$$
$$\therefore\ x = a(b \pm \sqrt{b^2 - b}).$$

15. Solve $$\frac{x}{a - x} + \frac{a + b}{x} = \frac{a}{a - x}.$$
$$x^2 + (a + b)(a - x) = ax.$$
$$x^2 - (2a + b)x = -a^2 - ab.$$
$$x^2 - (\ \ ) + \left(\frac{2a + b}{2}\right)^2 = \frac{b^2}{4}.$$
$$x - \frac{2a + b}{2} = \pm\frac{b}{2}.$$
$$\therefore\ x = a + b, \text{ or } a.$$

16. Solve $$\frac{x^2 - ab}{x - b} = \frac{x + a}{2}.$$
$$2x^2 - 2ab = x^2 + (a - b)x - ab.$$
$$x^2 - (a - b)x = ab.$$
$$4x^2 - 4(a - b)x = 4ab.$$
$$4x^2 - 4(a - b)x + (a - b)^2 = a^2 + 2ab + b^2.$$
$$2x - (a - b) = \pm(a + b).$$
$$2x = 2a, \text{ or } -2b.$$
$$\therefore\ x = a, \text{ or } -b.$$

17. Solve $$\frac{a+b}{x-2a}+\frac{2a+b}{a}=\frac{x}{a}.$$

$$a^2+ab+2a+bx-4a^2-2ab=x^2-2ax.$$

$$x^2-(4a+b)x=-ab-3a^2.$$

$$4x^2-4(4a+b)x=-4ab-12a^2.$$

$$4x^2-4(4a+b)x+(4a+b)^2=4a^2+4ab+b^2.$$

$$2x-4a-b=\pm(2a+b).$$

$$2x=2a, \text{ or } 6a+2b.$$

$$\therefore\ x=a, \text{ or } 3a+b.$$

18. Solve $$\frac{ax}{b^2}+\frac{a+x}{x}=\frac{5a+x}{2b}.$$

$$2ax^2+2ab^2+2b^2+2b^2x=5abx+bx^2.$$

$$(2a-b)x^2+(2b^2-5ab)x=-2ab^2.$$

Multiply by $4(2a-b)$,

$$4(2a-b)^2x^2+4(2a-b)(2b^2-5ab)x=-16a^2b^2+8ab^3.$$

$$4(2a-b)^2x^2+(\quad)+(2b^2-5ab)^2=4b^4-12ab^3+9a^2b^2.$$

$$2(2a-b)x+2b^2-5ab=\pm(2b^2-3ab).$$

$$2(2a-b)x=2ab, \text{ or } -4b^2+8ab.$$

$$\therefore\ x=\frac{ab}{2a-b}, \text{ or } 2b.$$

19. Solve $$\frac{ab}{ax-bx}=a+b-(a-b)x.$$

$$ab=(a^2-b^2)x-(a-b)^2x^2.$$

$$(a-b)^2x^2-(a^2-b^2)x=-ab.$$

$$4(a-b)^2x^2-4(a^2-b^2)x=-4ab.$$

$$4(a-b)^2x^2-4(a^2-b^2)x+(a+b)^2=a^2-2ab+b^2.$$

$$2(a-b)x-(a+b)=\pm(a-b).$$

$$2(a-b)x=2a, \text{ or } 2b.$$

$$\therefore\ x=\frac{a}{a-b}, \text{ or } \frac{b}{a-b}.$$

20. Solve $$\frac{5ab-3b^2-ax}{2a-x}=\frac{2a+x}{3}.$$

$$15ab-9b^2-3ax=4a^2-x^2.$$

$$x^2-3ax=4a^2-15ab+9b^2.$$

$$4x^2-12ax=16a^2-60ab+36b^2.$$

$$4x^2-12ax+9a^2=25a^2-60ab+36b^2.$$

$$2x-3a=\pm(5a-6b).$$

$$2x=8a-6b, \text{ or } 6b-2a.$$

$$\therefore\ x=4a-3b, \text{ or } 3b-a.$$

21. Solve
$$x^2 - ax = \frac{(3a + 2x)b}{2} + \frac{3(a^2 + b^2)}{4}.$$
$$4x^2 - 4ax = 6ab + 4bx + 3a^2 + 3b^2.$$
$$4x^2 - 4(a + b)x = 3a^2 + 6ab + 3b^2.$$
$$4x^2 - 4(a + b)x + (a + b)^2 = 4a^2 + 8ab + 4b^2.$$
$$2x - (a + b) = \pm 2(a + b).$$
$$2x = 3(a + b), \text{ or } -(a + b).$$
$$\therefore\ x = \tfrac{3}{2}(a + b), \text{ or } -\tfrac{1}{2}(a + b).$$

22. Solve
$$\frac{3a}{x + a} + \frac{2a}{x + 2a} = \frac{4a}{x} + \frac{a}{x + 3a}.$$

Divide by a and simplify,
$$3(x + 2a)x(x + 3a) + 2(x + a)x(x + 3a)$$
$$= 4(x + a)(x + 2a)(x + 3a) + x(x + a)(x + 2a).$$
$$3x^3 + 15ax^2 + 18a^2x + 2x^3 + 8ax^2 + 6a^2x$$
$$= 4x^3 + 24ax^2 + 44a^2x + 24a^3 + x^3 + 3ax^2 + 2a^2x.$$
$$-4ax^2 - 22a^2x = 24a^3.$$
$$4x^2 + 22ax = -24a^2.$$
$$4x^2 + 22ax + \tfrac{121}{4}a^2 = \tfrac{25}{4}a^2.$$
$$2x + \tfrac{11}{2}a = \pm\tfrac{5}{2}a.$$
$$2x = -3a, \text{ or } -8a.$$
$$\therefore\ x = -\frac{3a}{2}, \text{ or } -4a.$$

23. Solve
$$\frac{a - b + x}{a + b + x} + \frac{a + b}{x + b} = 2.$$
$$ax - bx + x^2 + ab - b^2 + bx + a^2 + 2ab + b^2 + ax + bx$$
$$= 2ax + 2bx + 2x^2 + 2ab + 2b^2 + 2bx.$$
$$x^2 + 3bx = ab - 2b^2 + a^2.$$
$$x^2 + 3bx + \tfrac{9}{4}b^2 = a^2 + ab + \frac{b^2}{4}.$$
$$x + \tfrac{3}{2}b = \pm\left(a + \frac{b}{2}\right).$$
$$\therefore\ x = a - b, \text{ or } -(a + 2b).$$

24. Solve
$$\frac{a+4b}{x+2b}-\frac{a-4b}{x-2b}=\frac{4b}{a}.$$
$$a^2x-2a^2b+4abx-8ab^2-a^2x-2a^2b+4abx+8ab^2$$
$$=4bx^2-16b^3.$$
$$4bx^2-8abx=-4a^2b+16b^3.$$
$$4x^2-8ax+4a^2=16b^2.$$
$$2x-2a=\pm 4b.$$
$$\therefore\ x=a\pm 2b.$$

25. Solve
$$\frac{4(x+a)}{a+b}-\frac{3(a+b)}{x+a}=4.$$
$$4x^2+8ax+4a^2-3a^2-6ab-3b^2=4ax+4bx+4a^2+4ab.$$
$$4x^2+4(a-b)x=3a^2+10ab+3b^2.$$
$$4x^2+4(a-b)x+(a-b)^2=4a^2+8ab+4b^2.$$
$$2x+a-b=\pm(2a+2b).$$
$$2x=a+3b,\text{ or }-(3a+b).$$
$$\therefore\ x=\frac{a+3b}{2},\text{ or }-\frac{3a+b}{2}.$$

26. Solve
$$\frac{(4a^2-9b^2)(x^2+1)}{4a^2+9b^2}=2x.$$
$$(4a^2-9b^2)x^2-2(4a^2+9b^2)x=9b^2-4a^2.$$

Multiply by $4a^2-9b^2$,
$$(4a^2-9b^2)^2x^2-2(4a^2-9b^2)(4a^2+9b^2)=72a^2b^2-81b^4-16a^4.$$
$$(4a^2-9b^2)^2x^2-(\quad)+(4a^2+9b^2)^2=144a^2b^2.$$
$$(4a^2-9b^2)x-(4a^2+9b^2)=\pm 12ab.$$
$$(4a^2-9b^2)x=(2a\pm 3b)^2,\text{ or }(2a-3b)^2.$$
$$\therefore\ x=\frac{2a+3b}{2a-3b},\text{ or }\frac{2a-3b}{2a+3b}.$$

27. Solve
$$(3a^2+b^2)(x^2-x+1)=(a^2+3b^2)(x^2+x+1).$$
$$2(a^2-b^2)x^2-4(a^2+b^2)x=2b^2-2a^2.$$

Multiply by $\frac{a^2-b^2}{2}$,
$$(a^2-b^2)^2x^2-2(a^2-b^2)(a^2+b^2)x=-a^4+2a^2b^2-b^4.$$
$$(a^2-b^2)^2x^2-(\quad)+(a^2+b^2)^2=4a^2b^2.$$
$$(a^2-b^2)x-(a^2+b^2)=\pm 2ab.$$
$$(a^2-b^2)x=(a+b)^2,\text{ or }(a-b)^2.$$
$$\therefore\ x=\frac{a+b}{a-b},\text{ or }\frac{a-b}{a+b}.$$

28. Solve
$$\frac{4a^2}{x+2}-\frac{b^2}{x-2}=\frac{4a^2-b^2}{x(4-x^2)}.$$
$$4a^2x(x-2)-b^2x(x+2)=b^2-4a^2.$$
$$(4a^2-b^2)x^2-2(4a^2+b^2)x=b^2-4a^2.$$
Multiply by $4a^2-b^2$,
$$(4a^2-b^2)^2x^2-2(4a^2-b^2)(4a^2+b^2)x=-(4a^2-b^2)^2.$$
$$(4a^2-b^2)^2x^2-(\quad)+(4a^2+b^2)^2=16a^2b^2.$$
$$(4a^2-b^2)x-(4a^2+b^2)=\pm 4ab.$$
$$(4a^2-b^2)x=(2a+b)^2, \text{ or } (2a-b)^2.$$
$$\therefore x=\frac{2a+b}{2a-b}, \text{ or } \frac{2a-b}{2a+b}.$$

29. Solve
$$\frac{a+2b}{a-2b}=\frac{a^2}{(a-2b)x}-\frac{4b^2}{x^2}.$$
$$(a+2b)x^2=a^2x-4ab^2+8b^3.$$
$$(a+2b)x^2-a^2x=-4ab^2+8b^3.$$
Multiply by $a+2b$,
$$(a+2b)^2x^2-a^2(a+2b)x=-4a^2b^2+16b^4.$$
$$(a+2b)^2x^2-a^2(a+2b)x+\frac{a^4}{4}=\frac{a^4}{4}-4a^2b^2+16b^4.$$
$$(a+2b)x-\frac{a^2}{2}=\pm\left(\frac{a^2}{2}-4b^2\right).$$
$$(a+2b)x=a^2-4b^2, \text{ or } 4b^2.$$
$$x=a-2b, \text{ or } \frac{4b^2}{a+2b}.$$

30. Solve
$$\frac{x+1}{c}-\frac{2}{cx}=\frac{x+2}{ax-bx}.$$
$$(a-b)\,x\,(x+1)-2(a-b)=c(x+2).$$
$$(a-b)x^2+(a-b-c)x=2(a-b+c).$$
Multiply by $4(a-b)$,
$$4(a-b)^2x^2+4(a-b)(a-b-c)x=8a^2-16ab+8b^2+8ac-8bc.$$
$$4(a-b)^2x^2+(\quad)+(a-b-c)^2=9a^2-18ab+9b^2+6ac-6bc+c^2.$$
$$4(a-b)^2x^2+(\quad)+(a-b-c)^2=9(a-b)^2+6(a-b)c+c^2.$$
$$2(a-b)x+a-b-c=\pm[3(a-b)+c].$$
$$2(a-b)x=2(a-b+c), \text{ or } -4(a-b).$$
$$x=\frac{a-b+c}{a-b}, \text{ or } -2.$$

31. Solve $$\frac{a-c}{x-a}-\frac{x-a}{a-c}=\frac{3b(x-c)}{(a-c)(x-a)}.$$

$$(a-c)^2-(x-a)^2=3b(x-c).$$

$$-2ac+c^2-x^2+2ax=3bx-3bc.$$

$$x^2-(2a-3b)x=c^2-2ac+3bc.$$

$$4x^2-4(2a-3b)x=4c^2-8ac+12bc.$$

$$4x^2-(\quad)+(2a-3b)^2=4a^2-12ab+9b^2-8ac+12bc+4c^2$$

$$=(2a-3b)^2-4(2a-3b)c+4c^2.$$

$$2x-(2a-3b)=\pm(2a-3b-2c).$$

$$2x=4a-6b-2c, \text{ or } 2c.$$

$$x=2a-3b-c, \text{ or } c.$$

32. Solve $$x(x+b^2-b)=ax(a+1)-(a+b)^2(a-b).$$

$$x^2+(b^2-b-a^2-a)x=-(a+b)^2(a-b).$$

$$x^2-(a+b)(a-b+1)x=-(a+b)^2(a-b).$$

$$4x^2-4(a+b)(a-b+1)x=-4(a+b)^2(a-b).$$

$$4x^2-(\quad)+(a+b)^2(a-b+1)^2=(a+b)^2[(a-b)^2-2(a-b)+1].$$

$$2x-(a+b)(a-b+1)=\pm(a+b)(a-b-1).$$

$$2x=2(a+b)(a-b), \text{ or } 2(a+b).$$

$$x=a^2-b^2, \text{ or } a+b.$$

33. Solve $$\frac{x}{2}+\frac{(4m^2-n^2)mn}{x}=\frac{4m^2+n^2}{2}.$$

$$x^2-(4m^2+n^2)x=-2(4m^2-n^2)mn.$$

$$4x^2-4(4m^2+n^2)x=-8(4m^2-n^2)mn.$$

$$4x^2-4(4m^2+n^2)x+(4m^2+n^2)^2=16m^4-32m^3n+8m^2n^2+8mn^3+n^4.$$

$$4x^2-4(4m^2+n^2)x+(4m^2+n^2)^2=16(m^2-mn)^2-8(m^2-mn)n^2+n^4.$$

$$2x-(4m^2+n^2)=\pm[4(m^2-mn)-n^2].$$

$$2x=8m^2-4mn, \text{ or } 4mn+2n^2.$$

$$x=2m(2m-n), \text{ or } n(2m+n).$$

34. Solve $$\frac{x^2}{m+n}-\left(1+\frac{1}{mn}\right)x+\frac{1}{m}+\frac{1}{n}=0.$$

$$mnx^2-(m^2n+mn^2+m+n)x+(m+n)^2=0.$$

Multiply by $4mn$,

$$4m^2n^2x^2-4mn(m+n)(mn+1)x=-4mn(m+n)^2.$$

$$4m^2n^2x^2-(\quad)+(m+n)^2(mn+1)^2=(m+n)^2(mn-1)^2.$$

$$2mnx-(m+n)(mn+1)=\pm(m+n)(mn-1).$$

$$2mnx=2(m+n)mn, \text{ or } 2(m+n).$$

$$\therefore\ x=m+n, \text{ or } \frac{m+n}{mn}.$$

35. Solve $$\frac{2ab}{3x+1}+\frac{(3x-1)b^2}{2x+1}=\frac{(2x+1)a^2}{3x+1}.$$

$$2ab(2x+1)+(9x^2-1)b^2=(2x+1)^2a^2.$$

$$(9b^2-4a^2)x^2+(4ab-4a^2)x=a^2+b^2-2ab.$$

Multiply by $9b^2-4a^2$,

$$(9b^2-4a^2)^2x^2-4a(a-b)(9b^2-4a^2)x=(a-b)^2(9b^2-4a^2).$$

$$(9b^2-4a^2)^2x^2-(\quad)+4a^2(a-b)^2=9b^2(a-b)^2.$$

$$(9b^2-4a^2)x-2a(a-b)=\pm 3b(a-b).$$

$$(9b^2-4a^2)x=(2a+3b)(a-b),\text{ or }(2a-3b)(a-b).$$

$$\therefore\ x=\frac{a-b}{3b-2a},\text{ or }-\frac{a-b}{2a+3b},$$

or $$x=\frac{b-a}{2a-3b},\text{ or }\frac{b-a}{2a+3b}.$$

36. Solve

$$\frac{x+2a-4b}{2bx}-\frac{8b-7a}{ax-2bx}+\frac{x-4a}{2(ab-2b^2)}=0.$$

$$(a-2b)(x+2a-4b)-16b^2+14ab+x^2-4ax=0.$$

$$x^2-3ax-2bx=8b^2-6ab-2a^2.$$

$$4x^2-4(3a+2b)x=32b^2-24ab-8a^2.$$

$$4x^2-4(3a+2b)x+(3a+2b)^2=a^2-12ab+36b^2.$$

$$2x-(3a+2b)=\pm(a-6b).$$

$$2x=4(a-b),\text{ or }2a+8b.$$

$$\therefore\ x=2(a-b),\text{ or }a+4b.$$

37. Solve

$$\frac{1}{a+2b}-\frac{x}{a^2-4b^2}+\frac{x-5b}{(a+2b)x}=\frac{x+19b-2a}{2bx-ax}.$$

$$(a-2b)x-x^2+(a-2b)(x-5b)=-(a+2b)(x+19b-2a).$$

$$x^2-(3a-2b)x=-2a^2+10ab+48b^2.$$

$$4x^2-4(3a-2b)x=-8a^2+40ab+192b.$$

$$4x^2-4(3a-2b)x+(3a-2b)^2=a^2+28ab+196b^2.$$

$$2x-(3a-2b)=\pm(a+14b).$$

$$2x=4a+12b,\text{ or }2a-16b.$$

$$\therefore\ x=2a+6b,\text{ or }a-8b.$$

38. Solve $$\frac{a-2b}{x+2b}+\frac{2(x+4a+3b)}{x-5a+3b}=0.$$

$$(a-2b)x-5a^2+3ab+2b(5a-3b)+2x^2+2(4a+3b)x$$
$$+4bx+4b(4a+3b)=0.$$

$$2x^2+(9a+8b)x=5a^2-29ab-6b^2.$$

$$16x^2+8(9a+8b)x=40a^2-232ab-48b^2.$$

$$16x^2+8(9a+8b)x+(9a+8b)^2=121a^2-88ab+16b^2.$$

$$4x+9a+8b=\pm(11a-4b).$$

$$4x=2a-12b,\text{ or }-20a-4b.$$

$$\therefore\ x=\frac{a-6b}{2},\text{ or }-(5a+b).$$

39. **Solve** $\dfrac{x+3b}{8a^2-12ab}+\dfrac{3b}{4a^2-9b^2}=\dfrac{a+3b}{(2a+3b)(x-3b)}.$

$$(2a+3b)(x^2-9b^2)+12ab(x-3b)=4(a+3b)(2a-3b)a.$$

$$(2a+3b)x^2+12abx=8a^3+12a^2b+18ab^2+27b^3.$$

$$(2a+3b)x^2+12abx=(2a+3b)(4a^2+9b^2).$$

Multiply by $2a+3b$,

$$(2a+3b)^2x^2+12ab(2a+3b)x=(2a+3b)^2(4a^2+9b^2).$$

$$(2a+3b)^2x^2+(\quad)+36a^2b^2=(4a^2+9b^2)^2+12ab(4a^2+9b^2)+36a^2b^2.$$

$$(2a+3b)x+6ab=\pm(4a^2+9b^2+6ab).$$

$$(2a+3b)x=4a^2+9b^2,\text{ or }-4a^2-9b^2-12ab.$$

$$\therefore\ x=\frac{4a^2+9b^2}{2a+3b},\text{ or }-(2a+3b).$$

40. Solve $\dfrac{1}{2x^2+x-1}+\dfrac{1}{2x^2-3x+1}=\dfrac{a}{2bx-b}+\dfrac{2bx+b}{a-ax^2}.$

$$2x^2+x-1=(2x-1)(x+1).$$

$$2x^2-3x+1=(2x-1)(x-1).$$

$$2bx-b=b(2x-1).$$

$$a-ax^2=a(1-x)(1+x).$$

$$\therefore\ \text{L.C.D.}=ab(x-1)(x+1)(2x-1).$$

$$ab(x^2-1)+ab(x+1)=a^2(x^2-1)-b^2(4x^2-1).$$

$$(a^2-4b^2)x^2-2abx=a^2-b^2.$$

Multiply by (a^2-4b^2),

$$(a^2-4b^2)^2x^2-2ab(a^2-4b^2)x=a^4-5a^2b^2+4b^4.$$

$$(a^2-4b^2)^2x^2-2ab(a^2-4b^2)x+a^2b^2=a^4-4a^2b^2+4b^4.$$

$$(a^2-4b^2)x-ab=\pm(a^2-2b^2).$$

$$(a^2-4b^2)x=ab+a^2-2b^2,$$

$$\text{or }ab-a^2+2b^2.$$

$$x=\frac{a^2+ab-2b^2}{a^2-4b^2},\text{ or }-\frac{a^2-ab-2b^2}{a^2-4b^2}.$$

$$\therefore\ x=\frac{a-b}{a-2b},\text{ or }-\frac{a+b}{a+2b}.$$

41. Solve $$\frac{1}{x}+\frac{4ax^2+3b(2-x)}{2ax^2+2a+3b}=2.$$

$$2ax^2+2a+3b+4ax^3+6bx-3bx^2=4ax^3+4ax+6bx.$$

$$(2a-3b)x^2-4ax=-2a-3b.$$

Multiply by $2a-3b$,

$$(2a-3b)^2x^2-4a(2a-3b)x=-4a^2+9b^2.$$

$$(2a-3b)^2x^2-4a(2a-3b)x+4a^2=9b^2.$$

$$(2a-3b)x-2a=\pm 3b.$$

$$(2a-3b)x=2a+3b,\text{ or }2a-3b.$$

$$\therefore\ x=\frac{2a+3b}{2a-3b},\text{ or }1.$$

42. Solve

$$\frac{x-a}{2b(x+a)}+\frac{2(ab-ax+2b^2)}{a(x+a)^2}=\frac{1}{a}.$$

$$a(x^2-a^2)+4ab^2-4abx+8b^3=2bx^2+4abx+2a^2b.$$

$$(a-2b)x^2-8abx=a^3-4ab^3+2a^2b-8b^3.$$

$$(a-2b)x^2-8abx=(a-2b)(a+2b)^2.$$

Multiply by $a-2b$,

$$(a-2b)^2x^2-8ab(a-2b)x=(a-2b)^2(a+2b)^2$$
$$=(a^2-4b^2)^2.$$

$$(a-2b)^2x^2-8ab(a-2b)x+16a^2b^2=(a^2+4b^2)^2.$$

$$(a-2b)x-4ab=\pm(a^2+4b^2).$$

$$(a-2b)x=(a+2b)^2,\text{ or }-(a-2b)^2.$$

$$\therefore\ x=\frac{(a+2b)^2}{a-2b},\text{ or }2b-a.$$

43. Solve $$\frac{2ax+b}{ax+b}+\frac{2ax-b}{ax-b}=\frac{9b^2x^2+(4a^2-6b^2)x-(a^2+b^2)}{a^2x^2-b^2}.$$

$$2a^2x^2-abx-b^2+2a^2x^2+abx-b^2=9b^2x^2+(4a^2-6b^2)x-(a^2+b^2).$$

$$(4a^2-9b^2)x^2-(4a^2-6b^2)x=b^2-a^2.$$

Multiply by $4a^2-9b^2$,

$$(4a^2-9b^2)^2x^2-2(4a^2-9b^2)(2a^2-3b^2)=13a^2b^2-4a^4-9b^4.$$

$$(4a^2-9b^2)^2x^2-2(4a^2-9b^2)(2a^2-3b^2)+(2a^2-3b^2)^2=a^2b^2.$$

$$(4a^2-9b^2)x-(2a^2-3b^2)=\pm ab.$$

$$(4a^2-9b^2)x=2a^2+ab-3b^2,\text{ or }2a^2-ab-3b^2.$$

$$x=\frac{2a^2+ab-3b^2}{4a^2-9b^2},\text{ or }\frac{2a^2-ab-3b^2}{4a^2-9b^2}.$$

$$\therefore\ x=\frac{a-b}{2a-3b},\text{ or }\frac{a+b}{2a+3b}.$$

44. Solve $\frac{x+a+b}{x-3a+b}+\frac{3(a+c)}{x+b+c}=2$.

$$x^2+(a+2b+c)x+ab+ac+b^2+bc+3(a+c)x-9a^2-9ac+3ab+3bc$$
$$=2x^2+2(2b-3a+c)x-6ab-6ac+2b^2+2bc.$$
$$x^2+(2b-10a-2c)x=10ab-2ac-b^2+2bc-9a^2.$$
$$x^2+2(b-5a-c)x+(b-5a-c)^2=16a^2+8ac+c^2.$$
$$x+b-5a-c=\pm(4a+c).$$
$$x=9a-b+2c, \text{ or } a-b.$$

Exercise 21.

1. Find all the roots of

$$(x-1)(x-2)(x^2-4x+8)=0.$$
$$x=1,\ x=2, \text{ or } x^2-4x+8=0.$$
$$x^2-4x+8=0,$$
$$x^2-4x+4=-4.$$
$$x-2=\pm 2\sqrt{-1}.$$
$$x=2\pm 2\sqrt{-1}.$$
$$\therefore\ x=1,\ 2,\ 2\pm 2\sqrt{-1}.$$

2. Find all the roots of

$$(x^2-2x+2)(x^2-6x+7)=0.$$
$$x^2-2x+2=0,$$
or $$x^2-6x+7=0.$$
$$x^2-2x+2=0.$$
$$x^2-2x+1=-1.$$
$$x-1=\pm\sqrt{-1}.$$
$$x=1\pm\sqrt{-1}.$$
$$x^2-6x+7=0.$$
$$x^2-6x+9=2.$$
$$x-3=\pm\sqrt{2}.$$
$$x=3\pm\sqrt{2}.$$
$$\therefore\ x=1\pm\sqrt{-1},\ 3\pm\sqrt{2}.$$

3. Find all the roots of

$$x^3+27=0.$$
$$x^3=-27.$$
$$x=-3.$$
$\therefore\ x+3$ is a factor of x^3+27.

$$x^3+27=(x+3)(x^2-3x+9)=0.$$
$$x^2-3x+9=0.$$
$$x^2-3x+\tfrac{9}{4}=-\tfrac{27}{4}.$$
$$x-\tfrac{3}{2}=\pm\tfrac{3}{2}\sqrt{-3}.$$
$$x=\frac{3\pm 3\sqrt{-3}}{2}.$$
$$\therefore\ x=-3,\ \frac{3\pm 3\sqrt{-3}}{2}.$$

4. Find all the roots of

$$x^4-81=0.$$
$$(x^2-9)(x^2+9)=0.$$
$$x^2-9=0.$$
$$x=\pm 3.$$
$$x^2+9=0.$$
$$x=\pm 3\sqrt{-1}.$$
$$\therefore\ x=\pm 3,\ \pm 3\sqrt{-1}.$$

5. Find all the roots of

$$x^3-27+4(x^2-9)=0.$$
$$(x-3)\{(x^2+3x+9)+4(x+3)\}=0.$$
$$x-3=0,$$
or $$x^2+7x+21=0.$$
$$x^2+7x+\tfrac{49}{4}=-\tfrac{35}{4}.$$
$$x+\tfrac{7}{2}=\pm\tfrac{1}{2}\sqrt{-35}.$$
$$x=\frac{-7\pm\sqrt{-35}}{2}.$$
$$\therefore\ x=3,\ \frac{-7\pm\sqrt{-35}}{2}.$$

6. Find all the roots of

$$x^4 + 9x^2 - 16(x^2 + 9) = 0.$$
$$(x^2 - 16)(x^2 + 9) = 0.$$
$$x^2 = 16,$$
or $$x^2 = -9.$$
$$x = \pm 4,\ \pm 3\sqrt{-1}.$$

7. Find all the roots of

$$2x^3 + 3x^2 - 2x - 3 = 0.$$
$$(2x + 3)(x^2 - 1) = 0.$$
$$2x + 3 = 0,$$
or $$x^2 - 1 = 0.$$
$$x = -\tfrac{3}{2},\ \pm 1.$$

8. Find all the roots of

$$x^4 - 4x^3 + 8x^2 - 32x = 0.$$
$$x(x^3 - 4x^2 + 8x - 32) = 0.$$
$$x(x - 4)(x^2 + 8) = 0.$$
$$x = 0,\quad x = 4,$$
or $$x^2 + 8 = 0.$$
$$x = \pm 2\sqrt{-2}.$$
$$\therefore\ x = 0,\ 4,\ \pm 2\sqrt{-2}.$$

9. Find all the roots of

$$x^3 - x - 6 = 0.$$
$x = 2$ is a root.
$$x^3 - x - 6 = (x - 2)(x^2 + 2x + 3).$$
$$x^2 + 2x + 3 = 0.$$
$$x^2 + 2x + 1 = -2.$$
$$x + 1 = \pm\sqrt{-2}.$$
$$x = -1 \pm \sqrt{-2}.$$
$$\therefore\ x = 2, \text{ or } -1 \pm \sqrt{-2}.$$

10. Find all the roots of

$$x^3 - 6x^2 + 11x - 6 = 0.$$
$x = 2$ is a root.
$$x^3 - 6x^2 + 11x - 6$$
$$= (x - 2)(x^2 - 4x + 3).$$
$$x^2 - 4x + 3 = 0.$$
$$(x - 1)(x - 3) = 0.$$
$$x = 3, \text{ or } 1.$$
$$\therefore\ x = 1,\ 2,\ 3.$$

11. Find all the roots of

$$x^4 - 3x^3 - 8x^2 + 6x + 4 = 0.$$
$x = 1$, or -2, satisfies the equation.
$$x^4 - 3x^3 - 8x^2 + 6x + 4$$
$$= (x - 1)(x + 2)(x^2 - 4x - 2).$$
$$x^2 - 4x - 2 = 0.$$
$$x^2 - 4x + 4 = 6.$$
$$x - 2 = \pm\sqrt{6}.$$
$$x = 2 \pm \sqrt{6}.$$
$$\therefore\ x = 1,\ -2,\ 2 \pm \sqrt{6}.$$

12. Find all the roots of

$$x^3 + x^2 - 14x - 24 = 0.$$
$x = 4$ is a root.
$$(x - 4)(x^2 + 5x + 6) = 0.$$
$$x^2 + 5x + 6 = 0.$$
$$(x + 2)(x + 3) = 0.$$
$$x = -2, \text{ or } -3.$$
$$\therefore\ x = 4,\ -2,\ -3.$$

13. Find all the roots of

$$x^4 - 6x^3 + 9x^2 + 4x - 12 = 0.$$
$$x^2(x - 3)^2 + 4(x - 3) = 0.$$
$$(x - 3)\{x^2(x - 3) + 4\} = 0.$$
$$x = 3, \text{ or } x^3 - 3x^2 + 4 = 0.$$
$x = -1$ is a root.
$$x^3 - 3x + 4 = (x + 1)(x^2 - 4x + 4).$$
$$x = -1, \text{ or } x^2 - 4x + 4 = 0.$$
$$(x - 2)(x - 2) = 0.$$
$$x = 2,\ 2.$$
$$\therefore\ x = 3,\ -1,\ 2,\ 2.$$

14. Find all the roots of

$$x(x - 3)(x + 1) = a(a - 3)(a + 1).$$
$x = a$ is a root.
Simplify the equation,
$$x^3 - a^3 - 2x^2 + 2a^2 - 3x + 3a = 0.$$
$$(x - a)(x^2 + ax + a^2 - 2x - 2a - 3) = 0.$$
$$x = a,$$
or $$x^2 + ax + a^2 - 2x - 2a - 3 = 0.$$

$$x^2 + (a-2)x = 2a + 3 - a^2.$$

$$4x^2 + 4(a-2)x + (a-2)^2$$

$$= -3a^2 + 4a + 16.$$

$$2x + a - 2 = \pm\sqrt{16 + 4a - 3a^2}.$$

$$x = \frac{2 - a \pm \sqrt{16 + 4a - 3a^2}}{2}.$$

$$\therefore\ x = a,\ \frac{2 - a \pm \sqrt{16 + 4a - 3a^2}}{2}.$$

15. Find all the roots of

$$x(x-3)(x+1) = 20.$$

$$x(x-3)(x+1) = 4 \times 1 \times 5.$$

By inspection, $x = 4.$

$$x^3 - 2x^2 - 3x - 20 = 0.$$

$$(x-4)(x^2 + 2x + 5) = 0.$$

$$x = 4,$$

or $x^2 + 2x + 5 = 0.$

$$x^2 + 2x + 1 = -4.$$

$$x + 1 = \pm 2\sqrt{-1},$$

$$x = -1 \pm 2\sqrt{-1}.$$

$$\therefore\ x = 4,\ -1 \pm 2\sqrt{-1}.$$

16. Find all the roots of

$$(x-1)(x-2)(x-3) = 24.$$

$$(x-1)(x-2)(x-3) = 4 \times 3 \times 2.$$

By inspection, $x = 5.$

$$x^3 - 6x^2 + 11x - 6 = 24.$$

$$x^3 - 6x^2 + 11x - 30 = 0.$$

$$(x-5)(x^2 - x + 6) = 0.$$

$$x = 5,$$

or $x^2 - x + 6 = 0.$

$$x^2 - x + \tfrac{1}{4} = -\tfrac{23}{4}.$$

$$x - \tfrac{1}{2} = \pm\tfrac{1}{2}\sqrt{-23}.$$

$$x = \frac{1 \pm \sqrt{-23}}{2}.$$

$$\therefore\ x = 5,\ \frac{1 \pm \sqrt{-23}}{2}.$$

17. Find all the roots of

$$(x+2)(x-3)(x+4) = 240.$$

$$(x+2)(x-3)(x+4) = 8 \times 3 \times 10.$$

By inspection, $x = 6.$

$$x^3 + 3x^2 - 10x - 24 = 240.$$

$$x^3 + 3x^2 - 10x - 264 = 0.$$

$$(x-6)(x^2 + 9x + 44) = 0.$$

$$x = 6,$$

or $x^2 + 9x + 44 = 0.$

$$x^2 + 9x + \tfrac{81}{4} = -\tfrac{95}{4}.$$

$$x + \tfrac{9}{2} = \pm\tfrac{1}{2}\sqrt{-95}.$$

$$x = \frac{-9 \pm \sqrt{-95}}{2}.$$

$$\therefore\ x = 6,\ \frac{-9 \pm \sqrt{-95.}}{2}$$

18. Find all the roots in

$$(x+1)(x+5)(x-6) = 96.$$

$$(x+1)(x+5)(x-6) = 8 \times 12 \times 1.$$

By inspection, $x = 7.$

$$x^3 - 31x - 30 = 96.$$

$$x^3 - 31x - 126 = 0.$$

$$(x-7)(x^2 + 7x + 18) = 0.$$

$$x = 7,$$

or $x^2 + 7x + 18 = 0.$

$$x^2 + 7x + \tfrac{49}{4} = -\tfrac{23}{4}.$$

$$x + \tfrac{7}{2} = \pm\tfrac{1}{2}\sqrt{-23}.$$

$$x = \frac{-7 \pm \sqrt{-23}}{2}.$$

$$\therefore\ x = 7,\ \frac{-7 \pm \sqrt{-23}}{2}.$$

Exercise 22.

Determine, without solving, the character of the roots of each of the following equations.

1. $x^2 - 6x + 8 = 0.$
$a = 1, b = -6, c = 8.$
$b^2 - 4ac = 4.$

$\therefore$ The roots are real, different, and exact.

2. $x^2 - 4x + 2 = 0.$
$a = 1, b = -4, c = 2.$
$b^2 - 4ac = 8.$

$\therefore$ The roots are real, different surds.

3. $x^2 + 6x + 13 = 0.$
$a = 1, b = 6, c = 13.$
$b^2 - 4ac = -16.$

$\therefore$ The roots are both imaginary.

4. $4x^2 - 12x + 7 = 0.$
$a = 4, b = -12, c = 7.$
$b^2 - 4ac = 32.$

$\therefore$ The roots are real, different surds.

5. $5x^2 - 9x + 6 = 0.$
$a = 5, b = -9, c = 6.$
$b^2 - 4ac = -49.$

$\therefore$ The roots are both imaginary.

6. $16x^2 - 56x + 49 = 0.$
$a = 16, b = -56, c = 49.$
$b^2 - 4ac = 0.$

$\therefore$ The roots are real and equal.

7. $3x^2 - 2x + 12 = 0.$
$a = 3, b = -2, c = 12.$
$b^2 - 4ac = -140.$

$\therefore$ The roots are both imaginary.

8. $2x^2 - 19x + 17 = 0.$
$a = 2, b = -19, c = 17.$
$b^2 - 4ac = 225.$

$\therefore$ The roots are real, different, and exact.

9. $9x^2 + 30x + 25 = 0.$
$a = 9, b = 30, c = 25.$
$b^2 - 4ac = 0.$

$\therefore$ The roots are real and equal.

10. $17x^2 - 12x + \frac{36}{17} = 0.$
$a = 17, b = -12, c = \frac{36}{17}.$
$b^2 - 4ac = 0.$

$\therefore$ The roots are real and equal.

Determine the values of *m* for which the two roots of each of the following equations are equal.

11. $(3m + 1)x^2 + (2m + 2)x + m = 0.$
$a = 3m + 1, b = 2m + 2, c = m.$

$$b^2 - 4ac = 4m^2 + 8m + 4 - 12m^2 - 4m.$$
$$b^2 - 4ac = 4 + 4m - 8m^2.$$

If the roots are equal, $b^2 - 4ac = 0.$

$$8m^2 - 4m - 4 = 0.$$
$$16m^2 - 8m + 1 = 9.$$
$$4m - 1 = \pm 3.$$
$$4m = 4, \text{ or } -2.$$
$$\therefore m = 1, \text{ or } -\tfrac{1}{2}. \quad \textit{Ans.}$$

12. $(m-2)x^2+(m-5)x+2m-5=0.$

$$a=m-2,\ b=m-5,\ c=2m-5.$$
$$b^2-4ac=m^2-10m+25-8m^2+36m-40.$$
$$b^2-4ac=-7m^2+26m-15.$$

If the roots are equal,

$$7m^2-26m+15=0.$$
$$(7m-5)(m-3)=0.$$
$$\therefore\ m=\tfrac{5}{7},\text{ or }3.\ \textit{Ans.}$$

13. $2mx^2+x^2-6mx-6x+6m+1=0.$

$$a=2m+1,\ b=-6m-6,\ c=6m+1.$$
$$b^2-4ac=36m^2+72m+36-48m^2-32m-4.$$
$$b^2-4ac=-12m^2+40m+32.$$

If the roots are equal,

$$12m^2-40m-32=0.$$
$$3m^2-10m-8=0.$$
$$(m-4)(3m+2)=0.$$
$$\therefore\ m=4,\text{ or }-\tfrac{2}{3}.\ \textit{Ans.}$$

14. $mx^2+2x^2+2m=3mx-9x+10.$

$$(m+2)x^2-(3m-9)x+(2m-10)=0.$$
$$a=m+2,\ b=-(3m-9),\ c=2m-10.$$
$$b^2-4ac=m^2-30m+161.$$

If the roots are equal,

$$m^2-30m+161=0.$$
$$(m-7)(m-23)=0.$$
$$\therefore\ m=7,\text{ or }23.\ \textit{Ans.}$$

Exercise 23.

1. The product of two consecutive numbers exceeds their sum by 181. Find the numbers.

Let x equal the first, and $x+1$ the second number.

Then $x(x+1)=$ their product,

and $2x+1=$ their sum.

$$\therefore\ x(x+1)=2x+1+181.$$
$$x^2+x=2x+182.$$
$$x^2-x=182.$$
$$x^2-x+\tfrac{1}{4}=182\tfrac{1}{4}.$$
$$x-\tfrac{1}{2}=\pm 13\tfrac{1}{2}.$$
$$x=14,\text{ or }-13.$$

$\therefore$ The numbers are 14 and 15.

2. The square of the sum of two consecutive numbers exceeds the sum of their squares by 220. Find the numbers.

Let x equal the first, and $x + 1$ the second number.

Then $2x + 1 =$ their sum,

and $x^2 + (x + 1)^2 =$ the sum of their squares.

$$\therefore\ (2x + 1)^2 = x^2 + (x + 1)^2 + 220.$$
$$4x^2 + 4x + 1 = 2x^2 + 2x + 221.$$
$$2x^2 + 2x = 220.$$
$$4x^2 + 4x + 1 = 441.$$
$$2x + 1 = \pm 21.$$
$$2x = 20, \text{ or } -22.$$
$$x = 10, \text{ or } -11.$$

$\therefore$ The numbers are 10 and 11.

3. The difference of the cubes of two consecutive numbers is 817. Find the numbers.

Let x equal the first, and $x + 1$ the second number.

Then $x^3 =$ cube of first,

and $(x + 1)^3 =$ cube of second.

$$\therefore\ (x + 1)^3 - x^3 = 817.$$
$$3x^2 + 3x + 1 = 817.$$
$$x^2 + x = 272.$$
$$x^2 + x + \tfrac{1}{4} = 272\tfrac{1}{4}.$$
$$x + \tfrac{1}{2} = \pm 16\tfrac{1}{2}.$$
$$x = 16, \text{ or } -17.$$

$\therefore$ The numbers are 16 and 17.

4. The difference of two numbers is 5 times the less, and the square of the less is twice the greater. Find the numbers.

Let $x =$ the lesser number.

Then $x^2 =$ twice the greater number.

$$\therefore\ \frac{x^2}{2} = \text{the greater number.}$$
$$\therefore\ \frac{x^2}{2} - x = 5x.$$
$$x^2 - 12x = 0.$$
$$x = 0, \text{ or } 12.$$
$$\frac{x^2}{2} = 0, \text{ or } 72.$$

$\therefore$ The numbers are 12 and 72.

5. The numerator of a certain fraction exceeds the denominator by 1. If the numerator and denominator be interchanged, the sum of the resulting fraction and the original fraction is $2\frac{1}{30}$. What was the original fraction?

Let $x =$ the denominator.
Then $x + 1 =$ the numerator.
$\frac{x+1}{x} =$ the fraction.
$\frac{x}{x+1} =$ the fraction after numerator and denominator are interchanged.

$$\therefore \frac{x+1}{x} + \frac{x}{x+1} = 2\tfrac{1}{30}.$$
$$30(2x^2 + 2x + 1) = 61(x^2 + x).$$
$$x^2 + x = 30.$$
$$x^2 + x + \tfrac{1}{4} = 30\tfrac{1}{4}.$$
$$x + \tfrac{1}{2} = \pm 5\tfrac{1}{2}.$$
$$x = 5, \text{ or } -6.$$
$$x + 1 = 6, \text{ or } -5.$$

$\therefore$ The fraction is $\frac{6}{5}$.

6. The denominator of a certain fraction exceeds twice the numerator by 3. If $3\frac{3}{14}$ be added to the fraction, the resulting fraction is the reciprocal of the original fraction. Find the original fraction.

Let $x =$ the numerator.
Then $2x + 3 =$ the denominator.
$\frac{x}{2x+3} =$ the fraction.
$\frac{2x+3}{x} =$ its reciprocal.

$$\therefore \frac{x}{2x+3} + 3\tfrac{3}{14} = \frac{2x+3}{x}.$$
$$14x^2 + 45x(2x+3) = 14(2x+3)^2.$$
$$48x^2 - 33x = 126.$$
$$16x^2 - 11x = 42.$$
$$16x^2 - 11x + \tfrac{121}{64} = \tfrac{2809}{64}.$$
$$4x - \tfrac{11}{8} = \pm \tfrac{53}{8}.$$
$$4x = 8, \text{ or } -\tfrac{42}{8}.$$
$$x = 2, \text{ or } -\tfrac{21}{16}.$$
$$2x + 3 = 7, \text{ or } \tfrac{3}{8}.$$

$\therefore$ The fraction is $\frac{2}{7}$.

7. A farmer bought a number of geese for \$24. Had he bought 2 more geese for the same money, he would have paid $\frac{2}{5}$ of a dollar less for each. How many geese did he buy, and what did he pay for each?

State the problem to which the negative solution applies.

Let $x =$ number of geese bought.

Then $\frac{24}{x} =$ price of each goose,

and $\frac{24}{x+2} =$ price he would have paid for each if he had bought two more for the same money.

$$\therefore \frac{24}{x+2} = \frac{24}{x} - \frac{2}{5}.$$

$$120x = 120(x+2) - 2x(x+2).$$

$$2x^2 + 4x = 240.$$

$$x^2 + 2x = 120.$$

$$x^2 + 2x + 1 = 121.$$

$$x + 1 = \pm 11.$$

$$x = 10, \text{ or } -12.$$

$$\frac{24}{x} = 2.4, \text{ or } -2.$$

∴ He bought 10 geese at \$2.40 apiece.

The second solution $x = -12$ is to be rejected. If however the problem read: Had he bought 2 geese *less* for the same money, he would have paid $\frac{2}{5}$ of a dollar *more* for each, the algebraic statement would be

$$\frac{24}{x-2} = \frac{24}{x} + \frac{2}{5},$$

which leads to the quadratic equation

$$x^2 - 2x = 120,$$

the roots of which are 12 and -10.

8. A laborer worked a number of days, and received for his labor \$36. Had his wages been 20 cents more per day, he would have received the same amount for 2 days' less labor. What were his daily wages, and how many days did he work?

State the problem to which the negative solution applies.

Let $x =$ number of days he worked.

Then $\frac{36}{x}$ = number of dollars he received per day.

$\frac{36}{x}+\frac{1}{5}$ = number of dollars he would have received per day if he had received 20 cents more.

$$\therefore\ (x-2)\left(\frac{36}{x}+\frac{1}{5}\right)=36.$$

$$(x-2)(180+x)=180x.$$

$$178x+x^2-360=180x.$$

$$x^2-2x=360.$$

$$x^2-2x+1=361.$$

$$x-1=\pm 19.$$

$$x=20, \text{ or } -18.$$

$$\frac{36}{x}=1.80, \text{ or } -2.$$

∴ He worked 20 days at $1.80 per day.

If the problem read: Had his wages been 20 cents *less* a day, he would have received the same amount for 2 days *more* labor, the algebraic statement would be

$$(x+2)\left(\frac{36}{x}-\frac{1}{5}\right)=36,$$

which leads to the quadratic

$$x^2+2x=360,$$

the roots of which are 18 and −20.

9. For a journey of 336 miles, 4 days less would have sufficed had I travelled 2 miles more per day. How many days did the journey take? State the problem to which the negative solution applies.

Let x = number of days which the journey took.

Then $\frac{336}{x}$ = number of miles travelled a day.

$x-4$ = number of days it would take if I travelled $\frac{336}{x}+2$ miles a day.

$$\therefore\ (x-4)\left(\frac{336}{x}+2\right)=336.$$

$$(x-4)(336+2x)=336x.$$

$$2x^2+328x-1344=336x.$$

$$2x^2-8x=1344.$$

$$x^2 - 4x = 672.$$
$$x^2 - 4x + 4 = 676.$$
$$x - 2 = \pm 26.$$
$$x = 28, \text{ or } -24.$$
$$\frac{336}{x} = 12, \text{ or } -14.$$

$\therefore$ The journey took 28 days.

The negative solutions would have been applicable if the problem had read: For a journey of 336 miles, 4 days *more* would have been necessary had I travelled 2 miles less per day. The algebraic statement would be

$$(x+4)\left(\frac{336}{x} - 2\right) = 336,$$

which reduces to

$$x^2 + 4x = 672,$$

the roots of which are 24 and -28.

10. A farmer hires a number of acres for $420. He lets all but 4 for $420, and receives for each acre $2.50 more than he pays for it. How many acres does he hire?

Let x = number of acres hired.

Then $\frac{420}{x}$ = rent paid per acre.

$x - 4$ = number let for $420.

$\frac{420}{x-4}$ = rent received per acre.

$$\therefore \frac{420}{x-4} - 2\tfrac{1}{2} = \frac{420}{x}.$$
$$840x - 5x^2 + 20x = 840x - 3360.$$
$$5x^2 - 20x = 3360.$$
$$x^2 - 4x = 672.$$
$$x^2 - 4x + 4 = 676.$$
$$x - 2 = \pm 26.$$
$$x = 28, \text{ or } -24.$$

$\therefore$ He hires 28 acres.

11. A broker sells a number of railway shares for $3240. A few days later, the price having fallen $9 per share, he buys, for the same sum, 5 more shares than he had sold. Find the number of shares transferred on each day, and the price paid.

Let x = number of shares sold on the first day.

Then $\frac{3240}{x}$ = number of dollars received per share on the first day.

$\frac{3240}{x} - 9$ = number of dollars paid per share on the second day.

$x + 5$ = number of shares bought on the second day.

$$\therefore\ (x+5)\left(\frac{3240}{x} - 9\right) = 3240.$$

$$(x+5)(3240 - 9x) = 3240x.$$

$$-9x^2 + 3240x - 45x + 16200 = 3240x.$$

$$9x^2 + 45x = 16200.$$

$$x^2 + 5x = 1800.$$

$$x^2 + 5x + \tfrac{25}{4} = \tfrac{7225}{4}.$$

$$x + \tfrac{5}{2} = \pm\tfrac{85}{2}.$$

$$x = 40, \text{ or } -45.$$

$$\frac{3240}{x} = 81. \quad \frac{3240}{x} - 9 = 72. \quad x + 5 = 45.$$

∴ He sold 40 shares at $81 per share, and bought 45 shares at $72 per share.

12. A man bought a number of sheep for $300. He kept 15, and sold the remainder for $270, gaining half a dollar on each sheep sold. How many sheep did he buy, and what did he pay for each?

Let x = number of sheep bought.

Then $\frac{300}{x}$ = number of dollars paid for each sheep.

$x - 15$ = number of sheep sold.

$\frac{270}{x-15}$ = number of dollars received for each sheep.

$$\therefore\ \frac{270}{x-15} = \frac{300}{x} + \frac{1}{2}.$$

$$540x = 600x - 9000 + x^2 - 15x.$$

$$x^2 + 45x = 9000.$$

$$x^2 + 45x + \tfrac{2025}{4} = \tfrac{38025}{4}.$$

$$x + \tfrac{45}{2} = \pm\tfrac{195}{2}.$$

$$x = 75, \text{ or } -120.$$

$$\frac{300}{x} = 4.$$

∴ He bought 75 sheep at $4 apiece.

13. The length of a rectangular lot exceeds its breadth by 20 yards. If each dimension be increased by 20 yards, the area of the lot will be doubled. Find the dimensions of the lot.

Let $x =$ number of yards in breadth.
Then $x + 20 =$ number of yards in length.
$x(x + 20) =$ number of square yards in the area.
$x + 20 =$ breadth increased by 20 yards.
$x + 40 =$ length increased by 20 yards.
$(x + 20)(x + 40) =$ number of square yards in area after the increase.
$\therefore\ (x + 20)(x + 40) = 2x(x + 20).$
$\therefore\ x + 20 = 0$, and $x = -20$;
or $x + 40 = 2x$, and $x = 40$.
$x + 20 = 60.$

$\therefore$ The lot is 40 yards wide and 60 yards long.

14. Twice the breadth of a rectangular lot exceeds the length by 2 yards; the area of the lot is 1200 square yards. Find the length and breadth.

Let $x =$ number of yards in breadth.
Then $2x - 2 =$ number of yards in length.
$x(2x - 2) =$ number of square yards in the area.
$\therefore\ x(2x - 2) = 1200.$
$x^2 - x = 600.$
$x^2 - x + \frac{1}{4} = \frac{2401}{4}.$
$x - \frac{1}{2} = \pm \frac{49}{2}.$
$x = 25$, or -24.
$2x - 2 = 48.$

$\therefore$ The lot is 25 yards wide and 48 yards long.

15. Three times the breadth of a rectangular field, of which the area is 2 acres, exceeds twice the length by 8 rods. At $5 per rod, what will it cost to fence the field?

Let $x =$ number of rods in breadth.
Then $3x - 8 =$ twice the number of rods in length.
$\frac{3x - 8}{2} =$ number of rods in length.
$\frac{x(3x - 8)}{2} =$ number of square rods in its area.

$$\therefore \frac{x(3x-1)}{2} = 320.$$
$$3x^2 - 8x = 640.$$
$$9x^2 - 24x + 16 = 1936.$$
$$3x - 4 = \pm 44.$$
$$3x = 48, \text{ or } -40.$$
$$x = 16, \text{ or } -\tfrac{40}{3}.$$
$$\frac{3x-8}{2} = 20.$$

∴ The field is 16 rods wide and 20 rods long.

Its perimeter is therefore $2 \times 16 + 2 \times 20 = 72$ rods.

At 5 dollars a rod the fencing would therefore cost 5×72 dollars, or $360.

16. Two pipes running together fill a cistern in $10\frac{2}{7}$ hours; the larger will fill the cistern in 6 hours less time than the smaller. How long will it take each, running alone, to fill the cistern?

Let x = number of hours it takes the smaller pipe.

Then $x - 6$ = number of hours it takes the larger pipe.

$\frac{1}{x}$ = part which the smaller pipe will fill in one hour.

$\frac{1}{x-6}$ = part which the larger pipe will fill in one hour.

$\frac{1}{x} + \frac{1}{x-6}$ = part which both pipes together will fill in one hour.

$$\therefore \frac{1}{x} + \frac{1}{x-6} = \frac{1}{10\frac{2}{7}} = \frac{7}{72}.$$
$$72(x - 6 + x) = 7x(x - 6).$$
$$7x^2 - 186x = -432.$$
$$49x^2 - 1302x = -3024.$$
$$49x^2 - 1302x + 8649 = 5625.$$
$$7x - 93 = \pm 75.$$
$$7x = 168, \text{ or } 18.$$
$$x = 24, \text{ or } \tfrac{18}{7}.$$
$$x - 6 = 18, \text{ or } -\tfrac{24}{7}.$$

∴ The smaller pipe will fill the cistern in 24 hours, the larger in 18 hours.

17. Three workmen, A, B, and C, dig a ditch. A can dig it alone in 6 days more time, B in 30 days more time, than the time it takes the three to dig the ditch together; C can dig the ditch in 3 times the time the three dig it in. How many days does it take the three, working together, to dig the ditch?

Let $x =$ number of days it takes the three together.

Then $x + 6 =$ number of days it takes A,

$x + 30 =$ number of days it takes B,

$3x =$ number of days it takes C.

$\frac{1}{x} =$ the part the three together can do in one day.

$\frac{1}{x+6} =$ the part that A can do in one day.

$\frac{1}{x+30} =$ the part that B can do in one day.

$\frac{1}{3x} =$ the part that C can do in one day.

$$\therefore \frac{1}{x+6} + \frac{1}{x+30} + \frac{1}{3x} = \frac{1}{x}.$$

$$3x(x+30) + 3x(x+6) + (x+6)(x+30) = 3(x+6)(x+30).$$

$$3x^2 + 90x + 3x^2 + 18x = 2x^2 + 72x + 360.$$

$$4x^2 + 36x = 360.$$

$$4x^2 + 36x + 81 = 441.$$

$$2x + 9 = \pm 21.$$

$$2x = 12, \text{ or } -30.$$

$$x = 6, \text{ or } -15.$$

∴ It will take the three together 6 days to dig the ditch.

18. A cistern holding 900 gallons can be filled by two pipes running together in as many hours as the larger pipe brings in gallons per minute; the smaller pipe brings in per minute one gallon less than the larger pipe. How long will it take each pipe by itself to fill the cistern?

Let $x =$ number of gallons the larger pipe brings in in 1 min.

Then $x - 1 =$ number of gallons the smaller pipe brings in in 1 min.

$2x - 1 =$ number of gallons both pipes bring in in 1 min.

$120x - 60 =$ number of gallons both pipes bring in in 1 hour.

$\frac{900}{120x - 60} =$ number of hours it would take both pipes to fill the cistern.

$$\therefore \frac{900}{120x - 60} = x.$$

$$900 = 120x^2 - 60x.$$

$$2x^2 - x = 15.$$

$$16x^2 - 8x + 1 = 121.$$

$$4x - 1 = \pm 11.$$
$$4x = 12, \text{ or } -10.$$
$$x = 3, \text{ or } -\tfrac{10}{4}.$$
$$x - 1 = 2.$$
$$\frac{900}{x} = 300. \quad 300 \text{ minutes} = 5 \text{ hours.}$$
$$\frac{900}{x-1} = 450. \quad 450 \text{ minutes} = 7\tfrac{1}{2} \text{ hours.}$$

$\therefore$ The larger pipe will fill the cistern in 5 hours, the smaller in $7\frac{1}{2}$ hours.

19. A number is formed by two digits, the second being less by 3 than one-half the square of the first. If 9 be added to the number, the order of the digits will be reversed. Find the number.

Let $x =$ first digit.
Then $x^2 =$ its square.

$$\frac{x^2}{2} - 3 = \text{second digit.}$$
$$10x + \frac{x^2}{2} - 3 = \text{the number.}$$
$$10\left(\frac{x^2}{2} - 3\right) + x = \text{the number reversed.}$$
$$\therefore \ 10x + \frac{x^2}{2} - 3 + 9 = 10\left(\frac{x^2}{2} - 3\right) + x.$$
$$\frac{9x^2}{2} - 9x = 36.$$
$$x^2 - 2x = 8.$$
$$x^2 - 2x + 1 = 9.$$
$$x - 1 = \pm 3.$$
$$x = 4, \text{ or } -2.$$
$$\frac{x^2}{2} - 3 = 5.$$

$\therefore$ The number is 45.

20. A number is formed by two digits; 5 times the second digit exceeds the square of the first digit by 4. If 3 times the first digit be added to the number, the order of the digits will be reversed. Find the number.

Let $x =$ first digit.
Then $x^2 + 4 = 5$ times second digit.

$$\frac{x^2+4}{5} = \text{second digit.}$$

$$10x + \frac{x^2+4}{5} = \text{the number.}$$

$$10\left(\frac{x^2+4}{5}\right) + x = \text{the number reversed.}$$

$$\therefore\ 10x + \frac{x^2+4}{5} + 3x = 10\left(\frac{x^2+4}{5}\right) + x.$$

$$12x = \tfrac{9}{5}(x^2+4).$$

$$9x^2 - 60x = -36.$$

$$9x^2 - 60x + 100 = 64.$$

$$3x - 10 = \pm 8.$$

$$3x = 18, \text{ or } 2.$$

$$x = 6, \text{ or } \tfrac{2}{3}.$$

$$\frac{x^2+4}{5} = 8.$$

∴ The number is 68.

21. A boat's crew row 3 miles down a river and back again in 1 hour and 15 minutes. Their rate in still water is 3 miles per hour faster than twice the rate of the current. Find the rates of the crew and the rate of the current.

Let $x =$ their rate in still water.

Then $x - 3 =$ twice the rate of the current.

$$\frac{x-3}{2} = \text{the rate of the current.}$$

$$x + \frac{x-3}{2} = \text{their rate down stream.}$$

$$x - \frac{x-3}{2} = \text{their rate up stream.}$$

$$\frac{3}{x + \frac{x-3}{2}} = \text{number of hours they take to row down three miles.}$$

$$\frac{3}{x - \frac{x-3}{2}} = \text{number of hours they take to row back three miles.}$$

1 hour and 15 minntes $= \frac{5}{4}$ hours.

$$\therefore\ \frac{3}{x + \frac{x-3}{2}} + \frac{3}{x - \frac{x-3}{2}} = \frac{5}{4}.$$

$$\frac{6}{3x-3}+\frac{6}{x+3}=\frac{5}{4}.$$
$$8(x+3)+24(x-1)=5(x-1)(x+3).$$
$$5x^2-22x=15.$$
$$25x^2-110x+121=196.$$
$$5x-11=\pm 14.$$
$$5x=25, \text{ or } -3.$$
$$x=5, \text{ and } \frac{x-3}{2}=1.$$

∴ The rate of the current is 1 mile per hour; and the rate of the crew in still water is 5 miles per hour.

22. A jeweller sold a watch for $22.75, and lost on the cost of the watch as many per cent as the watch cost dollars. What was the cost of the watch?

Let x = number of dollars paid for the watch.

Then $x - 22\frac{3}{4}$ = loss in dollars.

$$100\,\frac{x-22\frac{3}{4}}{x}=\text{loss in per cent.}$$
$$\therefore\ 100\,\frac{x-22\frac{3}{4}}{x}=x.$$
$$100x-2275=x^2.$$
$$x^2-100x=-2275.$$
$$x^2-100x+2500=225.$$
$$x-50=\pm 15.$$
$$x=65, \text{ or } 35.$$

∴ The watch cost either $65 or $35.

23. A farmer sold a horse for $138, and gained on the cost ⅛ as many per cent as the horse cost dollars. Find the cost of the horse.

Let x = number of dollars paid for the horse.

Then $138 - x$ = gain in dollars.

$$100\,\frac{138-x}{x}=\text{gain in per cent.}$$
$$\therefore\ 100\,\frac{138-x}{x}=\frac{x}{8}.$$
$$110400-800x=x^2.$$
$$x^2+800x=110400.$$
$$x^2+800x+160000=270400.$$
$$x+400=\pm 520.$$
$$x=120, \text{ or } -920.$$

∴ The horse cost $120.

24. A broker bought a number of $100 shares, when they were a certain per cent below par, for $8500. He afterwards sold all but 20, when they were the same per cent above par, for $9200. How many shares did he buy, and what did he pay for each share?

Let $x =$ number of shares bought.

Then $\frac{8500}{x} =$ number of dollars paid for each.

$x - 20 =$ number sold.

$\frac{9200}{x-20} =$ number of dollars received for each.

$$\therefore\ 100 - \frac{8500}{x} = \frac{9200}{x-20} - 100.$$

$$2 - \frac{85}{x} = \frac{92}{x-20}.$$

$$2x^2 - 40x - 85x + 1700 = 92x.$$

$$2x^2 - 217x = -1700.$$

$$16x^2 - 1736x = -13600.$$

$$16x^2 - 1736x + 47089 = 33489.$$

$$4x - 217 = \pm 183.$$

$$4x = 400, \text{ or } 34.$$

$$x = 100, \text{ or } \tfrac{17}{2}.$$

$$\frac{8500}{x} = 85.$$

$\therefore$ He bought 100 shares at $85 per share.

25. A drover bought a number of sheep for $110; 4 having died, he sold the remainder for $7.33⅓ a head, and made on his investment 4 times as many per cent as each sheep cost dollars. How many sheep did he buy, and how many dollars did he make?

Let $x =$ number of sheep bought.

Then $\frac{110}{x} =$ number of dollars paid for each sheep.

$x - 4 =$ number sold.

$7\frac{1}{3}(x-4) =$ number of dollars received for them.

$7\frac{1}{3}(x-4) - 110 =$ number of dollars gained.

$100\,\frac{7\frac{1}{3}(x-4) - 110}{110} =$ the amount gained in per cent.

$$\therefore\ 100\,\frac{7\frac{1}{3}(x-4) - 110}{110} = \frac{440}{x}.$$

$$\tfrac{22}{3}(x-4)-110=\frac{484}{x}.$$

$$22x^2-88ax-330x=1452.$$

$$22x^2-418x=1452.$$

$$x^2-19x=66.$$

$$x^2-19x-66=0.$$

$$(x-22)(x+3)=0.$$

$$\therefore\ x=22,\text{ or }-3.$$

$$7\tfrac{1}{3}(x-4)-110=22.$$

$\therefore$ He bought 22 sheep, and made $22 profit.

26. A certain train leaves A for B, distant 216 miles; 3 hours later another train leaves A to travel over the same route; the second train travels 8 miles per hour faster than the first, and arrives at B 45 minutes behind the first. Find the time each train takes to travel over the route.

Let x = number of miles per hour which the first train makes.

Then $x+8$ = number of miles per hour which the second train makes.

$\frac{216}{x}$ = number of hours it takes the first train to go from A to B.

$\frac{216}{x+8}$ = number of hours it takes the second train to go from A to B.

The second train gains 3 hours less 45 minutes, or $2\frac{1}{4}$ hours, on the first.

$$\therefore\ \frac{216}{x+8}=\frac{216}{x}-2\tfrac{1}{4}.$$

$$864x=864x+6912-9x^2-72x.$$

$$9x^2+72x=6912.$$

$$9x^2+72x+144=7056.$$

$$3x+12=\pm 84.$$

$$3x=72,\text{ or }-96.$$

$$x=24.$$

$$\frac{216}{x}=9.\quad \frac{216}{x+8}=6\tfrac{3}{4}.$$

$\therefore$ The first train travels over the route in 9 hours; the second, in $6\frac{3}{4}$ hours.

27. A coach, due at B twelve hours after it leaves A, after travelling from A as many hours as it travels miles per hour, breaks down; it then proceeds at a rate 1 mile per hour less than half its former rate, and arrives at B three hours late. Find the distance from A to B.

Let $x =$ number of miles from A to B.

Then $\frac{x}{12} =$ rate of the coach at first.

$\frac{x}{12} =$ number of hours travelled at this rate.

$\frac{x^2}{144} =$ number of miles travelled at this rate.

$x - \frac{x^2}{144} =$ number of miles remaining.

$\frac{x}{24} - 1 =$ rate of coach after the accident.

$\dfrac{x - \frac{x^2}{144}}{\frac{x}{24} - 1} =$ number of hours taken to travel the remaining distance.

$$\therefore \frac{x}{12} + \frac{x - \frac{x^2}{144}}{\frac{x}{24} - 1} = 12 + 3.$$

$$\frac{x}{12} + \frac{144x - x^2}{6x - 144} = 15.$$

$$6x^2 - 144x + 1728x - 12x^2 = 1080x - 25920.$$

$$6x^2 - 504x = 25920.$$

$$x^2 - 84x = 4320.$$

$$x^2 - 84x + 1764 = 6084.$$

$$x - 42 = \pm 78.$$

$$x = 120, \text{ or } -36.$$

$\therefore$ The distance from A to B is 120 miles.

Exercise 24.

1. $\begin{cases} x + y = 8 & (1) \\ xy = 15 & (2) \end{cases}$

Square (1), $x^2 + 2xy + y^2 = 64$ (3)

$4 \times (2)$ is $4xy = 60$ (4)

Subtract, $x^2 - 2xy + y^2 = 4$

Extract root, $x - y = \pm 2$ (5)

Add (1) and (5), $2x = 10, \text{ or } 6$

$\therefore x = 5, \text{ or } 3$

Substitute the value of x in (1), $y = 3, \text{ or } 5$

2. $\begin{cases} x+y=6 \\ xy+27=0 \end{cases}$ (1) (2)

$xy=-27$ (3)

Square (1), $x^2+2xy+y^2=36$ (4)

$4\times(3)$ is $4xy=-108$ (5)

Subtract, $x^2-2xy+y^2=144$

Extract root, $x-y=\pm 12$ (6)

Add (1) and (6), $2x=18, \text{ or } -6$

$\therefore x=9, \text{ or } -3$

Substitute the value of x in (1), $y=-3, \text{ or } 9$

3. $\begin{cases} x-y=5 \\ xy=24 \end{cases}$ (1) (2)

Square (1), $x^2-2xy+y^2=25$ (3)

$4\times(2)$ is $4xy=96$ (4)

Add, $x^2+2xy+y^2=121$

Extract root, $x+y=\pm 11$ (5)

Add (1) and (5), $2x=16, \text{ or } -6$

$\therefore x=8, \text{ or } -3$

Substitute the value of x in (1), $y=3, \text{ or } -8$

4. $\begin{cases} x-y=16 \\ xy+60=0 \end{cases}$ (1) (2)

$xy=-60$ (3)

Square (1), $x^2-2xy+y^2=256$ (4)

$4\times(3)$ is $4xy=-240$ (5)

Add, $x^2+2xy+y^2=16$

Extract root, $x+y=\pm 4$ (6)

Add (1) and (6), $2x=20, \text{ or } 12$

$\therefore x=10, \text{ or } 6$

Substitute the value of x in (1), $y=-6, \text{ or } -10$

5. $\begin{cases} x+2y=12 \\ xy=18 \end{cases}$ (1) (2)

Square (1), $x^2+4xy+4y^2=144$ (3)

$8\times(2)$ is $8xy=144$ (4)

Subtract, $x^2-4xy+4y^2=0$

Extract root, $x-2y=0$

Add (1) and (5), $2x=12$

$\therefore x=6$

Substitute the value of x in (1), $y=3$

6.

$$\begin{cases} 2x + 3y = 1 & (1) \\ xy + 15 = 0 & (2) \end{cases}$$

$$xy = -15 \qquad (3)$$

Square (1), $4x^2 + 12xy + 9y^2 = 1$ (4)

24 × (3) is $24xy = -360$ (5)

Subtract, $4x^2 - 12xy + 9y^2 = 361$

Extract root, $2x - 3y = \pm 19$ (6)

Add (1) and (6), $4x = 20, \text{ or } -18$

$$\therefore x = 5, \text{ or } -4\tfrac{1}{2}$$

Substitute the value of x in (1), $3y = -9, \text{ or } 10$

$$\therefore y = -3, \text{ or } 3\tfrac{1}{3}$$

7.

$$\begin{cases} y = 9 - 3x & (1) \\ x^2 = 10 - xy & (2) \end{cases}$$

$$x^2 = 10 - x(9 - 3x)$$

$$x^2 = 10 - 9x + 3x^2$$

$$2x^2 - 9x = -10$$

$$4x^2 - 18x + 20\tfrac{1}{4} = \tfrac{1}{4}$$

$$2x - 4\tfrac{1}{2} = \pm\tfrac{1}{2}$$

$$2x = 5, \text{ or } 4$$

$$\therefore x = \tfrac{5}{2}, \text{ or } 2$$

Substitute the value of x in (1),

$$y = \tfrac{3}{2}, \text{ or } 3$$

8.

$$\begin{cases} x + 2y = 12 & (1) \\ xy + y^2 = 35 & (2) \end{cases}$$

From (1), $x + y = 12 - y$ (3)

From (2),

$$y(x + y) = 35 \qquad (4)$$

Substitute the value of $x + y$ from (3) in (4),

$$y(12 - y) = 35$$

$$y^2 - 12y = -35$$

$$y^2 - 12y + 36 = 1$$

$$y - 6 = \pm 1$$

$$\therefore y = 7, \text{ or } 5$$

Substitute the value of y in (1),

$$x = -2, \text{ or } 2$$

9.

$$\begin{cases} x + 3y + 9 = 0 & (1) \\ xy - y^2 + 4 = 0 & (2) \end{cases}$$

From (1), $x - y = 2y - 9$ (3)

From (2), $y(x - y) = -4$ (4)

Substitute the value of $x - y$ from (3) in (4),

$$y(2y - 9) = -4$$

$$2y^2 - 9y = -4$$

$$16y^2 - 72y + 81 = 49$$

$$4y - 9 = \pm 7$$

$$4y = 16, \text{ or } 2$$

$$\therefore y = 4, \text{ or } \tfrac{1}{2}$$

Substitute the value of y in (1), $x = 3, \text{ or } -\tfrac{15}{2}$.

10. $\begin{cases} x^2+y^2=100 & (1) \\ x+y=14 & (2) \end{cases}$

Square (2), $x^2+2xy+y^2=196$ (3)

(1) is $x^2 \qquad +y^2=100$

Subtract, $2xy \qquad =96$ (4)

(1) is $x^2 \qquad +y^2=100$

(4) is $2xy \qquad =96$

Subtract, $x^2-2xy+y^2=4$

$x-y=\pm 2$ (5)

Add (2) and (5), $2x=16$, or 12

$\therefore x=8$, or 6

Substitute the value of x in (2), $y=6$, or 8

11. $\begin{cases} x^2+y^2=17 & (1) \\ 4x+y=15 & (2) \end{cases}$

From (2), $y=15-4x$

$\therefore x^2+(15-4x)^2=17$

$17x^2-120x=-208$

$17^2x^2-120\times 17x+3600=64$

$17x-60=\pm 8$

$17x=52$, or 68

$\therefore x=\frac{52}{17}$, or 4

Substitute the value of x in (2),

$y=\frac{47}{17}$, or -1

12. $\begin{cases} 2x^2-y^2+8=0 & (1) \\ 3x-y-2=0 & (2) \end{cases}$

From (2), $y=3x-2$ (3)

Substitute in (1),

$2x^2-(3x-2)^2+8=0$

$-7x^2+12x+4=0$

$49x^2-84x+36=64$

$7x-6=\pm 8$

$7x=14$, or -2

$\therefore x=2$, or $-\frac{2}{7}$

Substitute the value of x in (3),

$y=4$, or $-\frac{20}{7}$

13. $\begin{cases} x^2+xy=40 & (1) \\ 2x-3y=1 & (2) \end{cases}$

From (2), $y=\frac{2x-1}{3}$ (3)

Substitute in (1),

$x^2+\frac{(2x-1)x}{3}=40$

$5x^2-x=120$

$100x^2-20x+1=2401$

$10x-1=\pm 49$

$10x=50$, or -48

$\therefore x=5$, or $-\frac{24}{5}$

Substitute the value of x in (3),

$y=3$, or $-\frac{53}{15}$

14. $\begin{cases} x^2-y^2=13 & (1) \\ 3x-2y=9 & (2) \end{cases}$

From (2), $y=\frac{3x-9}{2}$ (3)

Substitute in (1),

$x^2-\left(\frac{3x-9}{2}\right)^2=13$

$5x^2-54x=-133$

$25x^2-270x+729=64$

$5x-27=\pm 8$

$5x=35$, or 19

$\therefore x=7$, or $\frac{19}{5}$

Substitute the value of x in (3),

$y=6$, or $\frac{6}{5}$

15.
$$\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=\dfrac{5}{18} & (1) \\ xy = 54 & (2) \end{cases}$$

Simplify (1),

$$y + x = \frac{5xy}{18} \qquad (3)$$

$$y + x = 15 \qquad (4)$$

Square (4),

$$\begin{array}{lrl} & y^2 + 2xy + x^2 = & 225 \\ 4 \times (2) \text{ is} & 4xy \quad = & 216 \\ \hline \text{Subtract,} & y^2 - 2xy + x^2 = & 9 \end{array}$$

$$y - x = \pm 3 \qquad (5)$$

Add (5) and (4),

$$2y = 18, \text{ or } 12$$

$$\therefore y = 9, \text{ or } 6$$

Substitute the value of y in (2),

$$x = 6, \text{ or } 9$$

16.
$$\begin{cases} \dfrac{1}{x}-\dfrac{1}{y}=\dfrac{1}{36} & (1) \\ x - 2y + 15 = 0 & (2) \end{cases}$$

Simplify (1),

$$y - x = \frac{xy}{36} \qquad (3)$$

From (2), $\quad x = 2y - 15 \quad (4)$

Substitute in (3),

$$y - (2y - 15) = \frac{(2y-15)y}{36}$$

$$36(-y + 15) = 2y^2 - 15y$$

$$2y^2 + 21y = 540$$

$$16y^2 + 168y = 4320$$

$$16y^2 + 168y + 441 = 4761$$

$$4y + 21 = \pm 69$$

$$4y = 48, \text{ or } -90$$

$$\therefore y = 12, \text{ or } -\tfrac{45}{2}$$

Substitute the value of x in (4),

$$x = 9, \text{ or } -60$$

17.
$$\begin{cases} x^2 + 4y + 11 = 0 & (1) \\ 3x + 2y + 1 = 0 & (2) \end{cases}$$

$2 \times (2)$ is

$$6x + 4y + 2 = 0$$

Subtract from (1),

$$x^2 - 6x + 9 = 0$$

$$x - 3 = 0$$

$$\therefore x = 3$$

Substitute the value of x in (2),

$$y = -5$$

18.
$$\begin{cases} x + 3y + 1 = 0 & (1) \\ x + \dfrac{4y+1}{x+2y} = 2(y+1) & (2) \end{cases}$$

Simplify (2),

$$x^2 + 2xy + 4y + 1 = 2xy + 4y^2 + 2x + 4y$$

$$x^2 - 2x + 1 = 4y^2$$

$$x - 1 = \pm 2y$$

Therefore either

$$x - 2y - 1 = 0 \qquad (3)$$

or $\quad x + 2y - 1 = 1 \quad (4)$

Subtract (3) from (1),

$$5y + 2 = 0$$

$$y = -\tfrac{2}{5}$$

Substitute the value of x in (3),

$$x = \tfrac{1}{5}$$

Subtract (4) from (1),

$$y + 2 = 0$$

$$\therefore y = -2$$

Substitute the value of y in (4),

$$x = 5$$

$$\therefore x = 5, \text{ or } \tfrac{1}{5}$$

$$y = -2, \text{ or } -\tfrac{2}{5}$$

19.

$$\begin{cases} x^2 + y^2 = 106 & (1) \\ xy = 45 & (2) \end{cases}$$

$$\begin{array}{ll} \text{(1) is} & x^2 + y^2 = 106 \\ 2 \times \text{(2) is} & 2xy = 90 \\ \hline \text{Subtract,} & x^2 - 2xy + y^2 = 16 \end{array}$$

$$x - y = \pm 4 \quad (3)$$

$$\begin{array}{ll} & x^2 + y^2 = 106 \\ & 2xy = 90 \\ \hline \text{Add,} & x^2 + 2xy + y^2 = 196 \end{array}$$

$$x + y = \pm 14 \quad (4)$$

Add (3) and (4),

$$2x = \pm 18, \text{ or } \pm 10$$

$$\therefore x = \pm 9, \text{ or } \pm 5$$

Substitute the value of x in (2),

$$y = \pm 5, \text{ or } \pm 9$$

20.

$$\begin{cases} x^2 + y^2 = 52 & (1) \\ xy + 24 = 0 & (2) \end{cases}$$

$$xy = -24$$

$$x^2 + 2xy + y^2 = 4$$

$$x + y = \pm 2 \quad (3)$$

$$x^2 - 2xy + y^2 = 100$$

$$x - y = \pm 10 \quad (4)$$

Add (3) and (4),

$$2x = \pm 12, \text{ or } \pm 8$$

$$x = \pm 6, \text{ or } \pm 4$$

Substitute the value of x in (2),

$$y = \mp 4, \text{ or } \mp 6$$

21.

$$\begin{cases} x^2 - xy = 3 & (1) \\ y^2 + xy = 10 & (2) \end{cases}$$

Substitute $y = vx$ in both equations,

$$x^2 - vx^2 = 3 \quad (3)$$

$$v^2x^2 + vx = 10 \quad (4)$$

Divide (3) by (4),

$$\frac{1 - v}{v^2 + v} = \frac{3}{10}$$

$$10 - 10v = 3v^2 + 3v$$

$$3v^2 + 13v - 10 = 0$$

$$(3v - 2)(v + 5) = 0$$

$$v = \tfrac{2}{3}, \text{ or } -5$$

Substituting the values of v in (3),

$$\tfrac{1}{3}x^2 = 3$$

$$x^2 = 9$$

$$x = \pm 3$$

or

$$6x^2 = 3$$

$$x^2 = \tfrac{1}{2}$$

$$x = \pm \sqrt{\tfrac{1}{2}}$$

$$y = vx$$

$$\therefore y = \pm 2, \text{ or } \mp 5\sqrt{\tfrac{1}{2}}$$

$$\therefore x = \pm 3, \text{ or } \pm \sqrt{\tfrac{1}{2}}$$

$$y = \pm 2, \text{ or } \mp 5\sqrt{\tfrac{1}{2}}$$

22.

$$\begin{cases} xy + y^2 = 4 & (1) \\ 2x^2 - y^2 = 17 & (2) \end{cases}$$

Substitute $y = vx$ in both equations,

$$vx^2 + v^2x^2 = 4 \quad (3)$$

$$2x^2 - v^2x^2 = 17 \quad (4)$$

Divide (3) by (4),

$$\frac{v + v^2}{2 - v^2} = \frac{4}{17}$$

$$17v + 17v^2 = 8 - 4v^2$$

$$21v^2 + 17v - 8 = 0$$

$$21^2v^2 + 21 \times 17v = 168$$

$$21^2v^2 + 21 \times 17v + \tfrac{289}{4} = \tfrac{961}{4}$$

$$21v + \tfrac{17}{2} = \pm \tfrac{31}{2}$$

$$21v = 7, \text{ or } -24$$

$$v = \tfrac{1}{3}, \text{ or } -\tfrac{8}{7}$$

Substitute the value of v in (4),

$$x^2 = 9, \text{ or } \tfrac{49}{2}$$

$$\therefore x = \pm 3, \text{ or } \pm 7\sqrt{\tfrac{1}{2}}$$

$$y = \pm 1, \text{ or } \mp 4\sqrt{2}$$

23. $\begin{cases} x^2 + 3xy = 27 & (1) \\ xy - y^2 = 2 & (2) \end{cases}$

Subtract, $x^2 + 2xy + y^2 = 25$

$x + y = \pm 5 \qquad (3)$

$x = 5 - y$

Substitute in (2),

$(5 - y)y - y^2 = 2$

$2y^2 - 5y + 2 = 0$

$(2y - 1)(y - 2) = 0$

$\therefore y = \frac{1}{2}, \text{ or } 2$

$x = \frac{9}{2}, \text{ or } 3$

or $x = -5 - y$

Substitute in (3),

$(-5 - y)y - y^2 = 2$

$2y^2 + 5y + 2 = 0$

$(2y + 1)(y + 2) = 0$

$y = -\frac{1}{2}, \text{ or } -2$

$x = -\frac{9}{2}, \text{ or } -3$

$\therefore x = \pm \frac{9}{2}, \text{ or } \pm 3$

$y = \pm \frac{1}{2}, \text{ or } \pm 2$

24. $\begin{cases} x^2 + xy = 60 & (1) \\ y^2 + xy = 40 & (2) \end{cases}$

Add, $x^2 + 2xy + y^2 = 100$

$x + y = \pm 10$

(1) is $x(x + y) = 60$

$\therefore x = \pm 6$

(2) is $y(y + x) = 40$

$\therefore y = \pm 4$

25. $\begin{cases} x^2 + 2xy - y^2 = 28 & (1) \\ 3x^2 + 2xy + 2y^2 = 72 & (2) \end{cases}$

Add, $4x^2 + 4xy + y^2 = 100$

$2x + y = \pm 10$

$y = \pm 10 - 2x \quad (3)$

Substitute in (2),

$3x^2 + 2x(\pm 10 - 2x)$

$+ 2(\pm 10 - 2x)^2 = 72$

$7x^2 \mp 60x = -128$

$49x^2 \mp 420x + 900 = 4$

$7x \mp 30 = \pm 2$

$7x = \pm 32, \text{ or } \pm 28$

$\therefore x = \pm \frac{32}{7}, \text{ or } \pm 4$

$y = \pm \frac{6}{7}, \text{ or } \pm 2$

26. $\begin{cases} x^2 - 4xy = 45 & (1) \\ y^2 - xy = 6 & (2) \end{cases}$

Substitute $y = vx$ in both equations,

$x^2 - 4vx^2 = 45 \qquad (3)$

$v^2x^2 - vx^2 = 6 \qquad (4)$

Divide (3) by (4),

$\frac{1 - 4v}{v^2 - v} = \frac{45}{6}$

$6 - 24v = 45v^2 - 45v$

$45v^2 - 21v - 6 = 0$

$(15v + 3)(3v - 2) = 0$

$v = -\frac{1}{5}, \text{ or } \frac{2}{3}$

Substitute the value of v in (3),

$x^2 = 25, \text{ or } -27$

$\therefore x = \pm 5, \text{ or } \pm 3\sqrt{-3}$

$y = \mp 1, \text{ or } \pm 2\sqrt{-3}$

27. $\begin{cases} x^2 + 3xy = 55 & (1) \\ 2y^2 + xy = 18 & (2) \end{cases}$

Substitute $y = vx$ in both equations, $x^2 + 3vx^2 = 55 \qquad (3)$

$2v^2x^2 + vx^2 = 18 \qquad (4)$

Divide (3) by (4),

$\frac{1 + 3v}{2v^2 + v} = \frac{55}{18}$

$18 + 54v = 110v^2 + 55v$

$110v^2 + v - 18 = 0$

$110^2v^2 + 110v + \frac{1}{4} = 1980\frac{1}{4}$

$110v + \frac{1}{2} = \pm 44\frac{1}{2}$

$110v = 44, \text{ or } 45$

$v = \frac{4}{10}, \text{ or } -\frac{9}{22}$

Substitute the value of v in (3),

$x^2 = 25, \text{ or } -242$

$\therefore x = \pm 5, \text{ or } \pm 11\sqrt{-2}$

$y = \pm 2, \text{ or } \mp \frac{9}{2}\sqrt{-2}$

28.

$$\begin{cases} x^2 - xy + y^2 = 37 & (1) \\ x^2 + 2xy + 8 = 0 & (2) \end{cases}$$

Substitute $y = vx$ in both equations,

$$x^2 - vx^2 + v^2x^2 = 37 \qquad (3)$$

$$x^2 + 2vx^2 = -8 \qquad (4)$$

Divide (3) by (4), $$\frac{1 - v + v^2}{1 + 2v} = -\frac{37}{8}$$

$$8 - 8v + 8v^2 = -37 - 74v$$

$$8v^2 + 66v = -45$$

$$16v^2 + 132v + 272\tfrac{1}{4} = 182\tfrac{1}{4}$$

$$4v + 16\tfrac{1}{2} = \pm 13\tfrac{1}{2}$$

$$4v = -3, \text{ or } -30$$

$$v = -\tfrac{3}{4}, \text{ or } -\tfrac{15}{2}$$

Substitute the value of v in (4), $$x^2 = 16, \text{ or } \tfrac{4}{7}$$

$$\therefore x = \pm 4, \text{ or } \pm 2\sqrt{\tfrac{1}{7}}$$

$$y = \mp 3, \text{ or } \mp 15\sqrt{\tfrac{1}{7}}$$

29.

$$\begin{cases} x^2 + xy + 2y^2 = 44 & (1) \\ 2x^2 - xy + y^2 = 16 & (2) \end{cases}$$

Substitute $y = vx$ in both equations,

$$x^2 + vx^2 + 2v^2x^2 = 44 \qquad (3)$$

$$2x^2 - vx^2 + v^2x^2 = 16 \qquad (4)$$

Divide (3) by (4), $$\frac{1 + v + 2v^2}{2 - v + v^2} = \frac{44}{16}$$

$$16 + 16v + 32v^2 = 88 - 44v + 44v^2$$

$$12v^2 - 60v = -72$$

$$v^2 - 5v + 6 = 0$$

$$(v - 2)(v - 3) = 0$$

$$\therefore v = 2, \text{ or } 3$$

Substitute the value of v in (3), $$x^2 = 4, \text{ or } 2$$

$$\therefore x = \pm 2, \text{ or } \pm \sqrt{2}$$

$$y = \pm 4, \text{ or } \pm 3\sqrt{2}$$

30.

$$\begin{cases} 8x^2 - 3xy - y^2 = 40 & (1) \\ 9x^2 + xy + 2y^2 = 60 & (2) \end{cases}$$

Substitute $y = vx$ in both equations,

$$8x^2 - 3vx^2 - v^2y^2 = 40 \qquad (3)$$

$$9x^2 + vx^2 + 2v^2x^2 = 60 \qquad (4)$$

Divide (3) by (4), $$\frac{8 - 3v - v^2}{9 + v + 2v^2} = \frac{2}{3}$$

$$24 - 9v - 3v^2 = 18 + 2v + 4v^2$$

$$7v^2 + 11v = 6$$
$$7v^2 + 11v - 6 = 0$$
$$(7v - 3)(v + 2) = 0$$
$$v = \tfrac{3}{7}, \text{ or } -2$$

Substitute the value of v in (3), $x^2 = \tfrac{49}{8}, \text{ or } 4$

$$\therefore\ x = \pm \tfrac{7}{2}\sqrt{\tfrac{1}{2}}, \text{ or } \pm 2$$
$$y = \pm \tfrac{3}{2}\sqrt{\tfrac{1}{2}}, \text{ or } \mp 4$$

31.
$$\begin{cases} 3x^2 + 3xy + y^2 = 52 & (1) \\ 5x^2 + 7xy + 4y^2 = 140 & (2) \end{cases}$$

Substitute $y = vx$ in both equations,

$$3x^2 + 3vx^2 + v^2x^2 = 52 \qquad (3)$$
$$5x^2 + 7vx^2 + 4v^2x^2 = 140 \qquad (4)$$

Divide (3) by (4), $\dfrac{3 + 3v + v^2}{5 + 7v + 4v^2} = \dfrac{52}{140}$

$$\frac{3 + 3v + v^2}{5 + 7v + 4v^2} = \frac{13}{35}$$
$$105v^2 + 105v + 35v^2 = 65 + 91v + 52v^2$$
$$17v^2 - 14v - 40 = 0$$
$$(17v + 20)(v - 2) = 0$$
$$\therefore\ v = -\tfrac{20}{17}, \text{ or } 2$$

Substitute the value of v in (3), $\tfrac{247}{289}x^2 = 52, \text{ or } 13x^2 = 52$

$$\therefore\ x = \pm \frac{34}{\sqrt{19}}, \text{ or } \pm 2$$
$$y = \mp \frac{40}{\sqrt{19}}, \text{ or } \pm 4$$

32.
$$\begin{cases} 4x^2 + 3xy + 5y^2 = 27 & (1) \\ 7x^2 + 5xy + 9y^2 = 47 & (2) \end{cases}$$

Substitute $y = vx$ in both equations,

$$4x^2 + 3vx^2 + 5v^2x^2 = 27 \qquad (3)$$
$$7x^2 + 5vx^2 + 9v^2x^2 = 47 \qquad (4)$$

Divide (3) by (4), $\dfrac{4 + 3v + 5v^2}{7 + 5v + 9v^2} = \dfrac{27}{47}$

$$188 + 141v + 235v^2 = 189 + 135v + 243v^2$$
$$8v^2 - 6v = -1$$
$$16v^2 - 12v + 2\tfrac{1}{4} = \tfrac{1}{4}$$
$$4v - \tfrac{3}{2} = \pm \tfrac{1}{2}$$
$$4v = 1, \text{ or } 2$$
$$v = \tfrac{1}{4}, \text{ or } \tfrac{1}{2}$$

Substitute the value of v in (3), $\frac{81}{16}x^2 = 27$, or $\frac{27}{4}x^2 = 27$

$$\therefore\ x = \pm\frac{4}{\sqrt{3}},\ \text{or} \pm 2$$

$$y = \pm\frac{1}{\sqrt{3}},\ \text{or} \pm 1$$

33.

$$\begin{cases} 5x^2 + 3xy + 2y^2 = 188 & (1) \\ x^2 - xy + y^2 = 19 & (2) \end{cases}$$

Subtract, $4x^2 + 4xy + y^2 = 169$

$$2x + y = \pm 13$$

$$\therefore\ y = \pm 13 - 2x \qquad (3)$$

Substitute in (2),

$$x^2 - x(\pm 13 - 2x) + (\pm 13 - 2x)^2 = 19$$

$$x^2 \mp 13x + 2x^2 + 169 \mp 52x + 4x^2 = 19$$

$$7x^2 \mp 65x = -150$$

$$7x^2 \mp 65x + 150 = 0$$

$$(7x \mp 30)(x \mp 5) = 0$$

$$\therefore\ x = \pm\frac{30}{7},\ \text{or} \pm 5$$

Substitute value of x in (3), $y = \pm\frac{31}{7}$, or ± 3

34.

$$\begin{cases} x^3 + y^3 = 65 & (1) \\ x + y = 5 & (2) \end{cases}$$

Divide (1) by (2), $x^2 - xy + y^2 = 13$ (3)

Square (2), $x^2 + 2xy + y^2 = 25$ (4)

Subtract, $-3xy = -12$

$$xy = 4 \qquad (5)$$

Subtract (5) from (3), $x^2 - 2xy + y^2 = 9$

$$x - y = \pm 3 \qquad (6)$$

Add (6) and (2), $2x = 8$, or 2

$$\therefore\ x = 4,\ \text{or}\ 1$$

Substitute value of x in (2), $y = 1$, or 4

35.

$$\begin{cases} x^3 - y^3 = 98 & (1) \\ x - y = 2 & (2) \end{cases}$$

Divide (1) by (2), $x^2 + xy + y^2 = 49$ (3)

Square (2), $x^2 - 2xy + y^2 = 4$ (4)

Subtract, $3xy = 45$

$$xy = 15 \qquad (5)$$

Add (5) and (3), $x^2+2xy+y^2=64$

$x+y=\pm 8$ (6)

Add (6) and (2), $2x=10, \text{ or } -6$

$\therefore x=5, \text{ or } -3$

Substitute value of x in (2), $y=3, \text{ or } -5$

36. $\begin{cases} x^3+y^3=279 & (1) \\ x+y=3 & (2) \end{cases}$

Divide (1) by (2), $x^2-xy+y^2=93$ (3)

Square (2), $x^2+2xy+y^2=9$ (4)

Subtract, $-3xy=84$

$xy=-28$ (5)

Subtract (5) from (3), $x^2-2xy+y^2=121$

$x-y=\pm 11$ (6)

Add (2) and (6), $2x=14, \text{ or } -8$

$\therefore x=7, \text{ or } -4$

Substitute values of x in (2), $y=-4, \text{ or } 7$

37. $\begin{cases} x^3-y^3=218 & (1) \\ x-y=2 & (2) \end{cases}$

Divide (1) by (2), $x^2+xy+y^2=109$ (3)

Square (2), $x^2-2xy+y^2=4$ (4)

Subtract, $3xy=105$

$xy=35$ (5)

Add (5) and (3), $x^2+2xy+y^2=144$

$x+y=\pm 12$ (6)

Add (2) and (6), $2x=14, \text{ or } -10$

$\therefore x=7, \text{ or } -5$

Substitute values of x in (2), $y=5, \text{ or } -7$

38. $\begin{cases} x^3+y^3=152 & (1) \\ x^2-xy+y^2=19 & (2) \end{cases}$

Divide (1) by (2), $x+y=8$ (3)

Square (3), $x^2+2xy+y^2=64$

(2) is $x^2-xy+y^2=19$

Subtract, $3xy=45$

$xy=15$ (4)

Subtract (4) from (2), $x^2-2xy+y^2=4$

$x-y=\pm 2$ (5)

Add (3) and (5), $2x=10, \text{ or } 6$

$\therefore x=5, \text{ or } 3$

Substitute value of x in (3), $y=3, \text{ or } 5$

39.

$$\begin{cases} x^3 - y^3 = 1304 & (1) \\ x^2 + xy + y^2 = 163 & (2) \end{cases}$$

Divide (1) by (2), $x - y = 8$ (3)

Square (3), $x^2 - 2xy + y^2 = 64$

(2) is $x^2 + xy + y^2 = 163$

Subtract, $-3xy = -99$

$xy = 33$ (4)

Add (2) and (4), $x^2 + 2xy + y^2 = 196$

$x + y = \pm 14$ (5)

Add (3) and (5), $2x = 22$, or -6

$\therefore x = 11$, or -3

Substitute value of x in (3), $y = 3$, or -11

40.

$$\begin{cases} x^3 + y^3 = 91 & (1) \\ xy(x + y) = 84 & (2) \end{cases}$$

$3 \times (2)$ is $3x^2y + 3xy^2 = 252$ (3)

Add (1) and (3), $x^3 + 3x^2y + 3xy^2 + y^3 = 343$

Extract cube root, $x + y = 7$ (4)

Substitute in (2), $xy = 12$ (5)

Square (4), $x^2 + 2xy + y^2 = 49$

$4 \times (5)$ is $4xy = 48$

Subtract, $x^2 - 2xy + y^2 = 1$

$x - y = \pm 1$ (6)

Add (4) and (6), $2x = 8$, or 6

$\therefore x = 4$, or 3

Substitute value of x in (4), $y = 3$, or 4

41.

$$\begin{cases} x^3 - y^3 = 98 & (1) \\ x - y = \dfrac{30}{xy} & (2) \end{cases}$$

Simplify (2), $x^2y - xy^2 = 30$

$3x^2y - 3xy^2 = 90$ (3)

Subtract (3) from (1),

$$x^3 - 3x^2y + 3xy^2 - y^3 = 8$$

Extract cube root, $x - y = 2$ (4)

Substitute in (2), $2 = \dfrac{30}{xy}$

$xy = 15$ (5)

Square (4), $x^2 - 2xy + y^2 = 4$

4 × (5) is $4xy = 60$

Add, $x^2 + 2xy + y^2 = 64$

$$x + y = \pm 8 \tag{6}$$

Add (4) and (6), $2x = 10, \text{ or } -6$

$$\therefore x = 5, \text{ or } -3$$

Substitute value of x in (4), $y = 3, \text{ or } -5$

42.

$$\begin{cases} \dfrac{x^2}{y} + \dfrac{y^2}{x} = \dfrac{27}{2} & (1) \\ \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{2} & (2) \end{cases}$$

Simplify both equations, $$x^3 + y^3 = \frac{27xy}{2} \tag{3}$$

$$x + y = \frac{xy}{2} \tag{4}$$

Divide (3) by (4), $$x^2 - xy + y^2 = 27 \tag{5}$$

Square (4), $x^2 + 2xy + y^2 = \dfrac{x^2y^2}{4}$

Subtract, $-3xy = 27 - \dfrac{x^2y^2}{4}$

$$x^2y^2 - 12xy - 108 = 0$$

$$(xy + 6)(xy - 18) = 0$$

$$xy = 18$$

From (4), $x + y = \dfrac{xy}{2} = 9$

$$x = 6, \text{ or } 3$$

$$y = 3, \text{ or } 6$$

$$xy = -6$$

From (4), $x + y = \dfrac{xy}{2} = -3$

$$x^2 + 2xy + y^2 = 9$$

$$4xy = -24$$

Subtract, $x^2 - 2xy + y^2 = 33$

$$x - y = \pm\sqrt{33}$$

$$x + y = -3$$

$$\therefore x = \frac{-3 \pm \sqrt{33}}{2}$$

$$y = \frac{-3 \mp \sqrt{33}}{2}$$

43.
$$\begin{cases} \dfrac{x^2}{y}+\dfrac{y^2}{x}=\dfrac{91}{12} & (1) \\ \dfrac{1}{x}+\dfrac{1}{y}=\dfrac{7}{12} & (2) \end{cases}$$

Simplify both equations, $x^3+y^3=\frac{91}{12}xy$ (3)

$x+y=\frac{7}{12}xy$ (4)

Divide (3) by (4), $x^2-xy+y^2=13$ (5)

$$\begin{array}{ll} \text{Square (4),} & x^2+2xy+y^2=\frac{49}{144}x^2y^2 \\ \hline \text{Subtract,} & -3xy=13-\frac{49}{144}x^2y^2 \end{array}$$

$$49x^2y^2-432xy-1872=0$$
$$(xy-12)(49xy+156)=0$$

$xy=12$, or $-\frac{156}{49}$ (6)

Substitute value of xy in (4), $x+y=7$, or $-\frac{13}{7}$ (7)

$$\begin{array}{ll} \text{Square (7),} & x^2+2xy+y^2=49, \text{ or } \frac{169}{49} \\ 4\times(6) \text{ is} & 4xy=48, \text{ or } -\frac{624}{49} \\ \hline \text{Subtract,} & x^2-2xy+y^2=1, \text{ or } \frac{793}{49} \end{array}$$

$x-y=\pm 1$, or $\pm\dfrac{\sqrt{793}}{7}$ (8)

Add (7) and (8), $2x=8$, or 6, or $\dfrac{-13\pm\sqrt{793}}{7}$

$\therefore\ x=4$, or 3, or $\dfrac{-13\pm\sqrt{793}}{14}$

Substitute value of x in (7), $y=3$, or 4, or $\dfrac{-13\mp\sqrt{793}}{14}$

44.
$$\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=\dfrac{1}{2} & (1) \\ \dfrac{1}{x^2}+\dfrac{1}{y^2}=\dfrac{5}{36} & (2) \end{cases}$$

$$\begin{array}{ll} \text{Square (1),} & \dfrac{1}{x^2}+\dfrac{2}{xy}+\dfrac{1}{y^2}=\dfrac{1}{4} \\ \text{(2) is} & \dfrac{1}{x^2}+\dfrac{1}{y^2}=\dfrac{5}{36} \\ \hline \text{Subtract,} & \dfrac{2}{xy}=\dfrac{4}{36} \end{array}$$

$\dfrac{2}{xy}=\dfrac{1}{9}$ (3)

Subtract (3) from (2), $\frac{1}{x^2}-\frac{2}{xy}+\frac{1}{y^2}=\frac{1}{36}$

$\frac{1}{x}-\frac{1}{y}=\pm\frac{1}{6}$ (4)

Add (1) and (4), $\frac{2}{x}=\frac{2}{3}, \text{ or } \frac{1}{3}$

$\therefore\ x=3, \text{ or } 6$

Substitute value of x in (1), $y=6, \text{ or } 3$

45. $\begin{cases} x^3-y^3=7xy & (1) \\ x-y=2 & (2) \end{cases}$

$$\begin{array}{ll}
\text{Cube (2),} & x^3-3x^2y+3xy^2-y^3=8 \\
\text{(1) is} & x^3 \qquad\qquad\quad -y^3=\quad 7xy \\
\hline
\text{Subtract,} & -3x^2y+3xy^2 \qquad =8-7xy \\
& -3xy(x-y)=8-7xy \qquad (3) \\
\text{From (2),} & x-y=2 \\
\text{Substitute in (3),} & -6xy=8-7xy \\
& xy=8 \qquad (4) \\
\text{Square (2),} & x^2-2xy+y^2=\ 4 \\
4\times(4) \text{ is} & 4xy \qquad\ =32 \\
\hline
\text{Add,} & x^2+2xy+y^2=36 \\
& x+y=\pm 6 \\
& \therefore\ x=4, \text{ or } -2 \\
& y=2, \text{ or } -4
\end{array}$$

46. $\begin{cases} x^3+y^3=\frac{27xy}{2} & (1) \\ x+y=9 & (2) \end{cases}$

$$\begin{array}{ll}
\text{Cube (2),} & x^3+3x^2y+3xy^2+y^3=729 \\
\text{(1) is} & x^3 \qquad\qquad\quad +y^3=\quad \tfrac{27}{2}xy \\
\hline
\text{Subtract,} & 3x^2y+3xy^2 \qquad =729-\tfrac{27}{2}xy \\
& 3xy(x+y)=729-\tfrac{27}{2}xy \qquad (3) \\
\text{From (2),} & x+y=9 \\
\text{Substitute in (3),} & 27xy=729-\tfrac{27}{2}xy \\
& xy=18 \qquad (4) \\
\text{Square (2),} & x^2+2xy+y^2=81 \\
4\times(4) \text{ is} & 4xy \qquad\ =72 \\
\hline
\text{Subtract,} & x^2-2xy+y^2=\ 9 \\
& x-y=\pm 3 \\
& \therefore\ x=6, \text{ or } 3 \\
& y=3, \text{ or } 6
\end{array}$$

47.

$$\begin{cases} x^2 + y^2 = \dfrac{5xy}{2} & (1) \\ x + y = \dfrac{5xy}{6} & (2) \end{cases}$$

Square (2), $x^2 + 2xy + y^2 = \dfrac{25x^2y^2}{36}$

(1) is $x^2 \qquad + y^2 = \qquad \dfrac{5xy}{2}$

Subtract, $2xy \qquad = \dfrac{25x^2y^2}{36} - \dfrac{5xy}{2}$

$$25x^2y^2 - 162xy = 0$$

$$\therefore xy = 0, \text{ or } \tfrac{162}{25}$$

(i) $xy = 0$ gives $x = 0$ and $y = 0$

(ii) $xy = \tfrac{162}{25}$ (3)

Substitute value of xy in (1),

$$x^2 \qquad + y^2 = \tfrac{81}{5}$$

2 × (3) is $2xy \qquad = \tfrac{324}{25}$

Subtract, $x^2 - 2xy + y^2 = \tfrac{81}{25}$

$$x - y = \pm \tfrac{9}{5} \qquad (4)$$

Substitute value of xy in (2), $x + y = \tfrac{27}{5}$

$$\therefore x = \tfrac{18}{5}, \text{ or } \tfrac{9}{5}$$

$$y = \tfrac{9}{5}, \text{ or } \tfrac{18}{5}$$

48.

$$\begin{cases} x^2y^2 - 16xy + 60 = 0 & (1) \\ x + y = 7 & (2) \end{cases}$$

$$x^2y^2 - 16xy + 64 = 4$$

$$xy - 8 = \pm 2$$

$$xy = 10, \text{ or } 6 \qquad (3)$$

Square (2), $x^2 + 2xy + y^2 = 49$

4 × (3) is $4xy \qquad = 40, \text{ or } 24$

Subtract, $x^2 - 2xy + y^2 = 9, \text{ or } 25$

$$x - y = \pm 3, \text{ or } \pm 5 \qquad (4)$$

$$\therefore x = 5, 2, 6, \text{ or } 1$$

$$y = 2, 5, 1, \text{ or } 6$$

49.

$$\begin{cases} x^2y^2 = 4xy + 12 & (1) \\ xy = x + y + 1 & (2) \end{cases}$$

$$x^2y^2 - 4xy + 4 = 16$$

$$xy - 2 = \pm 4$$

$$xy = 6, \text{ or } -2 \qquad (3)$$

Substitute in (2), $x + y + 1 = 6, \text{ or } -2$

$x + y = 5, \text{ or } -3$ (4)

Square (4), $x^2 + 2xy + y^2 = 25, \text{ or } 9$

$4 \times (3)$ is $4xy = 24, \text{ or } -8$

Subtract, $x^2 - 2xy + y^2 = 1, \text{ or } 17$

$x - y = \pm 1, \text{ or } \pm \sqrt{17}$ (5)

$\therefore\ x = 3, \text{ or } 2, \text{ or } \dfrac{-3 \pm \sqrt{17}}{2}$

$y = 2, \text{ or } 3, \text{ or } \dfrac{-3 \mp \sqrt{17}}{2}$

50.

$$\begin{cases} x^3 + y^3 = \dfrac{35x^2y^2}{36} & (1) \\[2ex] x + y = \dfrac{5xy}{6} & (2) \end{cases}$$

Divide (1) by (2), $x^2 - xy + y^2 = \dfrac{7xy}{6}$

Square (2), $x^2 + 2xy + y^2 = \dfrac{25x^2y^2}{36}$

Subtract, $-3xy = \dfrac{7xy}{6} - \dfrac{25x^2y^2}{36}$

$25x^2y^2 - 150xy = 0$

$\therefore\ xy = 0, \text{ or } 6$

$xy = 0$ gives $x = 0$ and $y = 0$

$xy = 6$ (4)

Substitute in (2), $x + y = 5$ (5)

Square (5), $x^2 + 2xy + y^2 = 25$

$4 \times (4)$ is $4xy = 24$

Subtract, $x^2 - 2xy + y^2 = 1$

$x - y = \pm 1$

$\therefore\ x = 3, \text{ or } 2$

$y = 2, \text{ or } 3$

51.

$$\begin{cases} x^2 + y^2 = 67 - xy & (1) \\ x + y = xy - 5 & (2) \end{cases}$$

(1) is $x^2 + y^2 = 67 - xy$

Square (2), $x^2 + 2xy + y^2 = x^2y^2 - 10xy + 25$

Subtract, $-2xy = -x^2y^2 + 9xy + 42$

$$x^2y^2 - 11\,xy - 42 = 0$$
$$(xy - 14)(xy + 3) = 0$$
$$xy = -3$$

Substitute in (2), $x + y = xy - 5 = -8$

$$\begin{array}{rl} x^2 + 2\,xy + y^2 = & 64 \\ 4\,xy \quad = & -12 \\ \hline \end{array}$$

Subtract, $x^2 - 2\,xy + y^2 = 76$

$$x - y = \pm 2\sqrt{19}$$
$$x + y = -8$$
$$x = -4 \pm \sqrt{19}$$
$$y = -4 \mp \sqrt{19}$$
$$xy = 14$$
$$x + y = xy - 5 = 9$$
$$\therefore\ x = 7, \text{ or } 2$$
$$y = 2, \text{ or } 7$$

52. $$\begin{cases} x^3 + y^3 = 1 - 3\,xy & (1) \\ x^2 + y^2 = xy + 37 & (2) \end{cases}$$

From (2), $x^2 - xy + y^2 = 37 \qquad (3)$

Divide (1) by (3) $x + y = \dfrac{1 - 3\,xy}{37} \qquad (4)$

Square (4), $x^2 + 2\,xy + y^2 = \dfrac{1 - 6\,xy + 9\,x^2y^2}{1369}$

(3) is $x^2 - \ \ xy + y^2 = \qquad 37$

Subtract, $3\,xy \quad = \dfrac{1 - 6\,xy + 9\,x^2y^2}{1369} - 37$

$$9\,x^2y^2 - 4113\,xy - 50652 = 0$$
$$x^2y^2 - 457\,xy - 5628 = 0$$
$$(xy + 12)(xy - 469) = 0$$
$$xy = -12, \text{ or } 469 \qquad (5)$$

Substitute value of xy in (4), $x + y = 1, \text{ or } -38 \qquad (6)$

(3) is $x^2 - \ \ xy + y^2 = \ 37$

(5) is $xy \quad = -12, \text{ or } \ 469$

Subtract, $x^2 - 2\,xy + y^2 = \ 49, \text{ or } -432$

$$x - y = \pm 7, \text{ or } \pm 12\sqrt{-3}$$
$$\therefore\ x = 4, \text{ or } -3, \text{ or } -19 \pm 6\sqrt{-3}$$
$$y = -3, \text{ or } 4, \text{ or } -19 \mp 6\sqrt{-3}$$

53.
$$\begin{cases} x^4 + y^4 = 706 & (1) \\ x + y = 2 & (2) \end{cases}$$

Raise (2) to 4th power,
$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = 16$$

(1) is
$$x^4 + y^4 = 706$$

Subtract,
$$4x^3y + 6x^2y^2 + 4xy^3 = -690$$
$$xy(2x^2 + 3xy + 2y^2) = -345 \tag{3}$$

Square (2),
$$x^2 + 2xy + y^2 = 4$$
$$2x^2 + 4xy + 2y^2 = 8$$
$$2x^2 + 2y^2 = 8 - 4xy$$

Substitute the value of $2x^2 + 2y^2$ in (3),
$$xy(8 - xy) = -345$$
$$x^2y^2 - 8xy = 345$$
$$x^2y^2 - 8xy + 16 = 361$$
$$xy - 4 = \pm 19$$
$$xy = 23, \text{ or } -15 \tag{4}$$

Square (2),
$$x^2 + 2xy + y^2 = 4$$

$4 \times (4)$ is
$$4xy = 92, \text{ or } -60$$

Subtract,
$$x^2 - 2xy + y^2 = -88, \text{ or } 64$$
$$x - y = \pm\sqrt{-88}, \text{ or } \pm 8$$
$$\therefore\ x = 1 \pm \sqrt{-22}, \text{ or } 5, \text{ or } -3$$
$$y = 1 \mp \sqrt{-22}, \text{ or } -3, \text{ or } 5$$

54.
$$\begin{cases} x^5 - y^5 = 211 & (1) \\ x - y = 1 & (2) \end{cases}$$

Divide (1) by (2),
$$x^4 + x^3y + x^2y^2 + xy^3 + y^4 = 211 \tag{3}$$

Raise (2) to 4th power,
$$x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4 = 1$$

Subtract,
$$5x^3y - 5x^2y^2 + 5xy^3 = 210$$
$$5xy(x^2 - xy + y^2) = 210$$
$$xy(x^2 - xy + y^2) = 42 \tag{4}$$

Square (2),
$$x^2 - 2xy + y^2 = 1$$
$$x^2 + y^2 = 1 + 2xy$$

Substitute value of $x^2 + y^2$ in (4),
$$xy(1 + xy) = 42$$
$$x^2y^2 + xy = 42$$
$$x^2y^2 + xy + \tfrac{1}{4} = 42\tfrac{1}{4}$$
$$xy + \tfrac{1}{2} = \pm 6\tfrac{1}{2}$$
$$xy = 6, \text{ or } -7 \tag{5}$$

Square (2), $x^2 - 2xy + y^2 = 1$

$4 \times (5)$ is $4xy = 24, \text{ or } -28$

Add, $x^2 + 2xy + y^2 = 25, \text{ or } -27$

$x + y = \pm 5, \text{ or } \pm 3\sqrt{-3}$ (6)

From (5) and (6), $x = 3, -2, \text{ or } \dfrac{1 \pm 3\sqrt{-3}}{2}$

$y = 2, -3, \text{ or } \dfrac{-1 \pm 3\sqrt{-3}}{2}$

55. $\begin{cases} x^5 + y^5 = 3368 & (1) \\ x + y = 8 & (2) \end{cases}$

Divide (1) by (2), $x^4 - x^3y + x^2y^2 - xy^3 + y^4 = 421$ (3)

Raise (2) to 4th power, $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = 4096$

Subtract, $-5x^3y - 5x^2y^2 - 5xy^3 = -3675$

$x^3y + x^2y^2 + xy^3 = 735$

$xy(x^2 + xy + y^2) = 735$ (4)

Square (2), $x^2 + 2xy + y^2 = 64$

$x^2 + xy + y^2 = 64 - xy$

Substitute in (4), $xy(64 - xy) = 735$

$x^2y^2 - 64xy = -735$

$x^2y^2 - 64xy + 1024 = 289$

$xy - 32 = \pm 17$

$xy = 49, \text{ or } 15$ (5)

Square (2), $x^2 + 2xy + y^2 = 64$

$4 \times (5)$ is $4xy = 196, \text{ or } 60$

Subtract, $x^2 - 2xy + y^2 = -132, \text{ or } 4$

$x - y = \pm 2\sqrt{-33}, \text{ or } \pm 2$

$\therefore x = 4 \pm \sqrt{-33}, \text{ or } 5, \text{ or } 3$

$y = 4 \mp \sqrt{-33}, \text{ or } 3, \text{ or } 5$

56. $\begin{cases} \dfrac{x^2}{y^2} + \dfrac{y^2}{x^2} = 17\left(\dfrac{xy}{16}\right)^2 & (1) \\ \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{3}{4} & (2) \end{cases}$

Simplify both equations, $x^4 + y^4 = \frac{17}{256}x^4y^4$ (3)

$x + y = \frac{3}{4}xy$ (4)

Raise (4) to 4th power, $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = \frac{81}{256}x^4y^4$

(3) is $x^4 + y^4 = \frac{17}{256}x^4y^4$

Subtract, $4x^3y + 6x^2y^2 + 4xy^3 = \frac{1}{4}x^4y^4$

$$2xy(2x^2+3xy+2y)=\tfrac{1}{4}x^4y^4 \qquad (5)$$

Square (4), $x^2+2xy+y^2=\tfrac{9}{16}x^2y^2$

$$2x^2+2y^2=\tfrac{9}{8}x^2y^2-4xy$$

Substitute in (5), $2xy(\tfrac{9}{8}x^2y^2-xy)=\tfrac{1}{4}x^4y^4$

$$\tfrac{1}{4}x^4y^4-\tfrac{9}{4}x^3y^3+2x^2y^2=0$$

$$x^4y^4-9x^3y^3+8x^2y^2=0$$

$$x^2y^2(xy-1)(xy-8)=0$$

$$\therefore\ xy=0,\text{ or }1,\text{ or }8 \qquad (6)$$

Substitute value of xy in (4), $x+y=0,\ \tfrac{3}{4},\text{ or }6$ (7)

Square (7), $x^2+2xy+y^2=0,\quad \tfrac{9}{16},\text{ or }36$

4 + (6) is $4xy\quad=0,\quad 4,\text{ or }32$

Subtract, $x^2-2xy+y^2=0,-\tfrac{55}{16},\text{ or }\ 4$

$$x-y=0,\ \pm\tfrac{1}{4}\sqrt{-55},\text{ or }\pm 2$$

$$x=0,\ \frac{3\pm\sqrt{-55}}{8},\ 4,\text{ or }2$$

$$y=0,\ \frac{3\mp\sqrt{-55}}{8},\ 2,\text{ or }4$$

87. $$\begin{cases} x^2+y^2=xy+19 & (1)\\ x+y=xy-7 & (2)\end{cases}$$

Square (2), $x^2+2xy+y^2=x^2y^2-14xy+49$

(1) is $x^2\quad+y^2=\quad xy+19$

Subtract, $2xy\quad=x^2y^2-15xy+30$

$$x^2y^2-17xy+30=0$$

$$(xy-2)(xy-15)=0$$

$$xy=2,\text{ or }15 \qquad (3)$$

Substitute value of xy in (2), $x+y=-5,\text{ or }8$ (4)

Square (4), $x^2+2xy+y^2=25,\text{ or }64$

4 × (3) is $4xy\quad=\ 8,\text{ or }60$

Subtract, $x^2-2xy+y^2=17,\text{ or }\ 4$

$$x-y=\pm\sqrt{17},\text{ or }2$$

$$\therefore\ x=\frac{-5\pm\sqrt{17}}{2},\text{ or }5,\text{ or }3$$

$$y=\frac{-5\mp\sqrt{17}}{2},\text{ or }3,\text{ or }5$$

58.

$$\begin{cases} \dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}=\dfrac{10}{3} & (1)\\ x^2+y^2=45 & (2)\end{cases}$$

Simplify (1),

$$\begin{aligned} x^2+2xy+y^2+x^2-2xy+y^2 &= \tfrac{10}{3}(x^2-y^2)\\ 2(x^2+y^2) &= \tfrac{10}{3}(x^2-y^2)\\ 3(x^2+y^2) &= 5(x^2-y^2)\end{aligned}$$

From (2),

$$\begin{aligned} x^2+y^2 &= 45\\ \therefore\ 5(x^2-y^2) &= 135\\ x^2-y^2 &= 27\\ 2x^2 &= 72\\ x^2 &= 36\\ \therefore\ x &= \pm 6\\ 2y^2 &= 18\\ y^2 &= 9\\ \therefore\ y &= \pm 3\\ x &= \pm 6, \text{ or } \mp 6\\ y &= \pm 3, \text{ or } \mp 3\end{aligned}$$

59.

$$\begin{cases} x^4+x^2y^2+y^4=133 & (1)\\ x^2-xy+y^2=\ \ 19 & (2)\end{cases}$$

$$\begin{aligned} x^4+x^2y^2+y^4 &= x^4+2x^2y^2+y^4-x^2y^2\\ x^4+x^2y^2+y^4 &= (x^2+y^2-xy)(x^2+y^2+xy)\\ \therefore\ 19(x^2+y^2+xy) &= 133\end{aligned}$$

$$x^2+y^2+xy = 7 \qquad (3)$$

(2) is

$$\begin{aligned} x^2+y^2-xy &= 19\\ \hline 2xy &= -12\\ xy &= -6 \qquad (4)\end{aligned}$$

Add (3) and (4),

$$\begin{aligned} x^2+2xy+y^2 &= 1\\ x+y &= \pm 1\end{aligned}$$

Subtract (4) from (2),

$$\begin{aligned} x^2-2xy+y^2 &= 25\\ x-y &= \pm 5\\ \therefore\ x &= \pm 3, \text{ or } \pm 2\\ y &= \mp 2, \text{ or } \mp 3\end{aligned}$$

60.

$$\begin{cases} x^4+x^2y^2+y^4=931 & (1)\\ x^2+xy+y^2=\ \ 49 & (2)\end{cases}$$

From (1), $(x^2+xy+y^2)(x^2-xy+y^2)=931$

Substituting from (2), $49(x^2-xy+y^2)=931$

$x^2 - xy + y^2 = 19$ (3)

(2) is $x^2 + xy + y^2 = 49$

Subtract, $2xy = 30$

$xy = 15$ (4)

Add (2) and (4), $x^2 + 2xy + y^2 = 64$

$x + y = \pm 8$

Subtract (4) from (3), $x^2 - 2xy + y^2 = 4$

$x - y = \pm 2$

$\therefore x = \pm 5$, or ± 3

$y = \pm 3$, or ± 5

61. $\begin{cases} x^2 + xy + y^2 = 84 & (1) \\ x + \sqrt{xy} + y = 6 & (2) \end{cases}$

From (1), $(x + \sqrt{xy} + y)(x - \sqrt{xy} + y) = 84$

Substitute from (2), $6(x - \sqrt{xy} + y) = 84$

$x - \sqrt{xy} + y = 14$ (3)

(2) is $x + \sqrt{xy} + y = 6$

Subtract, $-2\sqrt{xy} = 8$

$\sqrt{xy} = -4$ (4)

$xy = 16$ (5)

Substitute value of $\sqrt{xy}$ in (2), $x + y = 10$

(1) is $x^2 + xy + y^2 = 84$

$3 \times (5)$ is $3xy = 48$

Subtract, $x^2 - 2xy + y^2 = 36$

$x - y = \pm 6$

$\therefore x = 8$, or 2

$y = 2$, or 8

62. $\begin{cases} x^2 + y^2 = 819 - xy & (1) \\ x + y = 21 + \sqrt{xy} & (2) \end{cases}$

Rearrange both equations, $x^2 + xy + y^2 = 819$ (3)

$x - \sqrt{xy} + y = 21$ (4)

(3) is $(x - \sqrt{xy} + y)(x + \sqrt{xy} + y) = 819$

Substitute from (4), $21(x + \sqrt{xy} + y) = 819$

$x + \sqrt{xy} + y = 39$

(4) is $x - \sqrt{xy} + y = 21$

Subtract, $2\sqrt{xy} = 18$

$\sqrt{xy} = 9$ (5)

$xy = 81$ (6)

Substitute the value of $\sqrt{xy}$ in (4),

$$x + y = 30$$

(3) is $x^2 + xy + y^2 = 819$

$3 \times (6)$ is $3xy = 243$

Subtract, $x^2 - 2xy + y^2 = 576$

$$x - y = \pm 24$$

$$\therefore x = 27, \text{ or } 3$$

$$y = 3, \text{ or } 27$$

63.

$$\begin{cases} x^4 + y^4 = 97 & (1) \\ x^2 + y^2 = 49 - x^2y^2 & (2) \end{cases}$$

From (2), $x^2 + x^2y^2 + y^2 = 49$

$$2x^2 + 2x^2y^2 + 2y^2 = 98$$

(1) is $x^4 + y^4 = 97$

Add, $x^4 + 2x^2y^2 + y^4 + 2x^2 + 2y^2 = 195$

$$(x^2 + y^2)^2 + 2(x^2 + y^2) + 1 = 196$$

$$x^2 + y^2 + 1 = \pm 14$$

$$x^2 + y^2 = 13, \text{ or } -15 \qquad (3)$$

Substitute value of $x^2 + y^2$ in (2), $x^2y^2 = 36, \text{ or } 64$

$$xy = \pm 6, \text{ or } \pm 8 \qquad (4)$$

(3) is $x^2 + y^2 = 13, \text{ or } -15$

$2 \times (4)$ is $2xy = \pm 12, \text{ or } \pm 16$

$$x^2 + 2xy + y^2 = 25, 1, 1, \text{ or } -31$$

$$x + y = \pm 5, \pm 1, \pm 1, \text{ or } \pm\sqrt{-31}$$

Similarly, $x - y = \pm 1, \pm 5, \pm\sqrt{-31}, \text{ or } \pm 1$

$$\therefore x = \pm 3, \pm 3, \pm 2, \pm 2, \frac{1 \pm \sqrt{-31}}{2}, \frac{1 \pm \sqrt{-31}}{2}, \frac{-1 \pm \sqrt{-31}}{2}, \frac{-1 \pm \sqrt{-31}}{2}$$

$$y = \pm 2, \mp 2, \pm 3, \mp 3, \frac{1 \mp \sqrt{-31}}{2}, \frac{-1 \pm \sqrt{-31}}{2}, \frac{-1 \mp \sqrt{-31}}{2}, \frac{1 \pm \sqrt{-31}}{2}$$

In all 16 pairs of solutions.

64.

$$\begin{cases} 2x^2 + 3xy + 12 = 3y^2 & (1) \\ 3x + 5y + 1 = 0 & (2) \end{cases}$$

From (2), $x = -\dfrac{1 + 5y}{3}$

Substitute value of x in (1),

$$2\left(-\frac{1+5y}{3}\right)^2-(1+5y)y+12=3y^2$$

$$\tfrac{2}{9}+\tfrac{20}{9}y+\tfrac{50}{9}y^2-y-5y^2+12=3y^2$$

$$\tfrac{22}{9}y^2-\tfrac{11}{9}y=12\tfrac{2}{9}$$

$$22y^2-11y=110$$

$$2y^2-y-10=0$$

$$(y+2)(2y-5)=0$$

$$\therefore\ y=-2,\text{ or }\tfrac{5}{2}$$

Substitute value of y in (2), $x=3,\text{ or }-\tfrac{9}{2}$

65.

$$\begin{cases}\dfrac{x}{a}+\dfrac{y}{b}=1 & (1)\\[2ex] \dfrac{a}{x}+\dfrac{b}{y}=4 & (2)\end{cases}$$

From (1), $bx+ay=ab$ (3)

From (2), $bx+ay=4xy$ (4)

$$\therefore\ 4xy=ab$$

Square (3), $b^2x^2+2abxy+a^2y^2=a^2b^2$

$ab\times(4)$ is $4abxy\qquad=a^2b^2$

Subtract, $b^2x^2-2abxy+a^2y^2=0$

$$bx-ay=0$$

$$bx=ay$$

Substitute value of bx in (3), $2ay=ab$

$$y=\frac{b}{2}$$

$$\therefore\ x=\frac{a}{2}$$

66.

$$\begin{cases}x+y=a & (1)\\ 4xy=a^2-4b^2 & (2)\end{cases}$$

Square (1), $x^2+2xy+y^2=a^2$

(2) is $4xy\qquad=a^2-4b^2$

Subtract, $x^2-2xy+y^2=\qquad 4b^2$

$$x-y=\pm 2b$$

$$\therefore\ x=\frac{a\pm 2b}{2}$$

$$y=\frac{a\mp 2b}{2}$$

67.

$$\begin{cases} x^2 = ax + by & (1) \\ y^2 = bx + ay & (2) \end{cases}$$

Subtract, $$x^2 - y^2 = (a-b)(x-y)$$

$$(x+y)(x-y) = (a-b)(x-y)$$

$$\therefore\ x - y = 0,$$

or $$x + y = a - b$$

$$(1)\ x - y = 0$$

$$x = y$$

Substitute value of x in (1), $$y^2 = (a+b)y$$

$$\therefore\ y = 0, \text{ or } (a+b)$$

$$x = 0, \text{ or } (a+b)$$

$$(2)\ x + y = a - b$$

$$x = a - b - y$$

Substitute value of x in (1),

$$(a-b-y)^2 = a^2 - ab + (b-a)y$$

$$a^2 - 2ab + b^2 - 2(a-b)y + y^2 = a^2 - ab + (b-a)y$$

$$y^2 - (a-b)y = ab - b^2$$

$$y^2 - (a-b)y + \left(\frac{a-b}{2}\right)^2 = \frac{a^2}{4} + \frac{3ab}{2} - \frac{3b^2}{4}$$

$$y - \frac{a-b}{2} = \tfrac{1}{2}\sqrt{a^2 + 2ab - 3b^2}$$

$$y - \frac{a-b}{2} = \tfrac{1}{2}\sqrt{(a-b)(a+3b)}$$

$$\therefore\ y = \frac{a-b}{2} \pm \tfrac{1}{2}\sqrt{(a-b)(a+3b)}$$

$$x = \frac{a-b}{2} \mp \tfrac{1}{2}\sqrt{(a-b)(a+3b)}$$

68.

$$\begin{cases} x^2 - xy = a^2 + b^2 & (1) \\ xy - y^2 = 2ab & (2) \end{cases}$$

Subtract, $$x^2 - 2xy + y^2 = a^2 - 2ab + b^2$$

$$x - y = \pm(a-b) \tag{3}$$

(1) is $$x(x-y) = a^2 + b^2$$

Substitute value of $x - y$ from (3), $$x = \pm\frac{a^2+b^2}{a-b}$$

$$y = \pm\frac{2ab}{a-b}$$

69.

$$\begin{cases} x^2+y^2+x+y=18 & (1)\\ xy=6 & (2)\end{cases}$$

$$2xy=12$$

Add to (1),
$$x^2+2xy+y^2+x+y=30$$
$$(x+y)^2+(x+y)+\tfrac{1}{4}=30\tfrac{1}{4}$$
$$x+y+\tfrac{1}{2}=\pm 5\tfrac{1}{2}$$
$$x+y=5,\text{ or }-6 \qquad (3)$$

Square (3),
$$x^2+2xy+y^2=25,\text{ or }36$$
$4\times(2)$ is
$$4xy=24$$
Subtract,
$$x^2-2xy+y^2=1,\text{ or }12$$
$$x-y=\pm 1,\text{ or }\pm\sqrt{12}$$
$$\therefore\ x=3,\ 2,\ -3\pm\sqrt{3}$$
$$y=2,\ 3,\ -3\mp\sqrt{3}$$

70.

$$\begin{cases} x^4+y^4=10(x^2+y^2)+72 & (1)\\ 2(x^2+y^2)=5xy & (2)\end{cases}$$

Square (2),
$$4(x^4+2x^2y^2+y^4)=25x^2y^2$$
$$x^4+y^4=\tfrac{17}{4}x^2y^2$$
Substitute value of x^4+y^4 and x^2+y^2 in (1),
$$\tfrac{17}{4}x^2y^2=25xy+72$$
$$17x^2y^2-100xy-288=0$$
$$(xy-8)(17xy-36)=0$$
$$xy=8,\text{ or }\tfrac{36}{17} \qquad (3)$$
Substitute value of xy in (2),
$$2(x^2+y^2)=40,\text{ or }\tfrac{180}{17}$$
$$x^2+y^2=20,\text{ or }\tfrac{90}{17}$$
$2\times(3)$ is
$$2xy=16,\text{ or }\tfrac{72}{17}$$
Subtract,
$$x^2-2xy+y^2=4,\text{ or }\tfrac{18}{17}$$
$$x-y=\pm 2,\text{ or }\pm 3\sqrt{\tfrac{2}{17}}$$
Similarly,
$$x+y=\pm 6,\text{ or }\pm 9\sqrt{\tfrac{2}{17}}$$
$$\therefore\ x=\pm 4,\ \pm 2,\ \pm 6\sqrt{\tfrac{2}{17}},\ \pm 3\sqrt{\tfrac{2}{17}}$$
$$y=\pm 2,\ \pm 4,\ \pm 3\sqrt{\tfrac{2}{17}},\ \pm 6\sqrt{\tfrac{2}{17}}$$

71.

$$\begin{cases} x^2+y^2=2x^2y^2-15 & (1)\\ x+y=xy+1 & (2)\end{cases}$$

Square (2),
$$x^2+2xy+y^2=x^2y^2+2xy+1$$
(1) is
$$x^2+y^2=2x^2y^2-15$$
Subtract,
$$2xy=-x^2y^2+2xy+16$$

$$x^2y^2 = 16$$

$$\therefore\ xy = \pm 4 \tag{3}$$

Substitute values of xy in (1) and (2),

$$x^2 + y^2 = 17 \tag{4}$$

$$x + y = 5, \text{ or } -3 \tag{5}$$

$$\begin{array}{ll} \text{(4) is} & x^2 \qquad + y^2 = \quad 17 \\ 2 \times (3) \text{ is} & \qquad 2xy \qquad = \pm\ \ 8 \\ \hline \text{Subtract,} & x^2 - 2xy + y^2 = \quad 9, \text{ or } 25 \end{array}$$

$$x - y = \pm 3, \text{ or } \pm 5$$

$$\therefore\ x = 4,\ 1,\ 1,\ -4$$

$$y = 1,\ 4,\ -4,\ 1$$

or

$$x = \pm 4,\ 1$$

$$y = 1,\ \pm 4$$

72.

$$\begin{cases} ay^2 + bxy = b & (1) \\ bx^2 + axy = a & (2) \end{cases}$$

$$\begin{array}{ll} a \times (1) \text{ is} & a^2y^2 + \ abxy \qquad = \ ab \\ b \times (2) \text{ is} & \qquad\quad abxy + b^2x^2 = \ ab \\ \hline \text{Add,} & a^2y^2 + 2\,abxy + b^2x^2 = 2\,ab \end{array}$$

$$ay + bx = \pm \sqrt{2\,ab}$$

(1) is

$$(ay + bx)y = b$$

$$\therefore\ y = \frac{\pm b}{\sqrt{2\,ab}} = \pm\sqrt{\frac{b}{2\,a}}$$

(2) is

$$x(bx + ay) = a$$

$$\therefore\ x = \frac{\pm a}{\sqrt{2\,ab}} = \pm\sqrt{\frac{a}{2\,b}}$$

73.

$$\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{x + y} & (1) \\ \dfrac{1}{x^2} + \dfrac{1}{y^2} = \dfrac{1}{a^2} & (2) \end{cases}$$

Clear of fractions,

$$(x + y)^2 = xy \tag{3}$$

$$x^2 + y^2 = \frac{x^2y^2}{a^2} \tag{4}$$

$$\begin{array}{ll} \text{(3) is} & x^2 + y^2 = \qquad -xy \\ \hline \text{Subtract,} & \qquad 0 = \dfrac{x^2y^2}{a^2} + xy \end{array}$$

$$xy = 0, \text{ or } -a^2 \tag{5}$$

Substitute value of xy in (3), $(x+y)^2 = 0, \text{ or } -a^2$ (6)

$$x^2 + 2xy + y^2 = 0, \text{ or } -a^2$$

4 × (5) is $\quad 4xy = 0, \text{ or } -4a^2$

$$\overline{x^2 - 2xy + y^2 = 0, \text{ or } \quad 3a^2}$$

$$x - y = 0, \text{ or } \pm a\sqrt{3}$$

From (6), $\quad x + y = 0, \text{ or } \pm a\sqrt{-1}$

$$\therefore\ x = 0, \frac{a}{2}(\sqrt{-1} \pm \sqrt{3}), -\frac{a}{2}(\sqrt{-1} \pm \sqrt{3})$$

$$y = 0, \frac{a}{2}(\sqrt{-1} \mp \sqrt{3}), -\frac{a}{2}(\sqrt{-1} \mp \sqrt{-3})$$

74.
$$\begin{cases} \dfrac{(x+y)^2}{a^2} + \dfrac{(x-y)^2}{b^2} = 8 & (1) \\ x^2 + y^2 = 2(a^2 + b^2) & (2) \end{cases}$$

$$\frac{x^2 + 2xy + y^2}{a^2} + \frac{x^2 - 2xy + y^2}{b^2} = 8$$

Substitute value of $x^2 + y^2$ from (2),

$$\frac{2(a^2 + b^2 + xy)}{a^2} + \frac{2(a^2 + b^2 - xy)}{b^2} = 8$$

$$a^2b^2 + b^4 + b^2xy + a^4 + a^2b^2 - a^2xy = 4a^2b^2$$

$$(b^2 - a^2)xy = -a^4 + 2a^2b^2 - b^4$$

$$xy = a^2 - b^2 \qquad (3)$$

(2) is $\quad x^2 \quad + y^2 = 2(a^2 + b^2)$

2 × (3) is $\quad 2xy \quad = 2(a^2 - b^2)$

Add, $\quad x^2 + 2xy + y^2 = 4a^2$

$$x + y = \pm 2a$$

Similarly, $\quad x - y = \pm 2b$

$$\therefore\ x = a \pm b, -a \pm b$$

$$y = a \mp b, -a \mp b$$

75.
$$\begin{cases} x^2 + y^2 - 8ab = 5(a^2 + b^2) & (1) \\ xy - 5ab = 2(a^2 + b^2) & (2) \end{cases}$$

$$x^2 \quad + y^2 = 5a^2 + 8ab + 5b^2$$

From (2), $\quad 2xy \quad = 4a^2 + 10ab + 4b^2$

Add, $\quad x^2 + 2xy + y^2 = 9a^2 + 18ab + 9b^2$

$$x + y = \pm 3(a + b)$$

Similarly, $\quad x - y = \pm (a - b)$

$$\therefore\ x = \pm (2a + b), \pm (a + 2b)$$

$$y = \pm (a + 2b), \pm (2a + b)$$

76.

$$\begin{cases} x^2+y^2=axy & (1)\\ x+y=bxy & (2)\end{cases}$$

Square (2), $$x^2+2xy+y^2=b^2x^2y^2$$

(1) is $$x^2 \qquad +y^2=axy$$

Subtract, $$2xy \qquad =b^2x^2y^2-axy$$

$$b^2x^2y^2-(a+2)xy=0$$

$$\therefore\ xy=0,\ \text{or}\ \frac{a+2}{b^2} \qquad (3)$$

Substitute values of xy in (1) and (2),

$$x^2+y^2=0,\ \text{or}\ \frac{a(a+2)}{b^2} \qquad (4)$$

$$x+y=0,\ \text{or}\ \frac{(a+2)}{b} \qquad (5)$$

(4) is $$x^2 \qquad +y^2=0,\ \text{or}\ \frac{a(a+2)}{b^2}$$

$2\times(3)$ is $$2xy \qquad =0,\ \text{or}\ \frac{2(a+2)}{b^2}$$

Subtract, $$x^2-2xy+y^2=0,\ \text{or}\ \frac{a^2-4}{b^2}$$

$$x-y=0,\ \text{or}\ \pm\frac{\sqrt{a^2-4}}{b}$$

$$\therefore\ x=0,\ \frac{a+2\pm\sqrt{a^2-4}}{2b}$$

$$y=0,\ \frac{a+2\mp\sqrt{a^2-4}}{2b}$$

77.

$$\begin{cases} 2(x^2+y^2)=5xy-9ab & (1)\\ 2(a+b)(x+y)=3(xy-ab) & (2)\end{cases}$$

Subtract $3\times(2)$ from (1),

$$2(x^2+y^2)-6(a+b)(x+y)=-4xy$$

$$x^2+2xy+y^2-3(a+b)(x+y)=0$$

$$(x+y)[x+y-3(a+b)]=0$$

$$\left.\begin{aligned} \therefore\ x+y&=0,\\ \text{or}\ x+y&=3(a+b)\end{aligned}\right\} \qquad (3)$$

Substitute values of $x+y$ in (2),

$$3(xy-ab)=0,\ \text{or}\ 6(a+b)^2$$

$$xy-ab=0,\ \text{or}\ 2(a+b)^2$$

$$xy=ab,\ \text{or}\ 2a^2+5ab+2b^2 \quad (4)$$

$$\begin{array}{ll}
\text{Square (3),} & x^2 + 2xy + y^2 = 0, \text{ or } 9(a+b)^2 \\
4 \times (4) \text{ is} & \underline{\quad 4xy \qquad = \quad 4ab, \text{ or } 8a^2 + 20ab + 8b^2} \\
\text{Subtract,} & x^2 - 2xy + y^2 = -4ab, \text{ or } \quad a^2 - \ 2ab + \ b^2
\end{array}$$

$$x - y = \pm 2\sqrt{-ab}, \text{ or } \pm (a-b)$$

$$\therefore\ x = \pm\sqrt{-ab},\ 2a+b,\ a+2b$$

$$y = \mp\sqrt{-ab},\ a+2b,\ 2a+b$$

78.

$$\begin{cases} x^2 + y^2 + z^2 = 49 & (1) \\ x + y + z = 11 & (2) \\ 2x + 3y - 4z = 6 & (3) \end{cases}$$

$$\begin{array}{ll}
2 \times (2) \text{ is} & 2x + 2y + 2z = 22 \\
(3) \text{ is} & \underline{2x + 3y - 4z = \ 6} \\
\text{Subtract,} & -y + 6z = 16
\end{array}$$

$$y = 6z - 16 \qquad (4)$$

Substitute in (2), $x + 6z - 16 + z = 11$

$$x = 27 - 7z \qquad (5)$$

Substitute value of x and y in (1),

$$(27 - 7z)^2 + (6z - 16)^2 + z^2 = 49$$

$$729 - 378z + 49z^2 + 36z^2 - 192z + 256 + z^2 = 49$$

$$86z^2 - 570z + 936 = 0$$

$$43z^2 - 285z + 468 = 0$$

$$(43z - 156)(z - 3) = 0$$

$$\therefore\ z = 3, \text{ or } \tfrac{156}{43}$$

Substitute values of z in (4) and (5), $y = 2, \text{ or } \tfrac{248}{43}$

$$x = 6, \text{ or } \tfrac{69}{43}$$

79.

$$\begin{cases} xy + yz + xz = 40 & (1) \\ 4x = 3y = 2z + 4 & (2) \end{cases}$$

$$x = \frac{z+2}{2}$$

$$y = \frac{2z+4}{3}$$

Substitute values of x and y in (1),

$$\frac{2z^2 + 8z + 8}{6} + \frac{2z^2 + 4z}{3} + \frac{z^2 + 2z}{2} = 40$$

$$2z^2 + 8z + 8 + 4z^2 + 8z + 3z^2 + 6z = 240$$

$$9z^2 + 22z - 232 = 0$$

$$(z-4)(9z+58)=0$$

$$\therefore\ z=4, \text{ or } -\tfrac{58}{9}$$

$$y=4, \text{ or } -\tfrac{80}{27}$$

$$x=3, \text{ or } -\tfrac{20}{9}$$

80.

$$\begin{cases} x^2+y^2+z^2=84 & (1)\\ x+y+z=14 & (2)\\ y^2=xz & (3)\end{cases}$$

$$y=\pm\sqrt{xz}$$

Substitute values of y^2 and y in (1) and (2),

$$x^2+xz+z^2=84 \qquad (4)$$

$$x+\sqrt{xz}+z=14 \qquad (5)$$

(4) is $\qquad (x+\sqrt{xz}+z)(x-\sqrt{xz}+z)=84$

Substitute from (5), $\qquad 14(x\mp\sqrt{xz}+z)=84$

$$x\mp\sqrt{xz}+z=\ 6 \qquad (6)$$

(5) is $\qquad x+\sqrt{xz}+z=14$

Subtract, $\qquad 2\sqrt{xz}=\ 8$

$$\sqrt{xz}=4$$

$$xz=16 \qquad (7)$$

Substitute values of xz in (4),

$$x^2\qquad +z^2=68$$

$2\times(7)$ is $\qquad 2xz\qquad =32$

Subtract, $\qquad x^2-2xz+z^2=36$

$$x-z=\pm 6$$

Substitute value of $\sqrt{xz}$ in (5), $\qquad x+z=10$

$$\therefore\ x=8, \text{ or } 2$$

$$z=2, \text{ or } 8$$

and from (2), $\qquad y=4$

$$\therefore\ x=2,\ 8$$

$$y=4,\ 4$$

$$z=8,\ 2$$

81.

$$\begin{cases} 2xy+x+y=22 & (1)\\ 2yz+y+z=58 & (2)\\ 2xz+x+z=32 & (3)\end{cases}$$

Multiply (1) by 2, $\qquad 2xyz+xz+yz=22z$

Multiply (2) by x, $\qquad 2xyz+xy+xz=\qquad 58x$

Subtract, $\qquad yz-xy=22z-58x \qquad (4)$

Subtract (1) from (2),

$$2yz - 2xy + z - x = 36$$

$2 \times (4)$ is

$$\underline{2yz - 2xy \qquad = \qquad 44z - 116x}$$

Subtract,

$$\begin{aligned} z - x &= 36 - 44z + 116x \\ 45z - 117x &= 36 \\ 5z - 13x &= 4 \\ z &= \frac{4 + 13x}{5} \qquad (5) \end{aligned}$$

Substitute value of z in (3),

$$\begin{aligned} \frac{8x + 26x^2}{5} + x + \frac{4 + 13x}{5} &= 32 \\ 8x + 26x^2 + 5x + 4 + 13x &= 160 \\ 26x^2 + 26x &= 156 \\ x^2 + x - 6 &= 0 \\ (x + 3)(x - 2) &= 0 \\ \therefore\ x &= 2, \text{ or } -3 \end{aligned}$$

Substitute value of x in (5), $z = 6, \text{ or } -7$

Substitute value of x in (1), $4y + 2 + y = 22$

$$\begin{aligned} 5y &= 20 \\ y &= 4 \end{aligned}$$

or

$$\begin{aligned} -6y - 3 + y &= 22 \\ -5y &= 25 \\ y &= -5 \\ \therefore\ x &= 2, \text{ or } -3 \\ y &= 4, \text{ or } -5 \\ z &= 6, \text{ or } -7 \end{aligned}$$

82.

$$\begin{cases} x^2 + xy + xz = a^2 & (1) \\ y^2 + yz + xy = 2ab & (2) \\ z^2 + xz + yz = b^2 & (3) \end{cases}$$

Add,

$$\begin{aligned} x^2 + 2xy + y^2 + 2xz + 2yz + z^2 &= a^2 + 2ab + b^2 \\ (x + y + z)^2 &= (a + b)^2 \\ x + y + z &= \pm (a + b) \end{aligned}$$

(1) is

$$\begin{aligned} x(x + y + z) &= a^2 \\ \pm x(a + b) &= a^2 \\ \therefore\ x &= \pm \frac{a^2}{a + b} \end{aligned}$$

(2) is $y(x+y+z)=2ab$

$\pm y(a+b)=2ab$

$\therefore\ y=\pm\dfrac{2ab}{a+b}$

(3) is $z(x+y+z)=b^2$

$\pm z(a+b)=b^2$

$\therefore\ z=\pm\dfrac{b^2}{a+b}$

Exercise 25.

1. If the length and breadth of a rectangle were each increased 1 foot, the area would be 48 square feet; if the length and breadth were each diminished 1 foot, the area would be 24 square feet. Find the length and breadth of the rectangle.

Let x = number of feet in length,

and y = number of feet in breadth.

Then $(x+1)(y+1)$ = number of square feet area in the increased rectangle.

$(x-1)(y-1)$ = number of square feet area in the diminished rectangle.

$$\therefore\ (x+1)(y+1)=48 \qquad (1)$$

$$(x-1)(y-1)=24 \qquad (2)$$

Simplify (1), $xy+x+y+1=48$ (3)

Simplify (2), $xy-x-y+1=24$ (4)

Add, $2xy+2=72$

$2xy=70$

$\therefore\ xy=35$ (5)

Subtract (4) from (3), $2(x+y)=24$

$\therefore\ x+y=12$ (6)

Square, $x^2+2xy+y^2=144$

$4\times(5)$ is, $4xy=140$

Subtract, $x^2-2xy+y^2=4$

Extract root, $x-y=\pm 2$ (7)

From (6) and (7), $2x=14$, or 10

$\therefore\ x=7$, or 5

$y=5$, or 7

$\therefore$ The rectangle is 7 feet long and 5 feet wide.

2. A farmer laid out a rectangular lot containing 1200 square yards. He afterwards increased the width $1\frac{1}{2}$ yards and diminished the length 3 yards, thereby increasing the area by 60 square yards. Find the dimensions of the original lot.

Let $x =$ number of yards in length,
and $y =$ number of yards in width.

Then, $$xy = 1200 \qquad (1)$$

$$(x-3)(y+\tfrac{3}{2}) = 1260 \qquad (2)$$

Simplify (2), $$xy + \tfrac{3}{2}x - 3y - \tfrac{9}{2} = 1260$$

Substitute value of xy from (1),

$$1200 + \tfrac{3}{2}x - 3y - \tfrac{9}{2} = 1260$$

$$\tfrac{3}{2}x - 3y = 64\tfrac{1}{2}$$

$$x - 2y = 43 \qquad (3)$$

$$\begin{array}{ll} \text{Square,} & x^2 - 4xy + 4y^2 = 1849 \\ 8 \times (1) \text{ is} & \underline{\quad 8xy \qquad\quad = 9600} \\ \text{Add,} & x^2 + 4xy + 4y^2 = 11449 \end{array}$$

Extract root, $$x + 2y = \pm 107 \qquad (4)$$

From (3) and (4), $$2x = 150, \text{ or } -64$$

$$x = 75$$

$$y = 16$$

∴ The original lot was 75 yards long and 16 yards wide.

3. The diagonal of a rectangle is 89 inches; if each side were 3 inches less, the diagonal would be 85 inches. Find the area of the rectangle.

Let $x =$ number of inches in one side,
and $y =$ number of inches in the other side.
Then $\sqrt{x^2+y^2} =$ number of inches in the diagonal,
and $\sqrt{(x-3)^2+(y-3)^2} =$ number of inches in the diagonal if each side were 3 inches shorter.

$$\therefore \sqrt{x^2+y^2} = 89 \qquad (1)$$

$$\sqrt{(x-3)^2+(y-3)^2} = 85 \qquad (2)$$

$$\begin{array}{lll} \text{Simplify (1),} & x^2 \qquad + y^2 \qquad = 7921 & (3) \\ \text{Simplify (2),} & \underline{x^2 - 6x + y^2 - 6y = 7207} & (4) \\ \text{Subtract,} & \quad 6x \qquad + 6y = \ 714 & \end{array}$$

$$x + y = 119 \qquad (5)$$

Square, $x^2 + 2xy + y^2 = 14161$

(3) is $x^2 \qquad + y^2 = 7921$

Subtract, $2xy \qquad = 6240$ (6)

(3) is $x^2 \qquad + y^2 = 7921$

(5) is $2xy \qquad = 6240$

Subtract, $x^2 - 2xy + y^2 = 1681$

Extract root, $x - y = \pm 41$ (7)

From (5) and (7), $2x = 160, \text{ or } 78$

$$\therefore\ x = 80, \text{ or } 39$$

$$y = 39, \text{ or } 80$$

$$xy = 3120$$

∴ The area of the rectangle is 3120 square inches.

4. The diagonal of a rectangle is 65 inches; if the rectangle were 3 inches shorter and 9 inches wider, the diagonal would still be 65 inches. Find the area of the rectangle.

Let $x =$ number of inches in length,

and $y =$ number of inches in width.

Then $\sqrt{x^2 + y^2} =$ number of inches in the diagonal,

and $\sqrt{(x-3)^2 + (y+9)^2} =$ number of inches in the diagonal if the rectangle were 3 inches shorter and 9 inches wider.

$$\therefore\ \sqrt{x^2 + y^2} = 65 \qquad (1)$$

$$\sqrt{(x-3)^2 + (y+9)^2} = 65 \qquad (2)$$

Simplify (1), $x^2 \qquad + y^2 \qquad = 4225$ (3)

Simplify (2), $x^2 - 6x + y^2 + 18y = 4135$ (4)

Subtract, $6x \qquad - 18y = 90$

$$x - 3y = 15 \qquad (5)$$

$$x = 15 + 3y$$

Substitute value of x in (3),

$$225 + 90y + 9y^2 + y^2 = 4225$$

$$10y^2 + 90y = 4000$$

$$y^2 + 9y = 400$$

$$y^2 + 9y + \tfrac{81}{4} = \tfrac{1681}{4}$$
$$y + \tfrac{9}{2} = \pm \tfrac{41}{2}$$
$$\therefore\ y = 16, \text{ or } -25$$
$$x = 63, \text{ or } -60$$
$$xy = 1008$$

∴ The area of the rectangle is 1008 square inches.

5. The difference of two numbers is $\frac{3}{8}$ of the greater, and the sum of their squares is 356. Find the numbers.

Let $x =$ the greater number,
and $y =$ the less.

$$\therefore\ x - y = \frac{3x}{8} \tag{1}$$
$$x^2 + y^2 = 356 \tag{2}$$

Simplify (1), $5x = 8y$
$$x = \tfrac{8}{5}y$$

Substitute value of x in (2),
$$\tfrac{64}{25}y^2 + y^2 = 356$$
$$\tfrac{89}{25}y^2 = 356$$
$$y^2 = 100$$
$$y = \pm 10$$
$$\therefore\ x = \pm 16$$

∴ The numbers are 16 and 10.

6. The sum, the product, and the difference of the squares of two numbers are all equal. Find the numbers.

Let $x + y =$ the one number,
and $x - y =$ the other.

$$\therefore\ x + y + x - y = (x + y)(x - y) = (x + y)^2 - (x - y)^2 \tag{1}$$

Simplify, $2x = x^2 - y^2 = 4xy$ (2)
$$1 = 2y$$
$$\therefore\ y = \tfrac{1}{2} \tag{3}$$

Substitute value of y in (2), $2x = x^2 - \frac{1}{4}$
$$x^2 - 2x = \tfrac{1}{4}$$
$$x^2 - 2x + 1 = \tfrac{5}{4}$$
$$x - 1 = \pm \tfrac{1}{2}\sqrt{5}$$
$$\therefore\ x = 1 \pm \tfrac{1}{2}\sqrt{5} \tag{4}$$

From (3) and (4), $x + y = \frac{3}{2} \pm \frac{1}{2}\sqrt{5}$
$$x - y = \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{5}$$

∴ The numbers are $\frac{3}{2} \pm \frac{1}{2}\sqrt{5}$ and $\frac{1}{2} \pm \frac{1}{2}\sqrt{5}$.

7. The sum of two numbers is 5, and the sum of their cubes is 335. Find the numbers.

Let $x =$ the one number,
and $y =$ the other.

Then $x + y = 5$ (1)

$x^3 + y^3 = 335$ (2)

Divide (2) by (1), $x^2 - xy + y^2 = 67$ (3)

Square (1), $x^2 + 2xy + y^2 = 25$

Subtract, $-3xy = 42$

$xy = -14$ (4)

(3) is $x^2 - xy + y^2 = 67$

(4) is $xy = -14$

Subtract, $x^2 - 2xy + y^2 = 81$

$x - y = \pm 9$ (5)

From (5) and (1), $2x = 14, \text{ or } -4$

$x = 7, \text{ or } -2$

$y = -2, \text{ or } 7$

$\therefore$ The numbers are 7 and -2.

8. The sum of two numbers is 11, and the cube of their sum exceeds the sum of their cubes by 792. Find the numbers.

Let $x =$ the one number,
and $y =$ the other.

Then $x + y = 11$ (1)

$(x + y)^3 - x^3 - y^3 = 792$ (2)

Simplify (2), $3x^2y + 3xy^2 = 792$

$x^2y + xy^2 = 264$

$xy(x + y) = 264$ (3)

Substitute value of $x + y$ from (1) in (3),

$11xy = 264$

$xy = 24$ (4)

Square (1), $x^2 + 2xy + y^2 = 121$

$4 \times (4)$ is $4xy = 96$

Subtract, $x^2 - 2xy + y^2 = 25$

$x - y = \pm 5$ (5)

From (1) and (5), $2x = 16, \text{ or } 6$

$\therefore x = 8, \text{ or } 3$

$y = 3, \text{ or } 8$

$\therefore$ The numbers are 3 and 8.

9. A number is formed by two digits. The second digit is less by 8 than the square of the first digit; if 9 times the first digit be added to the number, the order of the digits will be reversed. Find the number.

Let $x =$ the first digit,
and $y =$ the second.
Then $10x + y =$ the number.
$10y + x =$ the number after the order of the digits is reversed.

$$y = x^2 - 8 \tag{1}$$

$$10x + y + 9x = 10y + x \tag{2}$$

Simplify (2), $-9y + 18x = 0$

$$-y + 2x = 0$$

$$y = 2x \tag{3}$$

Substitute value of y in (1), $2x = x^2 - 8$

$$x^2 - 2x = 8$$

$$x^2 - 2x + 1 = 9$$

$$x - 1 = \pm 3$$

$$x = 4, \text{ or } -2$$

From (3), $y = 8, \text{ or } -4$

∴ The number is 48.

10. A number is formed by three digits, the third digit being the sum of the other two; the product of the first and third digits exceeds the square of the second by 5. If 396 be added to the number, the order of the digits will be reversed. Find the number.

Let $x =$ the first digit,
$y =$ the second digit,
and $z =$ the third digit.
Then $100x + 10y + z =$ the number.
$100z + 10y + x =$ the number if the order of the digits is reversed.

$$z = x + y \tag{1}$$

$$xz - y^2 = 5 \tag{2}$$

$$100x + 10y + z + 396 = 100z + 10y + x \tag{3}$$

Simplify (3), $99x - 99z = -396$

$$x - z = -4 \tag{4}$$

$$x = z - 4$$

Substitute value of x in (1), $z = z - 4 + y \quad (5)$

$$\therefore y = 4$$

Substitute values of x and y in (2),

$$z(z-4)-16=5$$
$$z(z-4)=21$$
$$z^2-4z+4=25$$
$$z-2=\pm 5$$
$$z=7, \text{ or } -3$$

From (4), $x=3$

∴ The number is 347.

11. The numerator and denominator of a certain fraction are each greater by 1 than those of a second fraction; the sum of the two fractions is $\frac{17}{12}$. If the numerators were interchanged, the sum of the fractions would be $\frac{3}{2}$. Find the fractions.

Let $x=$ the numerator,
and $y=$ the denominator.

Then $\frac{x}{y}=$ the fractions,

and $\frac{x-1}{y-1}=$ the second fraction.

$$\therefore \frac{x}{y}+\frac{x-1}{y-1}=\frac{17}{12} \qquad (1)$$

$$\frac{x-1}{y}+\frac{x}{y-1}=\frac{3}{2} \qquad (2)$$

Subtract (2) from (1), $\frac{1}{y}-\frac{1}{y-1}=-\frac{1}{12}$ (3)

Simplify, $12(y-1-y)=-y^2+y$

$$y^2-y=12$$
$$y^2-y+\tfrac{1}{4}=12\tfrac{1}{4}$$
$$y-\tfrac{1}{2}=\pm 3\tfrac{1}{2}$$
$$y=4, \text{ or } -3$$

Substitute value of y in (1),

$$\frac{x}{4}+\frac{x-1}{3}=\frac{17}{12}, \qquad (4)$$

or

$$\frac{x}{-3}+\frac{x-1}{-4}=\frac{3}{2} \qquad (5)$$

From (4), $7x-4=17$

$$x=3$$

From (5), $-7x+3=18$

$$7x=-15$$

$x=3$, $y=4$ are the only available solutions.

∴ The fractions are $\frac{3}{4}$ and $\frac{2}{3}$.

12. There are two fractions. The numerator of the first is the square of the denominator of the second, and the numerator of the second is the square of the denominator of the first; the sum of the fractions is $\frac{35}{6}$, and the sum of their denominators 5. Find the fractions.

Let $x =$ the denominator of the first fraction,
and $y =$ the denominator of the second fraction.
Then $\frac{y^2}{x} =$ the first fraction,
and $\frac{x^2}{y} =$ the second fraction.

$$\therefore \frac{y^2}{x} + \frac{x^2}{y} = \frac{35}{6} \qquad (1)$$

$$x + y = 5 \qquad (2)$$

Simplify (1), $x^3 + y^3 = \frac{35}{6}xy \qquad (3)$

Divide (3) by (2), $x^2 - xy + y^2 = \frac{7}{6}xy \qquad (4)$

Square (2), $x^2 + 2xy + y^2 = 25$

Subtract, $-3xy = \frac{7}{6}xy - 25$

$$-\frac{25}{6}xy = -25$$

$$xy = 6 \qquad (5)$$

Substitute value of xy in right side of (4),

$$x^2 - xy + y^2 = 7$$

(5) is $xy = 6$

Subtract, $x^2 - 2xy + y^2 = 1$

$$x - y = \pm 1 \qquad (6)$$

From (2) and (6), $2x = 6$, or 4

$$\therefore x = 3, \text{ or } 2$$

$$y = 2, \text{ or } 3$$

$$x^2 = 9, \text{ or } 4$$

$$y^2 = 4, \text{ or } 9$$

$\therefore$ The fractions are $\frac{4}{3}$ and $\frac{9}{2}$.

13. The sum of two numbers which are formed by the same two digits is $\frac{55}{18}$ of their difference; the difference of the squares of the numbers is 3960. Find the numbers.

Let $x =$ the one digit,
and $y =$ the other.
Then $10x + y =$ the one number,
and $10y + x =$ the other.

$$\therefore\ 10x+y+10y+x=\tfrac{55}{18}(10x+y-10y-x) \qquad (1)$$

$$(10x+y)^2-(10y+x)^2=3960 \qquad (2)$$

Simplify (1), $18(11x+11y)=55(9x-9y)$

$$2x+2y=5x-5y$$

$$-3x=-7y$$

$$x=\tfrac{7}{3}y \qquad (3)$$

Simplify (2),

$$100x^2+20xy+y^2-100y^2-20xy-x^2=3960$$

$$99x^2-99y^2=3960$$

$$x^2-y^2=40 \qquad (4)$$

Substitute value of x from (3) in (4),

$$\tfrac{49}{9}y^2-y^2=40$$

$$\tfrac{40}{9}y^2=40$$

$$y^2=9$$

$$\therefore\ y=\pm 3$$

From (3), $x=\pm 7$

∴ The numbers are 73 and 37.

14. The fore wheel of a carriage turns in a mile 132 times more than the hind wheel; if the circumference of each were increased 2 feet, the fore wheel would turn only 88 times more. Find the circumference of each wheel.

Let x = number of feet in circumference of fore wheel,
and y = number of feet in circumference of hind wheel.
$5280 = 1$ mile.

Then $\frac{5280}{x}$ = number of times the fore wheel will turn in 1 mile.

$\frac{5280}{y}$ = number of times the hind wheel will turn in 1 mile.

$\frac{5280}{x+2}$ = number of times the fore wheel would turn in 1 mile if its circumference were increased by 2 feet.

$\frac{5280}{y+2}$ = number of times the hind wheel would turn in 1 mile if its circumference were increased by 2 feet.

$$\therefore\ \frac{5280}{x}-\frac{5280}{y}=132 \qquad (1)$$

$$\frac{5280}{x+2}-\frac{5280}{y+2}=88 \qquad (2)$$

Simplify (1), $5280y-5280x=132xy$

$$40y-40x=xy \qquad (3)$$

Simplify (2),
$$5280y - 5280x = 88xy + 176x + 176y + 352$$
$$60y - 60x = xy + 2x + 2y + 4$$
$$58y - 62x = xy + 4 \qquad (4)$$

Subtract (4) from (3),
$$22x - 18y = -4$$
$$11x - 9y = -2$$
$$x = \frac{9y-2}{11} \qquad (5)$$

Substitute value of x in (3),
$$40y - \frac{40(9y-2)}{11} = \frac{(9y-2)y}{11}$$
$$440y - 360y + 80 = 9y^2 - 2y$$
$$9y^2 - 82y = 80$$
$$9y^2 - 82y + (\tfrac{41}{3})^2 = \tfrac{2401}{9}$$
$$3y - \tfrac{41}{3} = \pm\tfrac{49}{3}$$
$$3y = 30, \text{ or } -\tfrac{8}{3}$$
$$y = 10$$

From (5),
$$x = 8$$

∴ The fore wheel is 8 feet in circumference, the hind wheel is 10 feet in circumference.

15. Two travellers, A and B, set out from two distant towns, A to go from the first town to the second, and B from the second town to the first, and both travel at uniform rates. When they meet, A has travelled 30 miles farther than B. A finishes his journey 4 days, and B 9 days, after they meet. Find the distance between the towns, and the number of miles A and B each travel per day.

Let x = number of miles A travels per day,

and y = number of miles B travels per day.

Then $4x$ = number of miles A travels in 4 days.

$9y$ = number of miles B travels in 9 days.

$4x + 9y$ = number of miles between the two towns.

$\frac{4x}{y}$ = number of days B has travelled when they meet.

$\frac{9y}{x}$ = number of days A has travelled when they meet.

$$\therefore \frac{4x}{y} = \frac{9y}{x} \qquad (1)$$
$$4x = 9y - 30 \qquad (2)$$

Simplify (1),
$$4x^2 = 9y^2$$
$$2x = \pm 3y \qquad (3)$$

Substitute value of $2x$ in (2), $\pm 6y = 9y - 30$

$$-3y = -30,$$

or $$-15y = -30$$

$$\therefore\ y = 10, \text{ or } 2$$

Substitute value of y in (3), $x = 15, \text{ or } -3$

$$4x + 9y = 150, \text{ or } 46$$

∴ A travels 15 miles a day, B travels 10 miles a day, and the towns are 150 miles apart.

16. Two boys run in opposite directions around a rectangular field, of which the area is one acre; they start from one corner, and meet 13 yards from the opposite corner. One boy runs only $\frac{5}{6}$ as fast as the other. Find the length and breadth of the field.

Let $x =$ number of yards in length,

and $y =$ number of yards in breadth.

$x + y =$ half the perimeter.

4840 sq. yds. = 1 acre.

Then $$xy = 4840 \qquad (1)$$

$$x + y - 13 = \tfrac{5}{6}(x + y + 13) \qquad (2)$$

Simplify (2), $$x + y = 143 \qquad (3)$$

Square, $$x^2 + 2xy + y^2 = 20449$$

4 × (1) is $$4xy = 19360$$

Subtract, $$x^2 - 2xy + y^2 = 1089$$

$$x - y = \pm 33 \qquad (4)$$

From (3) and (4), $$2x = 176, \text{ or } 110$$

$$\therefore\ x = 88, \text{ or } 55$$

$$y = 55, \text{ or } 88$$

∴ The field is 88 yards long and 55 yards wide.

17. A man walks from the base of a mountain to the summit, reaching the summit in $5\frac{1}{2}$ hours; during the last half of the distance he walks $\frac{1}{2}$ mile less per hour than during the first half. He descends in $3\frac{3}{4}$ hours, walking 1 mile per hour faster than during the first half of the ascent. Find the distance from the base to the summit and the rates of walking.

Let $y =$ number of miles from the base to the summit,

and $x =$ number of miles the man walks per hour at first.

Then $\frac{y}{2x} =$ number of hours it takes him to walk half way up.

$\frac{y}{2(x-\frac{1}{2})}=$ number of hours it takes him to walk up the 2d half.

$\frac{y}{(x+1)}=$ number of hours it takes him to walk down.

$$\therefore \frac{y}{2x}+\frac{y}{2(x-\frac{1}{2})}=5\tfrac{1}{2} \qquad (1)$$

$$\frac{y}{x+1}=3\tfrac{3}{4} \qquad (2)$$

Simplify (1), $2xy-y+2xy=22x^2-11x$

$$4xy-22x^2=y-11x \qquad (3)$$

Simplify (2), $4y=15x+15 \qquad (4)$

$$y=\frac{15x+15}{4} \qquad (5)$$

Substitute value of y in (3),

$$x(15x+15)-22x^2=\frac{15x+15}{4}-11x$$

$$60x^2+60x-88x^2=15x+15-44x$$

$$-28x^2+89x=15$$

$$196x^2-623x=-105$$

$$196x^2-623x+(\tfrac{89}{4})^2=\tfrac{6241}{16}$$

$$14x-\tfrac{89}{4}=\pm\tfrac{79}{4}$$

$$14x=42, \text{ or } \tfrac{5}{2}$$

$$x=3, \text{ or } \tfrac{5}{28}$$

Substitute value of x in (5), $y=15, \text{ or } 4\tfrac{47}{112}$

$$x-\tfrac{1}{2}=2\tfrac{1}{2}, \text{ or } -\tfrac{9}{28}$$

The second set of solutions is therefore to be neglected.

∴ The distance from the base to the summit is 15 miles. The man walks 3 miles an hour for $2\frac{1}{2}$ hours, and $2\frac{1}{2}$ miles an hour for 3 hours; on the return he walks 4 miles an hour.

18. A besieged garrison had bread for 11 days. If there had been 400 more men, each man's daily share would have been 2 ounces less; if there had been 600 less men, each man's daily share could have been increased by 2 ounces, and the bread would then have lasted 12 days. How many pounds of bread did the garrison have, and what was each man's daily share?

Let $x=$ number of ounces of bread on hand,

and $y=$ number of men.

Then $\frac{x}{11y}=$ each man's daily share.

$$\therefore \frac{x}{11(y+400)} = \frac{x}{11y} - 2 \qquad (1)$$

$$\frac{x}{12(y+600)} = \frac{x}{11y} + 2 \qquad (2)$$

Simplify (1), $xy = xy + 400x - 22y^2 - 8800y$

$$22y^2 + 8800y - 400x = 0 \qquad (3)$$

Simplify (2), $11xy = 12xy - 7200x + 264y^2 - 158400y$

$$264y^2 + xy - 7200x - 158400y = 0 \qquad (4)$$

18 × (3) is $396y^2 \quad - 7200x + 158400y = 0$

Subtract, $-132y^2 + xy \quad - 316800y = 0$

$$y(132y - x + 316800) = 0$$

$$\therefore y = 0,$$

or

$$132y - x + 316800 = 0$$

$$x = 132y + 316800 \qquad (5)$$

Substitute $y = 0$ in (3), $x = 0$

Substitute $x = 132y + 316800$ in (3),

$$22y^2 + 8800y - 52800y - 126720000 = 0$$

$$y^2 + 400y - 2400y - 5760000 = 0$$

$$y^2 - 2000y = 5760000$$

$$y^2 - 2000y + 1000000 = 6760000$$

$$y - 1000 = \pm 2600$$

$$\therefore y = 3600$$

Substitute value of y in (5), $x = 792000$

$$\frac{x}{11y} = 20$$

∴ The garrison had 792000 ounces, or 49500 pounds, of bread, and each man's daily share was 20 ounces.

19. Three students, A, B, and C, agree to work out a set of problems in preparation for an examination; each is to do all the problems. A solves 9 problems per day, and finishes the set 4 days before B; B solves 2 more problems per day than C, and finishes the set 6 days before C. Find the number of problems in the set.

Let x = number of problems,

and y = number C can do in one day.

Then $\frac{x}{9}$ = number of days it takes A to do all the problems.

$\frac{x}{9} + 4$ = number of days it takes B to do all the problems.

$\frac{x}{\frac{x}{9}+4}$ = number of problems B can do in one day.

$\frac{x}{y}$ = number of days it takes C to do all the problems.

$$\therefore \frac{x}{y} = \frac{x}{9} + 4 + 6 \qquad (1)$$

$$\frac{x}{\frac{x}{9}+4} = y + 2 \qquad (2)$$

Simplify (1), $9x = xy + 90y \qquad (3)$

Simplify (2), $9x = xy + 36y + 2x + 72$

Subtract, $0 = 54y - 2x - 72$

$$x - 27y + 36 = 0$$

$$x = 27y - 36 \qquad (4)$$

Substitute value of x in (3),

$$243y - 324 = 27y^2 - 36y + 90y$$

$$27y^2 - 189y = -324$$

$$y^2 - 7y = -12$$

$$y^2 - 7y + \tfrac{49}{4} = \tfrac{1}{4}$$

$$y - \tfrac{7}{2} = \pm\tfrac{1}{2}$$

$$y = 4, \text{ or } 3$$

Substitute value of y in (4), $x = 72$, or 45

∴ There are either 45 or 72 problems in the set.

20. A cistern can be filled by two pipes; one of these pipes can fill the cistern in 2 hours less time than the other; the cistern can be filled by both pipes running together in 1⅞ hours. Find the time in which each pipe will fill the cistern.

Let x = number of hours in which the greater pipe can fill the cistern,

and y = number of hours in which the lesser pipe can fill the cistern.

Then $\frac{1}{x}$ = part the greater pipe can fill in one hour.

$\frac{1}{y}$ = part the lesser pipe can fill in one hour.

$\frac{1}{x} + \frac{1}{y}$ = part both pipes can fill in one hour.

Hence, $$y - x = 2 \qquad (1)$$

and $$\left(\frac{1}{x} + \frac{1}{y}\right)1\tfrac{7}{8} = 1 \qquad (2)$$

Simplify (2), $$15x + 15y = 8xy \qquad (3)$$

From (1), $$y = x + 2 \qquad (4)$$

Substitute value of y in (3),

$$15x + 15x + 30 = 8x^2 + 16x$$

$$8x^2 - 14x = 30$$

$$16x^2 - 28x + \tfrac{49}{4} = \tfrac{289}{4}$$

$$4x - \tfrac{7}{2} = \pm\tfrac{17}{2}$$

$$4x = 12, \text{ or } -5$$

$$x = 3$$

From (4), $$y = 5$$

∴ The pipes can fill the cistern in 3 and 5 hours respectively.

21. A and B have a certain manuscript to copy between them. At A's rate of work he would copy the whole manuscript in 18 hours; B copies 9 pages per hour. A finishes his portion in as many hours as he copies pages per hour; B is occupied with his portion 2 hours longer than A is with his. Find the number of pages copied by each.

Let $x =$ number of pages A copies,

and $y =$ number of pages B copies.

Then $x + y =$ number of pages in the manuscript.

$\frac{x+y}{18} =$ number of pages A copies in one hour.

$\frac{18x}{x+y} =$ number of hours it takes A to finish his part.

$\frac{y}{9} =$ number of hours it takes B to finish his part.

$$\therefore \frac{18x}{x+y} = \frac{x+y}{18} \qquad (1)$$

$$\frac{y}{9} = \frac{18x}{x+y} + 2 \qquad (2)$$

Simplify (1), $$x^2 + 2xy + y^2 = 324x \qquad (3)$$

Simplify (2), $$xy + y^2 = 162x + 18x + 18y$$

$$xy + y^2 = 180x + 18y \qquad (4)$$

(3) is $$x^2 + 2xy + y^2 = 324x$$

2 × (4) is $$2xy + 2y^2 = 360x + 36y$$

Subtract, $$x^2 - y^2 = -36x - 36y$$

$$(x+y)(x-y)=-36(x+y)$$

$$\therefore\ x+y=0 \qquad (7)$$

or $$x-y+36=0 \qquad (8)$$

Substitute $x+y=0$ in (3), $x=0$

From (7), $y=0$

From (8), $x=y-36$

Substitute value of x in (3),

$$y^2-72y+1296+2y^2-72y+y^2=324y-11664$$

$$4y^2-468y=-12960$$

$$4y^2-468y+13689=729$$

$$2y-117=\pm 27$$

$$2y=144, \text{ or } 90$$

$$y=72, \text{ or } 45$$

From (8), $x=36$, or 9

$\therefore$ A copies 36 or 9 pages, and B copies 72 or 45 pages.

22. A and B have 4800 circulars to stamp, and intend to finish them in two days, 2400 each day. The first day A, working alone, stamps 800, and then A and B stamp the remaining 1600, A working altogether 3 hours. The second day A works 3 hours and B 1 hour, and they accomplish only $\frac{9}{10}$ of their task for that day. Find the number of circulars each stamps per minute, and the number of hours B works on the first day.

Let x = number A stamps per hour,

and y = number B stamps per hour.

Then, $\frac{800}{x}$ = number of hours A works alone on the first day.

$\frac{1600}{x+y}$ = number of hours A and B work together on the first day.

$$\therefore\ \frac{800}{x}+\frac{1600}{x+y}=3 \qquad (1)$$

$$3x+y=\tfrac{9}{10}\times 2400 \qquad (2)$$

Simplify (1), $800x+800y+1600x=3x^2+3xy$

$$3x^2+3xy=2400x+800y$$

800 × (2) is $$1728000=2400x+800y$$

Subtract, $$3x^2+3xy=1728000$$

$$x^2+xy=576000 \qquad (3)$$

From (2), $$y=2160-3x \qquad (4)$$

Substitute value of y in (3),

$$x^2 + 2160x - 3x^2 = 576000$$
$$-2x^2 + 2160x = 576000$$
$$x^2 - 1080x = -288000$$
$$x^2 - 1080x + 291600 = 3600$$
$$x - 540 = \pm 60$$
$$\therefore\ x = 600, \text{ or } 480$$

Substitute value of x in (4), $y = 360, \text{ or } 720$

$$\frac{1600}{x+y} = \frac{5}{3}, \text{ or } \frac{4}{3}$$

$\therefore$ A can stamp 600 per hour, or 10 per minute, B 360 per hour, or 6 per minute; and B works $\frac{5}{3}$ hours, or 1 hour and 40 minutes on the first day. Or, A can stamp 480 per hour, or 8 per minute, B 720 per hour, or 12 per minute, and B works $\frac{4}{3}$ hours, or 1 hour and 20 minutes, on the first day.

23. A, in running a race with B, to a post and back, meets him 10 yards from the post. To come in even with A, B must increase his pace from this point $41\frac{3}{7}$ yards per minute. If, without changing his pace, he turns back on meeting A, he will come in 4 seconds behind A. Find the distance to the post.

Let $x =$ number of yards to the post,

and $y =$ number of yards A runs per minute.

Then $x + 10 =$ number of yards A has run when he meets B.

$x - 10 =$ number of yards B has run in the same time.

$\frac{x+10}{y} =$ number of minutes elapsed up to the time of meeting.

$\dfrac{x-10}{\frac{x+10}{y}} = \dfrac{y(x-10)}{x+10} =$ number of yards B runs per minute.

$\dfrac{y(x-10)}{x+10} + 41\frac{3}{7} =$ number of yards B would run a minute after increasing his pace.

$$\therefore\ \frac{x+10}{\frac{y(x-10)}{x+10} + 41\frac{3}{7}} = \frac{x-10}{y} \qquad (1)$$

$$\frac{x-10}{\frac{y(x-10)}{x+10}} = \frac{x-10}{x} + \frac{1}{15} \qquad (2)$$

Simplify (1),

$$\frac{(x+10)^2}{y\,(x-10)+\tfrac{290}{7}(x+10)}=\frac{x-10}{y}$$

$$(x+10)^2y=y\,(x-10)^2+\tfrac{290}{7}(x^2-100)$$

$$40\,xy=\tfrac{290}{7}x^2-\tfrac{29000}{7} \qquad (3)$$

Simplify (2),

$$\frac{x^2-100}{xy-10\,y}=\frac{x-10}{y}+\frac{1}{15}$$

$$15\,y\,(x^2-100)=15\,y\,(x-10)^2+y^2(x-10)$$

$$300\,xy-1500\,y-1500\,y-y^2x+10\,y^2=0$$

$$300\,xy-3000\,y-y^2x+10\,y^2=0$$

$$y(300\,x-3000-yx+10\,y)=0$$

$$y(300-y)(x-10)=0 \qquad (4)$$

$$\therefore\ y=0,$$

or $$300-y=0,$$

or $$x-10=0$$

Substitute $y=0$ in (3), $$0=\tfrac{290}{7}x^2-\tfrac{29000}{7}$$

$$\therefore\ x^2=100$$

$$x=\pm 10$$

Substitute $y=300$ **in** (3), $$12000\,x=\tfrac{290}{7}x^2-\tfrac{29000}{7}$$

$$29\,x^2-8400\,x=2900$$

$$(29)^2x^2-29\times 8400\,x=84100$$

$$(29)^2x^2-(\ )+(4200)^2=17724100$$

$$29\,x-4200=\pm 4210$$

$$29\,x=8410,\text{ or }-10$$

$$\therefore\ x=290,\text{ or }-\tfrac{10}{29}$$

Substitute $x=10$ in (3), $$400\,y=0$$

$$y=0$$

$\therefore$ The algebraic solutions are $y=0$, $x=\pm 10$, and $y=300$, $x=290$, or $-\tfrac{10}{29}$. Of these only $y=300$, $x=290$ are available.

$\therefore$ The distance to the post is 290 yards, A runs 300 yards a minute, and B 280 yards a minute.

24. A boat's crew, rowing at half their usual speed, row 3 miles down stream and back again, accomplishing the distance in 2 hours and 40 minutes. At full speed they can go over the same course in 1 hour and 4 minutes. Find the rate of the crew and of the current.

Let $x=$ number of miles the crew row per hour at full speed,

and $y=$ velocity of the current in miles per hour.

Then $\frac{x}{2}+y=$ number of miles the boat will go per hour with the current if the crew row at half speed.

$\frac{x}{2}-y=$ number of miles the boat will go per hour against the current if the crew row at half speed.

2 hours and 40 minutes $= 2\frac{2}{3}$ hours.

1 hour and 4 minutes $= 1\frac{1}{15}$ hours.

$$\therefore \frac{3}{\frac{x}{2}+y}+\frac{3}{\frac{x}{2}-y}=2\frac{2}{3} \qquad (1)$$

$$\frac{3}{x+y}+\frac{3}{x-y}=1\frac{1}{15} \qquad (2)$$

Simplify (1), $\frac{3x}{2}-3y+\frac{3x}{2}+3y=\frac{8}{3}\left(\frac{x^2}{4}-y^2\right)$

$$9x=2x^2-8y^2 \qquad (3)$$

Simplify (2), $3x-3y+3x+3y=\frac{16}{15}(x^2-y^2)$

$$90x=16x^2-16y^2 \qquad (4)$$

$2\times(3)$ is $18x=4x^2-16y^2$

Subtract, $72x=12x^2$

$$12x^2-72x=0$$

$$x^2-6x=0$$

$$x(x-6)=0$$

$$\therefore x=0$$

or $x=6$

Substitute $x=0$ in (3), $8y^2=0$

$$\therefore y=0$$

Substitute $x=6$ in (3), $54=72-8y^2$

$$8y^2=18$$

$$y^2=\frac{9}{4}$$

$$y=\pm\frac{3}{2}$$

$\therefore$ The crew row 6 miles per hour, and the current is $\frac{3}{2}$ miles per hour.

25. A farmer sold a number of sheep for $286. He received for each sheep $2 more than he paid for it, and gained thereby on the cost of the sheep $\frac{1}{2}$ as many per cent as each sheep cost dollars. Find the number of sheep.

Let $x=$ number of sheep,

and $y=$ number of dollars paid for each.

Then, $y+2=$ number of dollars received for each,

and $\frac{286}{y+2}=$ number of sheep.

$$\frac{2}{y}100 = \text{gain per cent.}$$

$$\therefore \frac{286}{y+2} = x \tag{1}$$

$$\frac{2}{y}100 = \frac{y}{2} \tag{2}$$

From (2), $y^2 = 400$

$\therefore y = \pm 20$

Substitute value of y in (1), $\frac{286}{22} = x$

$\therefore x = 13$

$\therefore$ The number of sheep was 13.

26. A person has $1300, which he divides into two parts and loans at different rates of interest, in such a manner that the two portions produce equal returns. If the first portion had been loaned at the second rate of interest it would have yielded annually $36; if the second portion had been loaned at the first rate of interest it would have yielded annually $49. Find the two rates of interest.

Let x = first rate of interest,

and y = second rate of interest.

Then, $\frac{3600}{y}$ = number of dollars in the first part,

and $\frac{4900}{x}$ = number of dollars in the second part.

$$\therefore \frac{3600}{y} + \frac{4900}{x} = 1300 \tag{1}$$

$$\frac{3600}{y} \times \frac{x}{100} = \frac{4900}{x} \times \frac{y}{100} \tag{2}$$

Simplify (1), $3600x + 4900y = 1300xy$

$$36x + 49y = 13xy \tag{3}$$

Simplify (2), $36x^2 = 49y^2$

$6x = \pm 7y$

$$x = \pm \tfrac{7}{6}y \tag{4}$$

Substitute value of x in (3),

$\pm 42y + 49y = \pm \frac{91}{6}y^2$

$\frac{91}{6}y^2 = 91y$, or $-7y$

$y^2 = 6y$, or $-\frac{6}{13}y$

$\therefore y = 0, 6$, or $-\frac{6}{13}$

Substitute values of y in (4),

$x = 0, 7$, or $\frac{7}{13}$

$\therefore$ The first rate of interest was 7%, the second was 6%.

27. A person has $5000, which he divides into two portions and loans at different rates of interest in such a manner that the return from the first portion is double the return from the second portion. If the first portion had been loaned at the second rate of interest it would have yielded annually $245; if the second portion had been loaned at the first rate of interest it would have yielded annually $90. Find the two amounts and the two rates of interest.

Let $x =$ the first rate of interest,

and $y =$ the second rate of interest.

Then $\frac{24500}{y} =$ the number of dollars in the first portion,

and $\frac{9000}{x} =$ the number of dollars in the second portion.

$$\therefore \frac{24500}{y} + \frac{9000}{x} = 5000 \qquad (1)$$

$$\frac{24500}{y} \times \frac{x}{100} = 2 \times \frac{9000}{x} \times \frac{y}{100} \qquad (2)$$

Simplify (1),

$$245x + 90y = 50xy$$

$$49x + 18y = 10xy \qquad (3)$$

Simplify (2),

$$245x^2 = 180y^2$$

$$49x^2 = 36y^2 \qquad (4)$$

$$7x = \pm 6y$$

$$x = \pm \tfrac{6}{7}y \qquad (5)$$

Substitute value of x in (3),

$$\pm 42y + 18y = \pm \tfrac{60}{7}y^2$$

$$\tfrac{60}{7}y^2 = 60y, \text{ or } 24y$$

$$\therefore y = 0, 7, \text{ or } \tfrac{14}{5}$$

Substitute value of y in (5), $x = 0, 6, \text{ or } -\tfrac{12}{5}$

$$\tfrac{24500}{7} = 3500$$

$$\tfrac{9000}{6} = 1500$$

∴ The two amounts are $3500 and $1500, and the two rates of interest are 6% and 7%.

28. A number is formed by three digits; 10 times the middle digit exceeds the square of half the sum of the three digits by 21; if 99 be added to the number, the digits will be in reverse order; the number is 11 times the number formed by the first and third digit. Find the number.

Let $x =$ the first digit,

$y =$ the second digit,

and $z =$ the third digit.

Then, $100x + 10y + z =$ the number,

and $100z + 10y + x =$ the number with the order of the digits reversed.

$10x + z =$ the number formed by first and third digits.

$$\therefore\ 10y = \left(\frac{x+y+z}{2}\right)^2 + 21 \qquad (1)$$

$$100x + 10y + z + 99 = 100z + 10y + x \qquad (2)$$

$$100x + 10y + z = 11(10x + z) \qquad (3)$$

From (2),

$$99x + 99 = 99z$$

$$x + 1 = z \qquad (4)$$

From (3),

$$-10x + 10y - 10z = 0$$

$$x - y + z = 0 \qquad (5)$$

Substitute value of z from (4) in (5),

$$x - y + x + 1 = 0$$

$$2x + 1 = y \qquad (6)$$

Substitute values of y and z from (4) and (6) in (1),

$$20x + 10 = \left(\frac{4x+2}{2}\right)^2 + 21$$

$$20x - 11 = 4x^2 + 4x + 1$$

$$4x^2 - 16x = -12$$

$$x^2 - 4x + 4 = 1$$

$$x - 2 = \pm 1$$

$$x = 3, \text{ or } 1$$

Substitute values of x in (4) and (8),

$$y = 7, \text{ or } 3$$

$$z = 4, \text{ or } 2$$

$\therefore$ The number is either 374, or 132.

29. A number is formed by three digits; the sum of the last two digits is the square of the first digit; the last digit is greater by 2 than the sum of the first and second; if 396 be added to the number, the digits will be in reverse order. Find the number.

Let $x =$ the first digit,
$y =$ the second digit,
and $z =$ the third digit.

Then $100x + 10y + z =$ the number,

and $100z + 10y + x =$ the number with the order of the digits reversed.

$$\therefore\ y + z = x^2 \qquad (1)$$

$$z = x + y + 2 \qquad (2)$$

$$100x + 10y + z + 396 = 100z + 10y + x \qquad (3)$$

From (3), $99x + 396 = 99z$

$$x + 4 = z \tag{4}$$

Substitute values of z in (2), $x + 4 = x + y + 2$

$$\therefore\ y = 2$$

Substitute values of y and z in (1),

$$2 + x + 4 = x^2$$
$$x^2 - x = 6$$
$$x^2 - x + \tfrac{1}{4} = 6\tfrac{1}{4}$$
$$x - \tfrac{1}{2} = \pm 2\tfrac{1}{2}$$
$$x = 3$$

Substitute value of x in (4), $z = 7$

∴ The number is 327.

30. A railroad train, after travelling 1 hour from A, meets with an accident which delays it 1 hour; it then proceeds at a rate 8 miles per hour less than its former rate, and arrives at B 5 hours late. If the accident had happened 50 miles further on, the train would have been only $3\frac{1}{3}$ hours late. Find the distance from A to B.

Let x = number of miles distant from A to B,
and y = number of miles the train goes per hour at first.
Then y = number of miles it goes in the first hour.
$x - y$ = number of miles remaining.
$\dfrac{x-y}{y-8}$ = number of hours in which the train goes the remaining distance.
$\dfrac{x}{y}$ = number of hours it would go the whole distance at its first rate.

$$\therefore\ 1 + 1 + \frac{x-y}{y-8} = \frac{x}{y} + 5 \tag{1}$$

$$\frac{y+50}{y} + 1 + \frac{x-y-50}{y-8} = \frac{x}{y} + \frac{10}{3} \tag{2}$$

Simplify (1),

$$\frac{x-y}{y-8} = \frac{x}{y} + 3$$
$$xy - y^2 = xy - 8x + 3y^2 - 24y$$
$$4y^2 = 8x + 24y$$
$$y^2 = 2x + 6y \tag{3}$$

Simplify (2),

$$\frac{x-y-50}{y-8} = \frac{x-50}{y} + \frac{4}{3}$$
$$xy - y^2 - 50y = xy - 50y - 8x + 400 + \tfrac{4}{3}(y^2 - 8y)$$
$$\tfrac{7}{3}y^2 = 8x + \tfrac{32}{3}y - 400$$

$$7y^2 = 32y + 24x - 1200 \qquad (4)$$

12 × (3) is $$12y^2 = 72y + 24x$$

Subtract, $$5y^2 = 40y \qquad + 1200$$

$$y^2 - 8y + 16 = 256$$

$$y - 4 = \pm 16$$

$$\therefore\ y = 20$$

Substitute value of y in (3), $x = 140$

∴ The distance from A to B is 140 miles.

Exercise 26.

1. Solve $x^6 + 7x^3 = 8$

$$x^6 + 7x^3 + \tfrac{49}{4} = \tfrac{81}{4}$$

$$x^3 + \tfrac{7}{2} = \pm \tfrac{9}{2}$$

$$x^3 = 1, \text{ or } -8$$

Since, $x^3 - 1 = 0$

$$(x-1)(x^2+x+1) = 0$$

$$\therefore\ x = 1$$

or $x^2 + x + 1 = 0$

$$x^2 + x + \tfrac{1}{4} = -\tfrac{3}{4}$$

$$x + \tfrac{1}{2} = \pm \tfrac{1}{2}\sqrt{-3}$$

$$x = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{-3}$$

Since, $x^3 + 8 = 0$

$$(x+2)(x^2-3x+4) = 0$$

$$\therefore\ x = -2$$

or $x^2 - 2x + 4 = 0$

$$x^2 - 2x + 1 = -3$$

$$x - 1 = \pm\sqrt{-3}$$

$$\therefore\ x = 1 \pm \sqrt{-3}$$

$$\therefore\ x = 1,\ -2,\ 1 \pm \sqrt{-3},$$

$$\text{or } -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{-3}$$

2. Solve

$$x^4 - 5x^2 + 4 = 0$$

$$x^4 - 5x^2 + \tfrac{25}{4} = \tfrac{9}{4}$$

$$x^2 - \tfrac{5}{2} = \pm \tfrac{3}{2}$$

$$x^2 = 4, \text{ or } 1$$

$$\therefore\ x = \pm 2, \text{ or } \pm 1$$

3. Solve $x^6 + 4x^3 = 96$

$$x^6 + 4x^3 + 4 = 100$$

$$x^3 + 2 = \pm 10$$

$$x^3 = 8, \text{ or } -12$$

$$\therefore\ x = 2,\ -1 \pm \sqrt{-3},\ -\sqrt[3]{12},$$

$$\text{or } \sqrt[3]{12}\left(\frac{1 \pm \sqrt{-3}}{2}\right)$$

4. Solve $37x^2 - 9 = 4x^4$

$$4x^4 - 37x^2 = -9$$

$$4x^4 - 37x^2 + \frac{37^2}{16} = \frac{1225}{16}$$

$$2x^2 - \tfrac{37}{4} = \pm \tfrac{35}{4}$$

$$2x^2 = 18, \text{ or } \tfrac{1}{2}$$

$$x^2 = 9, \text{ or } \tfrac{1}{4}$$

$$x = \pm 3, \text{ or } \pm \tfrac{1}{2}$$

5. Solve $16x^8 = 17x^4 - 1$

$$16x^8 - 17x^4 = -1$$

$$16x^8 - 17x^4 + \frac{17^2}{8^2} = \frac{225}{64}$$

$$4x^4 - \tfrac{17}{8} = \pm \tfrac{15}{8}$$

$$4x^4 = 4, \text{ or } \tfrac{1}{4}$$

$$x^4 = 1, \text{ or } \tfrac{1}{16}$$

$$x^2 = 1,\ -1,\ \tfrac{1}{4},\ -\tfrac{1}{4}$$

$$\therefore\ x = \pm 1,\ \pm\sqrt{-1},\ \pm \tfrac{1}{2},\ \pm \tfrac{1}{2}\sqrt{-1}$$

6. Solve $32x^{10} = 33x^5 - 1$

$$32x^{10} - 33x^5 = -1$$
$$64x^{10} - 66x^5 = -2.$$
$$64x^{10} - 66x^5 + \frac{33^2}{8^2} = \frac{961}{64}$$
$$8x^5 - \tfrac{33}{8} = \pm\tfrac{31}{8}$$
$$8x^5 = 8, \text{ or } \tfrac{1}{4}$$
$$x^5 = 1, \text{ or } \tfrac{1}{32}$$
$$\therefore x = 1, \text{ or } \tfrac{1}{2}$$

The eight other roots may be found by methods explained later.

7. Solve

$$x^6 + 14x^3 + 24 = 0$$
$$x^6 + 14x^3 + 49 = 25$$
$$x^3 + 7 = \pm 5$$
$$x^3 = -2, \text{ or } -12$$
$$\therefore x = -\sqrt[3]{2}, -\sqrt[3]{12},$$
$$\sqrt[3]{2}\left(\frac{1 \pm \sqrt{-3}}{2}\right), \sqrt[3]{12}\left(\frac{1 \pm \sqrt{-3}}{2}\right)$$

8. Solve

$$19x^4 + 216x^7 = x$$
$$x(19x^3 + 216x^6 - 1) = 0$$
$$\therefore x = 0,$$
$$\text{or } 19x^3 + 216x^6 - 1 = 0$$
$$216x^6 + 19x^3 = 1$$
$$1296x^6 + 114x^3 + \frac{19^2}{12^2} = \frac{1225}{144}$$
$$36x^3 + \tfrac{19}{12} = \pm\tfrac{35}{12}$$
$$36x^3 = \tfrac{16}{12}, \text{ or } -\tfrac{54}{12}$$
$$x^3 = \tfrac{1}{27}, \text{ or } -\tfrac{1}{8}$$

Since $x^3 - \frac{1}{27} = 0$

$$(x - \tfrac{1}{3})(x^2 + \tfrac{1}{3}x + \tfrac{1}{9}) = 0$$
$$\therefore x = \tfrac{1}{3},$$

or $x^2 + \frac{1}{3}x + \frac{1}{9} = 0$

$$x^2 + \tfrac{1}{3}x + \tfrac{1}{36} = -\tfrac{1}{12}$$
$$x + \tfrac{1}{6} = \pm\tfrac{1}{2}\sqrt{-\tfrac{1}{3}}$$
$$x + \tfrac{1}{6} = \pm\tfrac{1}{6}\sqrt{-3}$$
$$x = \frac{-1 \pm \sqrt{-3}}{6}$$

Since, $x^3 + \frac{1}{8} = 0$

$$(x + \tfrac{1}{2})(x^2 - \tfrac{1}{2}x + \tfrac{1}{4}) = 0$$
$$\therefore x = -\tfrac{1}{2},$$

or $x^2 - \frac{1}{2}x + \frac{1}{4} = 0$

$$x^2 - \tfrac{1}{2}x + \tfrac{1}{16} = -\tfrac{3}{16}$$
$$x - \tfrac{1}{4} = \pm\tfrac{1}{4}\sqrt{-3}$$
$$\therefore x = \frac{1 \pm \sqrt{-3}}{4}$$
$$\therefore x = 0, \tfrac{1}{3}, -\tfrac{1}{2}, \frac{-1 \pm \sqrt{-3}}{6},$$
$$\text{or } \frac{1 \pm \sqrt{-3}}{4}$$

9. Solve

$$x^8 - 22x^4 + 21 = 0$$
$$x^8 - 22x^4 + 121 = 100$$
$$x^4 - 11 = \pm 10$$
$$x^4 = 1, \text{ or } 21$$
$$x^2 = \pm 1$$
$$x = \pm 1, \pm\sqrt{-1}$$
$$x^2 = \pm\sqrt{21}$$
$$x = \pm\sqrt[4]{21},$$
$$\pm\sqrt[4]{-21}$$

10. Solve

$$x^{2m} + 3x^m = 4$$
$$x^{2m} + 3x^m + \tfrac{9}{4} = \tfrac{25}{4}$$
$$x^m + \tfrac{3}{2} = \pm\tfrac{5}{2}$$
$$x^m = 1, \text{ or } -4$$
$$\therefore x = \sqrt[m]{1}, \text{ or } \sqrt[m]{-4}$$

11. Solve

$$x^{4n} - \frac{5x^{2n}}{3} = \frac{25}{12}$$
$$x^{4n} - \frac{5x^{2n}}{3} + \frac{25}{36} = \frac{100}{36}$$
$$x^{2n} - \tfrac{5}{6} = \pm\tfrac{10}{6}$$
$$x^{2n} = \tfrac{5}{2}, \text{ or } -\tfrac{5}{6}$$
$$\therefore x = \sqrt[2n]{\tfrac{5}{2}}, \text{ or } \sqrt[2n]{-\tfrac{5}{6}}$$

12. Solve

$$\begin{aligned} x^{6n} + 3x^{3n} &= 40 \\ x^{6n} + 3x^{3n} + \tfrac{9}{4} &= \tfrac{169}{4} \\ x^{3n} + \tfrac{3}{2} &= \pm \tfrac{13}{2} \\ x^{3n} &= 5, \text{ or } -8 \\ \therefore\ x &= \sqrt[3n]{5}, \text{ or } \sqrt[3n]{-8} \end{aligned}$$

13. Solve

$$\begin{aligned} x^{2m} + 2ax^{m} &= 8a^2 \\ x^{2m} + 2ax^{m} + a^2 &= 9a^2 \\ x^{m} + a &= \pm 3a \\ x^{m} &= 2a, \text{ or } -4a \\ x &= \sqrt[m]{2a} \\ &\text{or } \sqrt[m]{-4a} \end{aligned}$$

14. Solve

$$\begin{aligned} x^{-4} - 4x^{-2} &= 12 \\ x^{-4} - 4x^{-2} + 4 &= 16 \\ x^{-2} - 2 &= \pm 4 \\ x^{-2} &= 6, \text{ or } -2 \\ \frac{1}{x^2} &= 6, \text{ or } -2 \\ x^2 &= \tfrac{1}{6}, \text{ or } -\tfrac{1}{2} \\ \therefore\ x &= \pm\sqrt{\tfrac{1}{6}} \\ &\text{or } \pm\sqrt{-\tfrac{1}{2}} \end{aligned}$$

15. Solve

$$\begin{aligned} x^{-6} + 5x^{-3} - 36 &= 0 \\ x^{-6} + 5x^{-3} + \tfrac{25}{4} &= \tfrac{169}{4} \\ x^{-3} + \tfrac{5}{2} &= \pm \tfrac{13}{2} \\ x^{-3} &= -9, \text{ or } 4 \\ \frac{1}{x^3} &= -9, \text{ or } 4 \\ x^3 &= -\tfrac{1}{9}, \text{ or } \tfrac{1}{4} \end{aligned}$$

$$\therefore\ x = -\sqrt[3]{\tfrac{1}{9}},\ \sqrt[3]{\tfrac{1}{4}},\ \frac{-1 \pm \sqrt{-3}}{2\sqrt[3]{4}},\ \frac{1 \pm \sqrt{-3}}{2\sqrt[3]{9}}.$$

16. Solve

$$\begin{aligned} x^{-8} - 3x^{-4} - 154 &= 0 \\ x^{-8} - 3x^{-4} + \tfrac{9}{4} &= \tfrac{625}{4} \\ x^{-4} - \tfrac{3}{2} &= \pm \tfrac{25}{2} \\ x^{-4} &= 14, \text{ or } -11 \\ \frac{1}{x^4} &= 14, \text{ or } -11 \\ x^4 &= \tfrac{1}{14}, \text{ or } -\tfrac{1}{11} \\ \therefore\ x &= \pm\sqrt[4]{\tfrac{1}{14}} \\ &\pm\sqrt[4]{-\tfrac{1}{11}} \end{aligned}$$

17. Solve

$$\begin{aligned} 9x^{-4} + 4x^{-2} &= 5 \\ 9x^{-4} + 4x^{-2} + \tfrac{4}{9} &= \tfrac{49}{9} \\ 3x^{-2} + \tfrac{2}{3} &= \pm \tfrac{7}{3} \\ 3x^{-2} &= \tfrac{5}{3}, \text{ or } -3 \\ x^{-2} &= \tfrac{5}{9}, \text{ or } -1 \\ \frac{1}{x^2} &= \tfrac{5}{9}, \text{ or } -1 \\ x^2 &= \tfrac{9}{5}, \text{ or } -1 \\ \therefore\ x &= \pm 3\sqrt{-\tfrac{1}{5}} \\ &\text{or } \pm\sqrt{-1} \end{aligned}$$

18. Solve

$$\begin{aligned} 4x^{\frac{1}{2}} - 3x^{\frac{1}{4}} &= 10 \\ 4x^{\frac{1}{2}} - 3x^{\frac{1}{4}} + \tfrac{9}{16} &= \tfrac{169}{16} \\ 2x^{\frac{1}{4}} - \tfrac{3}{4} &= \pm \tfrac{13}{4} \\ 2x^{\frac{1}{4}} &= 4, \text{ or } -\tfrac{5}{2} \\ x^{\frac{1}{4}} &= 2, \text{ or } -\tfrac{5}{4} \\ \therefore\ x &= 16, \text{ or } \tfrac{625}{256} \end{aligned}$$

19. Solve

$$\begin{aligned} 2x^{\frac{1}{3}} - 3x^{\frac{1}{6}} &= 9 \\ 4x^{\frac{1}{3}} - 6x^{\frac{1}{6}} + \tfrac{9}{4} &= \tfrac{81}{4} \\ 2x^{\frac{1}{6}} - \tfrac{3}{2} &= \pm \tfrac{9}{2} \\ 2x^{\frac{1}{6}} &= 6, \text{ or } -3 \\ x^{\frac{1}{6}} &= 3, \text{ or } -\tfrac{3}{2} \\ \therefore\ x &= 729, \text{ or } \tfrac{729}{64} \end{aligned}$$

20. Solve $\sqrt{x^5} = \sqrt[4]{x^5} + 12$

$$x^{\frac{5}{2}} = x^{\frac{5}{4}} + 12$$

$$x^{\frac{5}{2}} - x^{\frac{5}{4}} + \tfrac{1}{4} = 12\tfrac{1}{4}$$

$$x^{\frac{5}{4}} - \tfrac{1}{2} = \pm \tfrac{7}{2}$$

$$x^{\frac{5}{4}} = 4, \text{ or } -3$$

$$x^5 = 256, \text{ or } 81$$

$$\therefore\ x = 2\sqrt[5]{8}, \text{ or } \sqrt[5]{81}$$

21. Solve $x = 9\sqrt{x} + 22$

$$x = 9x^{\frac{1}{2}} + 22$$

$$x - 9x^{\frac{1}{2}} + \tfrac{81}{4} = \tfrac{169}{4}$$

$$\sqrt{x} - \tfrac{9}{2} = \pm \tfrac{13}{2}$$

$$\sqrt{x} = 11, \text{ or } -2$$

$$\therefore\ x = 121, \text{ or } 4$$

22. Solve

$$\sqrt[3]{x^2} - 4\sqrt[3]{x} = 32$$

$$x^{\frac{2}{3}} - 4x^{\frac{1}{3}} = 32$$

$$x^{\frac{2}{3}} - 4x^{\frac{1}{3}} + 4 = 36$$

$$x^{\frac{1}{3}} - 2 = \pm 6$$

$$x^{\frac{1}{3}} = 8, \text{ or } -4$$

$$\therefore\ x = 512, \text{ or } -64$$

23. Solve

$$2\sqrt{x^3} - 3\sqrt[4]{x^3} = 35$$

$$2x^{\frac{3}{2}} - 3x^{\frac{3}{4}} = 35$$

$$4x^{\frac{3}{2}} - 6x^{\frac{3}{4}} + \tfrac{9}{4} = \tfrac{289}{4}$$

$$2x^{\frac{3}{4}} - \tfrac{3}{2} = \pm \tfrac{17}{2}$$

$$2x^{\frac{3}{4}} = 10, \text{ or } -7$$

$$x^{\frac{3}{4}} = 5, \text{ or } -\tfrac{7}{2}$$

$$x^3 = 625, \text{ or } \tfrac{2401}{16}$$

$$\therefore\ x = 5\sqrt[3]{5}, \text{ or } \tfrac{7}{2}\sqrt[3]{\tfrac{7}{2}}$$

24. Solve

$$\frac{1}{\sqrt[3]{x}} + \frac{1}{\sqrt[6]{x}} = \frac{3}{4}$$

$$\frac{1}{x^{\frac{1}{3}}} + \frac{1}{x^{\frac{1}{6}}} = \frac{3}{4}$$

$$x^{-\frac{1}{3}} + x^{-\frac{1}{6}} + \tfrac{1}{4} = 1$$

$$x^{-\frac{1}{6}} + \tfrac{1}{2} = \pm 1$$

$$x^{-\frac{1}{6}} = \tfrac{1}{2}, \text{ or } -\tfrac{3}{2}$$

$$\frac{1}{x^{\frac{1}{6}}} = \tfrac{1}{2}, \text{ or } -\tfrac{3}{2}$$

$$x^{\frac{1}{6}} = 2, \text{ or } -\tfrac{2}{3}$$

$$\therefore\ x = 64, \text{ or } \tfrac{64}{729}$$

25. Solve

$$x^{-\frac{1}{2}} + x^{-\frac{1}{4}} = \tfrac{4}{9}$$

$$x^{-\frac{1}{2}} + x^{-\frac{1}{4}} + \tfrac{1}{4} = \tfrac{25}{36}$$

$$x^{-\frac{1}{4}} + \tfrac{1}{2} = \pm \tfrac{5}{6}$$

$$x^{-\frac{1}{4}} = \tfrac{1}{3}, \text{ or } -\tfrac{4}{3}$$

$$\frac{1}{x^{\frac{1}{4}}} = \tfrac{1}{3}, \text{ or } -\tfrac{4}{3}$$

$$x^{\frac{1}{4}} = 3, \text{ or } -\tfrac{3}{4}$$

$$\therefore\ x = 81, \text{ or } \tfrac{81}{256}$$

26. Solve

$$3x^{-\frac{1}{3}} + 4x^{-\frac{1}{6}} = 20$$

$$9x^{-\frac{1}{3}} + 12x^{-\frac{1}{6}} + 4 = 64$$

$$3x^{-\frac{1}{6}} + 2 = \pm 8$$

$$3x^{-\frac{1}{6}} = 6, \text{ or } -10$$

$$x^{-\frac{1}{6}} = 2, \text{ or } -\tfrac{10}{3}$$

$$\frac{1}{x^{\frac{1}{6}}} = 2, \text{ or } -\tfrac{10}{3}$$

$$x^{\frac{1}{6}} = \tfrac{1}{2}, \text{ or } -\tfrac{3}{10}$$

$$\therefore\ x = \tfrac{1}{64},$$

$$\text{or } \tfrac{729}{1000000}$$

27. Solve

$$2x^{-\frac{2}{3}} - x^{-\frac{1}{3}} = 45$$

$$4x^{-\frac{2}{3}} - 2x^{-\frac{1}{3}} + \tfrac{1}{4} = 90\tfrac{1}{4}$$

$$2x^{-\frac{1}{3}} - \tfrac{1}{2} = \pm 9\tfrac{1}{2}$$

$$2x^{-\frac{1}{3}} = 10, \text{ or } -9$$

$$x^{-\frac{1}{3}} = 5, \text{ or } -\tfrac{9}{2}$$

$$x^{\frac{1}{3}} = \tfrac{1}{5}, \text{ or } -\tfrac{2}{9}$$

$$\therefore\ x = \tfrac{1}{125}, \text{ or } -\tfrac{8}{729}$$

28. Solve

$$4\sqrt[3]{x^{-2}} + 3\sqrt[3]{x^{-1}} = 27$$

$$4x^{-\frac{2}{3}} + 3x^{-\frac{1}{3}} = 27$$

$$4x^{-\frac{2}{3}} + 3x^{-\frac{1}{3}} + \tfrac{9}{16} = \tfrac{441}{16}$$

$$2x^{-\frac{1}{3}} + \tfrac{3}{4} = \pm \tfrac{21}{4}$$

$$2x^{-\frac{1}{3}} = \tfrac{9}{2}, \text{ or } -6$$

$$x^{-\frac{1}{3}} = \tfrac{9}{4}, \text{ or } -3$$

$$x^{\frac{1}{3}} = \tfrac{4}{9}, \text{ or } -\tfrac{1}{3}$$

$$\therefore\ x = \tfrac{64}{729}, \text{ or } -\tfrac{1}{27}$$

29. Solve

$$\sqrt[3]{2x} + \sqrt[3]{4x^2} = 72$$

$$(2x)^{\frac{2}{3}} + (2x)^{\frac{1}{3}} = 72$$

$$(2x)^{\frac{2}{3}} + (2x)^{\frac{1}{3}} + \tfrac{1}{4} = 72\tfrac{1}{4}$$

$$(2x)^{\frac{1}{3}} + \tfrac{1}{2} = \pm 8\tfrac{1}{2}$$

$$(2x)^{\frac{1}{3}} = 8, \text{ or } -9$$

$$2x = 512, \text{ or } -729$$

$$\therefore\ x = 256, \text{ or } -\tfrac{729}{2}$$

30. Solve

$$\sqrt{2x} + 4x = 1$$

$$2 \times 2x + (2x)^{\frac{1}{2}} = 1$$

$$4 \times 2x + 2(2x)^{\frac{1}{2}} + \tfrac{1}{4} = 2\tfrac{1}{4}$$

$$2\sqrt{2x} + \tfrac{1}{2} = \pm \tfrac{3}{2}$$

$$2\sqrt{2x} = 1, \text{ or } -2$$

$$\sqrt{2x} = \tfrac{1}{2}, \text{ or } -1$$

$$2x = \tfrac{1}{4}, \text{ or } 1$$

$$\therefore\ x = \tfrac{1}{8}, \text{ or } \tfrac{1}{2}$$

Exercise 27.

1. Solve $\sqrt{x+4} + \sqrt{2x-1} = 6$

Square,

$$x + 4 + 2\sqrt{2x^2 + 7x - 4} + 2x - 1 = 36$$

$$2\sqrt{2x^2 + 7x - 4} = 33 - 3x$$

Square, $8x^2 + 28x - 16 = 1089 - 198x + 9x^2$

$$x^2 - 226x = -1105$$

$$x^2 - 226x + 12769 = 11664$$

$$x - 113 = \pm 108$$

$$\therefore\ x = 221, \text{ or } 5$$

2. Solve $\sqrt{13x-1} - \sqrt{2x-1} = 5$

Square, $15x - 2 - 2\sqrt{26x^2 - 15x + 1} = 25$

$$-2\sqrt{26x^2 - 15x + 1} = 27 - 15x$$

Square, $104x^2 - 60x + 4 = 729 - 810x + 225x^2$

$$121x^2 - 750x = -725$$

$$121x^2 - 750x + (\tfrac{375}{11})^2 = \tfrac{52900}{121}$$

$$11x - \tfrac{375}{11} = \pm\tfrac{230}{11}$$

$$11x = 55, \text{ or } \tfrac{145}{11}$$

$$\therefore x = 5, \text{ or } \tfrac{155}{121}$$

3. Solve $\sqrt{x} + \sqrt{4+x} = 3$

$$\sqrt{4+x} = 3 - \sqrt{x}$$

Square, $4 + x = 9 - 6\sqrt{x} + x$

$$6\sqrt{x} = 5$$

$$36x = 25$$

$$\therefore x = \tfrac{25}{36}$$

4. Solve $\sqrt{x^2-9} + 21 = x^2$

$$x^2 - \sqrt{x^2-9} = 21$$

$$x^2 - 9 - \sqrt{x^2-9} = 12$$

Let $\sqrt{x^2-9} = y$

Then $y^2 - y = 12$

$$y^2 - y + \tfrac{1}{4} = 12\tfrac{1}{4}$$

$$y - \tfrac{1}{2} = \pm 3\tfrac{1}{2}$$

$$y = 4, \text{ or } -3$$

$$\sqrt{x^2-9} = 4, \text{ or } -3$$

$$x^2 - 9 = 16, \text{ or } 9$$

$$x^2 = 25, \text{ or } 18$$

$$x = \pm 5, \text{ or } \pm 3\sqrt{2}$$

5. Solve $\sqrt{x+1} + \sqrt{x+16} = \sqrt{x+25}$

Square,

$$x + 1 + 2\sqrt{x^2 + 17x + 16} + x + 16 = x + 25$$

$$2\sqrt{x^2 + 17x + 16} = 8 - x$$

Square, $4x^2 + 68x + 64 = 64 - 16x + x^2$

$$3x^2 + 84x = 0$$

$$3x(x + 28) = 0$$

$$\therefore x = 0, \text{ or } -28$$

6. Solve $\sqrt{2x+1} - \sqrt{x+4} = \dfrac{\sqrt{x-3}}{3}$

Square,

$$2x + 1 - 2\sqrt{2x^2 + 9x + 4} + x + 4 = \frac{x-3}{9}$$

$$-2\sqrt{2x^2+9x+4}=-\frac{26x+48}{9}$$

$$9\sqrt{2x^2+9x+4}=13x+24$$

Square, $162x^2+729x+324=169x^2+624x+576$

$$7x^2-105x=-252$$

$$x^2-15x=-36$$

$$x^2-15x+36=0$$

$$(x-3)(x-12)=0$$

$$\therefore\ x=3, \text{ or } 12$$

7. Solve $\sqrt{x+3}+\sqrt{x+8}=5\sqrt{x}$

Square,

$$x+3+2\sqrt{x^2+11x+24}+x+8=25x$$

$$2\sqrt{x^2+11x+24}=23x-11$$

Square, $4x^2+44x+96=529x^2-506x+121$

$$525x^2-550x=-25$$

$$21x^2-22x=-1$$

$$21x^2-22x+1=0$$

$$(x-1)(21x-1)=0$$

$$\therefore\ x=1, \text{ or } \tfrac{1}{21}$$

8. Solve

$$\sqrt{x+7}+\sqrt{x-5}+\sqrt{3x+9}=0$$

$$\sqrt{x+7}+\sqrt{x-5}=-\sqrt{3x+9}$$

Square,

$$x+7+2\sqrt{x^2+2x-35}+x-5=3x+9$$

$$2\sqrt{x^2+2x-35}=x+7$$

Square, $4x^2+8x-140=x^2+14x+49$

$$3x^2-6x-189=0$$

$$x^2-2x-63=0$$

$$(x-9)(x+7)=0$$

$$\therefore\ x=9, \text{ or } -7$$

9. Solve

$$\sqrt{x+5}+\sqrt{8-2x}+\sqrt{9-4x}=0$$

$$\sqrt{x+5}+\sqrt{8-2x}=-\sqrt{9-4x}$$

Square,

$$-x+13+2\sqrt{-2x^2-2x+40}=9-4x$$

$$2\sqrt{-2x^2-2x+40}=-4-3x$$

Square, $-8x^2 - 8x + 160 = 16 + 24x + 9x^2$

$17x^2 + 32x - 144 = 0$

$(17x - 36)(x + 4) = 0$

$\therefore x = -4, \text{ or } \frac{36}{17}$

10. Solve $\sqrt{7 - x} + \sqrt{3x + 10} + \sqrt{x + 3} = 0$

$\sqrt{7 - x} + \sqrt{3x + 10} = -\sqrt{x + 3}$

Square, $17 + 2x + 2\sqrt{-3x^2 + 11x + 70} = x + 3$

$2\sqrt{-3x^2 + 11x + 70} = -x - 14$

Square, $-12x^2 + 44x + 280 = x^2 + 28x + 196$

$13x^2 - 16x - 84 = 0$

$(13x - 42)(x + 2) = 0$

$\therefore x = -2, \text{ or } \frac{42}{13}$

11. Solve $\sqrt{2x^2 + 3x + 7} = 2x^2 + 3x - 5$

$2x^2 + 3x - 5 - \sqrt{2x^2 + 3x + 7} = 0$

$2x^2 + 3x + 7 - \sqrt{2x^2 + 3x + 7} = 12$

Let $\sqrt{2x^2 + 3x + 7} = y$

Then $y^2 - y = 12$

$y^2 - y - 12 = 0$

$(y - 4)(y + 3) = 0$

$\therefore y = 4, \text{ or } -3$

Since $\sqrt{2x^2 + 3x + 7} = 4$

$2x^2 + 3x + 7 = 16$

$2x^2 + 3x - 9 = 0$

$(2x - 3)(x + 3) = 0$

$\therefore x = \frac{3}{2}, \text{ or } -3$

Since $\sqrt{2x^2 + 3x + 7} = -3$

$2x^2 + 3x + 7 = 9$

$2x^2 + 3x - 2 = 0$

$(2x - 1)(x + 2) = 0$

$\therefore x = \frac{1}{2}, \text{ or } -2$

$x = \frac{3}{2}, -3, \frac{1}{2}, \text{ or } -2$

12. Solve $x^2 - 3x + 2 = 6\sqrt{x^2 - 3x - 3}$

$x^2 - 3x + 2 - 6\sqrt{x^2 - 3x - 3} = 0$

$x^2 - 3x - 3 - 6\sqrt{x^2 - 3x - 3} = -5$

Let $\sqrt{x^2 - 3x - 3} = y$

Then $y^2 - 6y = -5$

$y^2 - 6y + 5 = 0$

$$(y-5)(y-1)=0$$

$$\therefore\ y=5, \text{ or } 1$$

Since $$\sqrt{x^2-3x-3}=5$$

$$x^2-3x-3=25$$

$$x^2-3x-28=0$$

$$(x-7)(x+4)=0$$

$$\therefore\ x=7, \text{ or } -4$$

Since $$\sqrt{x^2-3x-3}=1$$

$$x^2-3x-3=1$$

$$x^2-3x-4=0$$

$$(x+1)(x-4)=0$$

$$\therefore\ x=-1, \text{ or } 4$$

$$x=7, -4, -1, \text{ or } 4$$

13. Solve $$6x^2-3x-2=\sqrt{2x^2-x}$$

$$3(2x^2-x)-\sqrt{2x^2-x}=2$$

Let $$\sqrt{2x^2-x}=y$$

Then $$3y^2-y=2$$

$$3y^2-y-2=0$$

$$(3y+2)(y-1)=0$$

$$\therefore\ y=1, \text{ or } -\tfrac{2}{3}$$

Since $$\sqrt{2x^2-x}=1$$

$$2x^2-x=1$$

$$2x^2-x-1=0$$

$$(x-1)(2x+1)=0$$

$$\therefore\ x=1, \text{ or } -\tfrac{1}{2}$$

Since $$\sqrt{2x^2-x}=-\tfrac{2}{3}$$

$$2x^2-x=\tfrac{4}{9}$$

$$4x^2-2x+\tfrac{1}{4}=\tfrac{41}{36}$$

$$2x-\tfrac{1}{2}=\pm\tfrac{1}{6}\sqrt{41}$$

$$2x=\tfrac{1}{2}\pm\tfrac{1}{6}\sqrt{41}$$

$$\therefore\ x=\tfrac{1}{4}\pm\tfrac{1}{12}\sqrt{41}$$

$$x=1, -\tfrac{1}{2}, \tfrac{1}{4}\pm\tfrac{1}{12}\sqrt{41}$$

14. Solve $$15x-3x^2-16=4\sqrt{x^2-5x+5}$$

$$3(x^2-5x)+4\sqrt{x^2-5x+5}=-16$$

$$3(x^2-5x+5)+4\sqrt{x^2-5x+5}=-1$$

Let $$\sqrt{x^2-5x+5}=y$$

Then $$3y^2+4y=-1$$

$$3y^2+4y+1=0$$

$(3y+1)(y+1)=0$

$\therefore\ y=-\frac{1}{3}, \text{ or } -1$

Since $\sqrt{x^2-5x+5}=-\frac{1}{3}$

$x^2-5x+5=\frac{1}{9}$

$x^2-5x+\frac{44}{9}=0$

$(x-\frac{11}{3})(x-\frac{4}{3})=0$

$\therefore\ x=\frac{11}{3}, \text{ or } \frac{4}{3}$

Since $\sqrt{x^2-5x+5}=-1$

$x^2-5x+5=1$

$x^2-5x+4=0$

$(x-4)(x-1)=0$

$\therefore\ x=4, \text{ or } 1$

$x=\frac{11}{3}, \frac{4}{3}, 4, \text{ or } 1$

15. Solve $6x^2-21x+20=\sqrt{4x^2-14x+16}$

$\frac{3}{2}(4x^2-14x)-\sqrt{4x^2-14x+16}=-20$

$\frac{3}{2}(4x^2-14x+16)-\sqrt{4x^2-14x+16}=4$

Let $\sqrt{4x^2-14x+16}=y$

Then $\frac{3}{2}y^2-y=4$

$3y^2-2y-8=0$

$(3y+4)(y-2)=0$

$\therefore\ y=2, \text{ or } -\frac{4}{3}$

Since $\sqrt{4x^2-14x+16}=2$

$4x^2-14x+16=4$

$4x^2-14x+12=0$

$(4x-6)(x-2)=0$

$\therefore\ x=2, \text{ or } \frac{3}{2}$

Since $\sqrt{4x^2-14x+16}=-\frac{4}{3}$

$4x^2-14x+16=\frac{16}{9}$

$4x^2-14x+\frac{49}{4}=-\frac{71}{36}$

$2x-\frac{7}{2}=\pm\frac{1}{6}\sqrt{-71}$

$2x=\frac{7}{2}\pm\frac{1}{6}\sqrt{-71}$

$\therefore\ x=\frac{7}{4}\pm\frac{1}{12}\sqrt{-71}$

$x=2, \frac{3}{2}, \frac{7}{4}\pm\frac{1}{12}\sqrt{-71}$

16. Solve $\sqrt{36x^2+12x+33}=41-8x-24x^2$

$24x^2+8x+\sqrt{36x^2+12x+33}=41$

$\frac{2}{3}(36x^2+12x)+\sqrt{36x^2+12x+33}=41$

$\frac{2}{3}(36x^2+12x+33)+\sqrt{36x^2+12x+33}=63$

Let $\sqrt{36x^2+12x+33}=y$

Then $\frac{2}{3}y^2+y=63$

$$2y^2 + 3y - 189 = 0$$
$$(2y + 21)(y - 9) = 0$$
$$\therefore\ y = 9, \text{ or } -\tfrac{21}{2}$$

Since $$\sqrt{36x^2 + 12x + 33} = 9$$
$$36x^2 + 12x + 33 = 81$$
$$36x^2 + 12x - 48 = 0$$
$$3x^2 + x - 4 = 0$$
$$(3x + 4)(x - 1) = 0$$
$$\therefore\ x = 1, \text{ or } -\tfrac{4}{3}$$

Since $$\sqrt{36x^2 + 12x + 33} = -\tfrac{21}{2}$$
$$36x^2 + 12x + 33 = \tfrac{441}{4}$$
$$36x^2 + 12x + 1 = \tfrac{313}{4}$$
$$6x + 1 = \pm \tfrac{1}{2}\sqrt{313}$$
$$6x = -1\ \tfrac{1}{2}\sqrt{313}$$
$$\therefore x = -\tfrac{1}{6} \pm \tfrac{1}{12}\sqrt{313}$$
$$x = 1, -\tfrac{4}{5}, -\tfrac{1}{6} \pm \tfrac{1}{12}\sqrt{313}$$

17. $$4x^4 - 12x^3 + 5x^2 + 6x - 15 = 0$$

Attempt to extract the square root,

$$\begin{array}{r|l|l}
 & 4x^4 - 12x^3 + 5x^2 + 6x - 15 & \underline{2x^2 - 3x - 1} \\
 & 4x^4 & \\
4x^2 - 3x & -12x^3 + 5x^2 & \\
 & \underline{-12x^3 + 9x^2} & \\
4x^2 - 6x - 1 & -4x^2 + 6x - 15 & \\
 & \underline{-4x^2 + 6x + \ \ 1} & \\
 & \qquad\qquad -16 &
\end{array}$$

$$\therefore\ 4x^4 - 12x^3 + 5x^2 + 6x - 15 = (2x^2 - 3x - 1)^2 - 16$$
$$\therefore\ (2x^2 - 3x - 1)^2 - 16 = 0$$
$$(2x^2 - 3x - 1)^2 = 16$$
$$2x^2 - 3x - 1 = \pm 4$$

Since $$2x^2 - 3x - 1 = 4$$
$$2x^2 - 3x - 5 = 0$$
$$(2x - 5)(x + 1) = 0$$
$$\therefore\ x = -1, \text{ or } \tfrac{5}{2}$$

Since $$2x^2 - 3x - 1 = -4$$
$$2x^2 - 3x = 3$$
$$4x^2 - 6x = -6$$
$$4x^2 - 6x + \tfrac{9}{4} = -\tfrac{15}{4}$$
$$2x - \tfrac{3}{2} = \pm \tfrac{1}{2}\sqrt{-15}$$

$$2x = \frac{3 \pm \sqrt{-15}}{2}$$

$$x = \frac{3 \pm \sqrt{-15}}{4}$$

$$\therefore\ x = -1,\ \tfrac{5}{2},\ \frac{3 \pm \sqrt{-15}}{4}$$

$$\begin{array}{r|l} & -12x^5 + 9x^3 \\ 4x^2 - 6x + 2 & 8x^2 - 12x - 12 \\ & \underline{8x^2 - 12x + 4} \\ & \qquad\qquad -16 \end{array}$$

Given equation is $(2x^2 - 3x + 2)^2 - 16 = 0$

$$\therefore\ 2x^2 - 3x + 2 = \pm 4$$

$$2x^2 - 3x + 2 = 4$$

$$2x^2 - 3x = 2$$

$$x^2 - \tfrac{3}{2}x + \tfrac{9}{16} = \tfrac{25}{16}$$

$$x = \tfrac{3}{4} \pm \tfrac{5}{4}$$

$$x = 2, \text{ or } -\tfrac{1}{2}$$

$$2x^2 - 3x + 2 = -4$$

$$2x^2 - 3x = -6$$

$$x^2 - \tfrac{3}{2}x + \tfrac{9}{16} = -\tfrac{39}{16}$$

$$\therefore\ x = \frac{3 \pm \sqrt{-39}}{4}$$

22. Solve $\sqrt{x} + \sqrt{x+3} = \dfrac{6}{\sqrt{x+3}}$

Simplify, $\sqrt{x^2 + 3x} + x + 3 = 6$

$$\sqrt{x^2 + 3x} = 3 - x$$

Square, $x^2 + 3x = 9 - 6x + x^2$

19. Solve $x^4 - 4x^3 - 10x^2 + 28x - 15 = 0$

Attempt to extract the square root,

$$\begin{array}{r|l} & x^4 - 4x^3 - 10x^2 + 28x - 15 \;\underline{|\; x^2 - 2x - 7} \\ & x^4 \\ 2x^2 - 2x & -4x^3 - 10x^2 \\ & \underline{-4x^3 + 4x^2} \\ 2x^2 - 4x + 7 & -14x^2 + 28x - 15 \\ & \underline{-14x^2 + 28x + 49} \\ & \qquad\qquad -64 \end{array}$$

$$\therefore\ x^4 - 4x^3 - 10x^2 + 28x - 15 = (x^2 - 2x - 7)^2 - 64$$

$$\therefore\ (x^2 - 2x - 7)^2 - 64 = 0$$

$$(x^2-2x-7)^2=64$$

$$x^2-2x-7=\pm 8$$

Since $$x^2-2x-7=8$$

$$x^2-2x-15=0$$

$$(x-5)(x+3)=0$$

$$\therefore\ x=5, \text{ or } -3$$

$$3x^2+x-4=0$$

$$(3x+4)(x-1)=0$$

$$\therefore\ x=1, \text{ or } -\tfrac{4}{3}$$

Since $$\sqrt{36x^2+12x+33}=-\tfrac{21}{2}$$

$$36x^2+12x+33=\tfrac{441}{4}$$

$$36x^2+12x+1=\tfrac{313}{4}$$

$$6x+1=\pm\tfrac{1}{2}\sqrt{313}$$

$$6x=-1\,\tfrac{1}{2}\sqrt{313}$$

$$\therefore\ x=-\tfrac{1}{6}\pm\tfrac{1}{12}\sqrt{313}$$

$$x=1,\ -\tfrac{4}{5},\ -\tfrac{1}{6}\pm\tfrac{1}{12}\sqrt{313}$$

17. $$4x^4-12x^3+5x^2+6x-15=0$$

Attempt to extract the square root,

$$\begin{array}{r|l|l}
 & 4x^4-12x^3+5x^2+6x-15 & \underline{2x^2-3x-1} \\
 & 4x^4 & \\
4x^2-3x & -12x^3+5x^2 & \\
 & -12x^3+9x^2 & \\
4x^2-6x-1 & -4x^2+6x-15 & \\
 & \underline{-4x^2+6x+\ \ 1} & \\
 & -16 &
\end{array}$$

$$\therefore\ 4x^4-12x^3+5x^2+6x-15=(2x^2-3x-1)^2-16$$

$$\therefore\ (2x^2-3x-1)^2-16=0$$

Since $$6x^2+4x=16$$

$$6x^2+4x-16=0$$

$$(3x-4)(2x+4)=0$$

$$\therefore\ x=-2, \text{ or } \tfrac{4}{3}$$

Since $$6x^2+4x=-11$$

$$36x^2+24x+4=-62$$

$$6x+2=\pm\sqrt{-62}$$

$$6x=-2\pm\sqrt{-62}$$

$$x=\frac{-2\pm\sqrt{-62}}{6}$$

$$\therefore\ x=-2,\ \tfrac{4}{3}, \text{ or } \frac{-2\pm\sqrt{-62}}{6}$$

21. Solve $4x^4 - 12x^3 + 17x^2 - 12x - 12 = 0$

Attempt to extract the square root,

$$\begin{array}{r|l|l}
 & 4x^4 - 12x^3 + 17x^2 - 12x - 12 & \underline{2x^2 - 3x + 2} \\
 & 4x^4 & \\
4x^2 - 3x & -12x^3 + 17x^2 & \\
 & -12x^3 + 9x^2 & \\
4x^2 - 6x + 2 & 8x^2 - 12x - 12 & \\
 & 8x^2 - 12x + 4 & \\
 & \hline -16 &
\end{array}$$

Given equation is $(2x^2 - 3x + 2)^2 - 16 = 0$

$$\begin{aligned}
\therefore 2x^2 - 3x + 2 &= \pm 4 \\
2x^2 - 3x + 2 &= 4 \\
2x^2 - 3x &= 2 \\
x^2 - \tfrac{3}{2}x + \tfrac{9}{16} &= \tfrac{25}{16} \\
x &= \tfrac{3}{4} \pm \tfrac{5}{4} \\
x &= 2, \text{ or } -\tfrac{1}{2} \\
2x^2 - 3x + 2 &= -4 \\
2x^2 - 3x &= -6 \\
x^2 - \tfrac{3}{2}x + \tfrac{9}{16} &= -\tfrac{39}{16} \\
\therefore x &= \frac{3 \pm \sqrt{-39}}{4}
\end{aligned}$$

22. Solve $\sqrt{x} + \sqrt{x+3} = \dfrac{6}{\sqrt{x+3}}$

Simplify, $\sqrt{x^2 + 3x} + x + 3 = 6$

$$\sqrt{x^2 + 3x} = 3 - x$$

Square, $x^2 + 3x = 9 - 6x + x^2$

$$\begin{aligned}
9x &= 9 \\
\therefore x &= 1
\end{aligned}$$

23. Solve $6 + \sqrt{x^2 - 1} = \dfrac{16}{\sqrt{x^2 - 1}}$

Simplify, $6\sqrt{x^2 - 1} + x^2 - 1 = 16$

$$6\sqrt{x^2 - 1} = 17 - x^2$$

Square, $36x^2 - 36 = 289 - 34x^2 + x^4$

$$\begin{aligned}
x^4 - 70x^2 + 325 &= 0 \\
(x^2 - 5)(x^2 - 65) &= 0 \\
\therefore x^2 &= 5, \text{ or } 65 \\
x &= \pm\sqrt{5}, \text{ or } \pm\sqrt{65}
\end{aligned}$$

24. Solve $$\frac{1}{\sqrt{x+1}}+\frac{1}{\sqrt{x-1}}=\frac{1}{\sqrt{x^2-1}}$$

Simplify, $$\sqrt{x-1}+\sqrt{x+1}=1$$

Square,
$$2x+2\sqrt{x^2-1}=1$$
$$2\sqrt{x^2-1}=1-2x$$
$$4x^2-4=1-4x+4x^2$$
$$4x=5$$
$$\therefore\ x=\tfrac{5}{4}$$

25. Solve $$\frac{\sqrt{x+2}-\sqrt{x-2}}{\sqrt{x+2}+\sqrt{x-2}}=\frac{x}{2}$$

Multiply both terms of the left fraction by $\sqrt{x+2}-\sqrt{x-2}$,
$$\frac{x+2-2\sqrt{x^2-4}+x-2}{4}=\frac{x}{2}$$
$$x-\sqrt{x^2-4}=x$$
$$\sqrt{x^2-4}=0$$
$$x^2-4=0$$
$$\therefore\ x=\pm 2$$

26. Solve $$\frac{3x+\sqrt{4x-x^2}}{3x-\sqrt{4x-x^2}}=2$$

Simplify, $$3x+\sqrt{4x-x^2}=6x-2\sqrt{4x-x^2}$$
$$3\sqrt{4x-x^2}=3x$$
$$\sqrt{4x-x^2}=x$$
$$4x-x^2=x^2$$
$$2x^2-4x=0$$
$$\therefore\ x=0,\text{ or }2$$

27. Solve $$\frac{\sqrt{3x^2+4}-\sqrt{2x^2+1}}{\sqrt{3x^2+4}+\sqrt{2x^2+1}}=\frac{1}{7}$$
$$7\sqrt{3x^2+4}-7\sqrt{2x^2+1}=\sqrt{3x^2+4}+\sqrt{2x^2+1}$$
$$6\sqrt{3x^2+4}=8\sqrt{2x^2+1}$$
$$3\sqrt{3x^2+4}=4\sqrt{2x^2+1}$$

Square, $$27x^2+36=32x^2+16$$
$$5x^2=20$$
$$x^2=4$$
$$\therefore\ x=\pm 2$$

28. Solve
$$\frac{\sqrt{7x^2+4}+2\sqrt{3x-1}}{\sqrt{7x^2+4}-2\sqrt{3x-1}}=7$$
$$\sqrt{7x^2+4}+2\sqrt{3x-1}=7\sqrt{7x^2+4}-14\sqrt{3x-1}$$
$$16\sqrt{3x-1}=6\sqrt{7x^2+4}$$
$$8\sqrt{3x-1}=3\sqrt{7x^2+4}$$
Square,
$$192x-64=63x^2+36$$
$$63x^2-192x+100=0$$
$$441x^2-1344x=-700$$
$$441x^2-1344x+1024=324$$
$$21x-32=\pm 18$$
$$21x=50,\text{ or }14$$
$$\therefore\ x=\tfrac{50}{21},\text{ or }\tfrac{2}{3}$$

29. Solve
$$\frac{\sqrt{5x-4}+\sqrt{5-x}}{\sqrt{5x-4}-\sqrt{5-x}}=\frac{2\sqrt{x}+1}{2\sqrt{x}-1}$$
Simplify,
$$2\sqrt{5x^2-4x}+2\sqrt{5x-x^2}-\sqrt{5x-4}-\sqrt{5-x}$$
$$=2\sqrt{5x^2-4x}-2\sqrt{5x-x^2}+\sqrt{5x-4}-\sqrt{5-x}$$
$$4\sqrt{5x-x^2}=2\sqrt{5x-4}$$
$$2\sqrt{5x-x^2}=\sqrt{5x-4}$$
Square,
$$20x-4x^2=5x-4$$
$$4x^2-15x=4$$
$$4x^2-15x+\tfrac{225}{16}=\tfrac{289}{16}$$
$$2x-\tfrac{15}{4}=\pm\tfrac{17}{4}$$
$$2x=8,\text{ or }-\tfrac{1}{2}$$
$$\therefore\ x=4,\text{ or }-\tfrac{1}{4}$$

30. Solve $\sqrt{(x+a)^2+2ab+b^2}+x+a=b$
$$\sqrt{(x+a)^2+2ab+b^2}=b-a-x$$
Square,
$$(x+a)^2+2ab+b^2=b^2-2ab-2bx+(x+a)^2$$
$$2bx=-4ab$$
$$\therefore\ x=-2a$$

31. Solve
$$\frac{\sqrt{3}}{\sqrt{2x-1}-\sqrt{x-2}}=\frac{1}{\sqrt{x-1}}$$
Invert,
$$\frac{\sqrt{2x-1}-\sqrt{x-2}}{\sqrt{3}}=\sqrt{x-1}$$

Square,
$$\frac{3x-3-2\sqrt{2x^2-5x+2}}{3}=x-1$$
$$-2\sqrt{2x^2-5x+2}=0$$
$$2x^2-5x+2=0$$
$$(2x-1)(x-2)=0$$
$$\therefore\ x=\tfrac{1}{2},\ \text{or}\ 2$$

32. Solve
$$\sqrt{\frac{x}{4}+3}+\sqrt{\frac{x}{4}-3}=\sqrt{\frac{2x}{3}}$$

Square,
$$\frac{x}{2}+2\sqrt{\frac{x^2}{16}-9}=\frac{2x}{3}$$
$$2\sqrt{\frac{x^2}{16}-9}=\frac{x}{6}$$

Square,
$$\frac{x^2}{4}-36=\frac{x^2}{36}$$
$$\frac{8x^2}{36}=36$$
$$x^2=\frac{36^2}{8}$$
$$x^2=162$$
$$\therefore\ x=\pm 9\sqrt{2}$$

33. Solve
$$\sqrt{1+\frac{x}{a}}-\sqrt{1-\frac{a}{x}}=1$$

Square,
$$2+\frac{x}{a}-\frac{a}{x}-2\sqrt{\frac{x}{a}-\frac{a}{x}}=1$$
$$\frac{x}{a}-\frac{a}{x}-2\sqrt{\frac{x}{a}-\frac{a}{x}}=-1$$

Let
$$\sqrt{\frac{x}{a}-\frac{a}{x}}=y$$

Then
$$y^2-2y=-1$$
$$y^2-2y+1=0$$
$$(y-1)(y-1)=0$$
$$y=1,\ 1$$
$$\frac{x}{a}-\frac{a}{x}=1$$
$$x^2-a^2=ax$$
$$x^2-ax=a^2$$

$$x^2 - ax + \frac{a^2}{4} = \frac{5a^2}{4}$$

$$x - \frac{a}{2} = \pm\frac{a}{2}\sqrt{5}$$

$$\therefore\ x = \frac{a}{2}(1 \pm \sqrt{5})$$

34. Solve

$$\sqrt{x^2 + a^2 + 3ax} + \sqrt{x^2 + a^2 - 3ax} = \sqrt{2a^2 + 2b^2}$$

Square,

$$2x^2 + 2a^2 + 2\sqrt{(x^2 + a^2)^2 - 9a^2x^2} = 2a^2 + 2b^2$$

$$\sqrt{x^4 + a^4 - 7a^2x^2} = b^2 - x^2$$

Square, $x^4 + a^4 - 7a^2x^2 = b^4 - 2b^2x^2 + x^4$

$$2b^2x^2 - 7a^2x^2 = b^4 - a^4$$

$$x^2 = \frac{a^4 - b^4}{7a^2 - 2b^2}$$

$$\therefore\ x = \pm\sqrt{\frac{a^4 - b^4}{7a^2 - 2b^2}}$$

35. Solve $4x^{\frac{1}{2}} - 3(x^{\frac{1}{2}} + 1)(x^{\frac{1}{2}} - 2) = x^{\frac{1}{2}}(10 - 3x^{\frac{1}{2}})$

$$4x^{\frac{1}{2}} - 3x + 3x^{\frac{1}{2}} + 6 = 10x^{\frac{1}{2}} - 3x$$

$$-3x^{\frac{1}{2}} = -6$$

$$x^{\frac{1}{2}} = 2$$

$$\therefore\ x = 4$$

36. Solve $(x^{\frac{2}{3}} - 2)(x^{\frac{4}{3}} - 4) = x^{\frac{2}{3}}(x^{\frac{2}{3}} - 1)^2 - 12$

$$x^2 - 2x^{\frac{4}{3}} - 4x^{\frac{2}{3}} + 8 = x^2 - 2x^{\frac{4}{3}} + x^{\frac{2}{3}} - 12$$

$$5x^{\frac{2}{3}} = 20$$

$$x^{\frac{2}{3}} = 4$$

$$x^{\frac{1}{3}} = \pm 2$$

$$\therefore\ x = \pm 8$$

37. Solve

$$3\sqrt{x^3 + 17} + \sqrt{x^3 + 1} + 2\sqrt{5x^3 + 41} = 0$$

$$3\sqrt{x^3 + 17} + \sqrt{x^3 + 1} = -2\sqrt{5x^3 + 41}$$

$$10x^3 + 154 + 6\sqrt{(x^3 + 17)(x^3 + 1)} = 20x^3 + 164$$

$$6\sqrt{(x^3 + 17)(x^3 + 1)} = 10x^3 + 10$$

$$6\sqrt{x^3 + 17}\sqrt{x^3 + 1} = 10(\sqrt{x^3 + 1})^2$$

$$\therefore \sqrt{x^3+1}=0,$$

$$\text{or } 6\sqrt{x^3+17}=10\sqrt{x^3+1}$$

Since $$\sqrt{x^3+1}=0$$

$$x^3+1=0$$

$$x^3=-1$$

$$x=-1,\ \tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{-3}$$

Since $$6\sqrt{x^3+17}=10\sqrt{x^3+1}$$

$$36x^3+612=100x^3+100$$

$$64x^3=512$$

$$x^3=8$$

$$\therefore x=2,\ -1\pm\sqrt{-3}$$

38. Solve $$\frac{1}{2}-\frac{3}{x}=\sqrt{\frac{1}{4}-\frac{1}{x}\sqrt{9-\frac{36}{x}}}$$

Square, $$\frac{1}{4}-\frac{3}{x}+\frac{9}{x^2}=\frac{1}{4}-\frac{1}{x}\sqrt{9-\frac{36}{x}}$$

$$-\frac{3}{x}+\frac{9}{x^2}=-\frac{3}{x}\sqrt{1-\frac{4}{x}}$$

Divide by $-\frac{3}{x}$, $$1-\frac{3}{x}=\sqrt{1-\frac{4}{x}}$$

Square, $$1-\frac{6}{x}+\frac{9}{x^2}=1-\frac{4}{x}$$

$$\frac{9}{x^2}-\frac{2}{x}=0$$

$$9-2x=0$$

$$\therefore x=\tfrac{9}{2}$$

39. Solve $$\frac{2}{x+\sqrt{2-x^2}}+\frac{2}{x-\sqrt{2-x^2}}=x$$

Reduce left side to a common denominator.

$$\frac{4x}{x^2-2+x^2}=x$$

$$\frac{4x}{2x^2-2}=x$$

$$\therefore x=0,$$

$$\text{or } \frac{2}{x^2-1}=1$$

$$x^2-1=2$$

$$x^2=3$$

$$x=\pm\sqrt{3}$$

$$\therefore x=0, \text{ or } \pm\sqrt{3}$$

40. Solve $$\frac{1}{1+\sqrt{1-x}}+\frac{1}{1-\sqrt{1-x}}=\frac{2x}{9}.$$

Reduce left side to a common denominator.

$$\frac{2}{x}=\frac{2x}{9}$$
$$x^2=9$$
$$\therefore\ x=\pm 3$$

41. Solve $$\frac{\sqrt{ax+b}+\sqrt{ax}}{\sqrt{ax+b}-\sqrt{ax}}=\frac{1+\sqrt{ax-b}}{1-\sqrt{ax-b}}$$

Simplify,

$$\sqrt{ax+b}+\sqrt{ax}-\sqrt{a^2x^2-b^2}-\sqrt{ax}\sqrt{ax-b}$$
$$=\sqrt{ax+b}-\sqrt{ax}+\sqrt{a^2x^2-b^2}-\sqrt{ax}\sqrt{ax-b}$$
$$2\sqrt{ax}=2\sqrt{a^2x^2-b^2}$$
$$ax=a^2x^2-b^2$$
$$a^2x^2-ax+\tfrac{1}{4}=b^2+\tfrac{1}{4}$$
$$ax-\tfrac{1}{2}=\pm\tfrac{1}{2}\sqrt{1+4b^2}$$
$$ax=\frac{1\pm\sqrt{1+4b^2}}{2}$$
$$x=\frac{1\pm\sqrt{1+4b^2}}{2a}$$

42. Solve $$\frac{\sqrt{a-x}+\sqrt{b-x}}{\sqrt{a-x}-\sqrt{b-x}}=\frac{\sqrt{x}+\sqrt{b}}{\sqrt{x}-\sqrt{b}}$$

Simplify, $$\sqrt{x}\sqrt{a-x}+\sqrt{x}\sqrt{b-x}-\sqrt{b}\sqrt{a-x}-\sqrt{b}\sqrt{b-x}$$
$$=\sqrt{x}\sqrt{a-x}-\sqrt{x}\sqrt{b-x}+\sqrt{b}\sqrt{a-x}-\sqrt{b}\sqrt{b-x}$$
$$2\sqrt{x}\sqrt{b-x}=2\sqrt{b}\sqrt{a-x}$$
$$bx-x^2=ba-bx$$
$$x^2-2bx=-ab$$
$$x^2-2bx+b^2=b^2-ab$$
$$x-b=\pm\sqrt{b^2-ab}$$
$$\therefore\ x=b\pm\sqrt{b^2-ab}$$

43. Solve $$\sqrt{x}+\sqrt{a-\sqrt{ax+x^2}}=\sqrt{a}$$
$$\sqrt{a-\sqrt{ax+x^2}}=\sqrt{a}-\sqrt{x}$$
$$a-\sqrt{ax+x^2}=a-2\sqrt{ax}+x$$
$$-\sqrt{ax+x^2}=x-2\sqrt{ax}$$

$$ax + x^2 = x^2 - 4x\sqrt{ax} + 4ax$$
$$3ax = 4x\sqrt{ax}$$
$$\therefore\ x = 0,$$
$$\text{or } 3a = 4\sqrt{ax}$$
$$9a^2 = 16ax$$
$$x = \tfrac{9}{16}a$$
$$\therefore\ x = 0, \text{ or } \tfrac{9}{16}a$$

44. Solve
$$\begin{cases} x^2 + y^2 + x + y = 48 & (1) \\ xy = 12 & (2) \end{cases}$$

$2 \times (2)$ is
$$2xy = 24 \qquad (3)$$

Add (1) and (3),
$$x^2 + 2xy + y^2 + x + y = 72$$
$$(x + y)^2 + (x + y) + \tfrac{1}{4} = 72\tfrac{1}{4}$$
$$x + y + \tfrac{1}{2} = \pm 8\tfrac{1}{2}$$
$$x + y = 8, \text{ or } -9$$

Substitute value of $x + y$ in (1),
$$x^2 + y^2 = 40, \text{ or } 57$$

$2 \times (2)$ is
$$2xy = 24$$

Subtract,
$$x^2 - 2xy + y^2 = 16, \text{ or } 33$$
$$x - y = \pm 4, \text{ or } \pm\sqrt{33}$$
$$\therefore\ 2x = 12,\ 4, \text{ or } -9 \pm \sqrt{33}$$
$$x = 6,\ 2, \text{ or } \frac{-9 \pm \sqrt{33}}{2}$$
$$y = 2,\ 6, \text{ or } \frac{-9 \mp \sqrt{33}}{2}$$

45. Solve
$$\begin{cases} x + y + \sqrt{x + y} = a & (1) \\ x - y + \sqrt{x - y} = b & (2) \end{cases}$$

From (1),
$$x + y + \sqrt{x + y} + \tfrac{1}{4} = a + \tfrac{1}{4}$$
$$\sqrt{x + y} + \tfrac{1}{2} = \pm\tfrac{1}{2}\sqrt{4a + 1}$$
$$\sqrt{x + y} = \frac{-1 \pm \sqrt{4a + 1}}{2}$$
$$x + y = \frac{2 + 4a \pm 2\sqrt{4a + 1}}{4}$$
$$x + y = \frac{1 + 2a \pm \sqrt{1 + 4a}}{2}$$

From (2),
$$x - y + \sqrt{x - y} + \tfrac{1}{4} = \frac{1 + 4b}{4}$$
$$\sqrt{x - y} + \tfrac{1}{2} = \pm\tfrac{1}{2}\sqrt{1 + 4b}$$

$$\sqrt{x-y}=\frac{-1\pm\sqrt{1+4b}}{2}$$

$$x-y=\frac{1+2b\pm\sqrt{1+4b}}{2}$$

$$\therefore\ 2x=\frac{2+2a+2b\pm\sqrt{1+4a}\pm\sqrt{1+4b}}{2}$$

$$x=\frac{2+2a+2b\pm\sqrt{1+4a}\pm\sqrt{1+4b}}{4}$$

$$y=\frac{2a-2b\pm\sqrt{1+4a}\mp\sqrt{1+4b}}{4}$$

46. Solve
$$\begin{cases} x^2+xy+y^2=a^2 & (1)\\ x+\sqrt{xy}+y=b \end{cases}$$

(1) is $(x+\sqrt{xy}+y)(x-\sqrt{xy}+y)=a^2$

$$b(x-\sqrt{xy}+y)=a^2$$

$$x-\sqrt{xy}+y=\frac{a^2}{b}$$

(1) is $x+\sqrt{xy}+y=b$

Subtract, $2\sqrt{xy}=b-\frac{a^2}{b}$

$$\sqrt{xy}=\frac{b^2-a^2}{2b}$$

$$xy=\frac{b^4-2a^2b^2+a^4}{4b^2} \qquad (3)$$

Substitute value of $\sqrt{xy}$ in (2), $x+y=b-\frac{b^2-a^2}{2b}$ (4)

$$x+y=\frac{a^2+b^2}{2b}$$

(1) is $x^2+xy+y^2=a^2$

$3\times(3)$ is $3xy=\frac{3b^4-6a^2b^2+3a^4}{4b^2}$

Subtract, $x^2-2xy+y^2=\frac{10a^2b^2-3a^4-3b^4}{4b^2}$

$$x-y=\pm\frac{1}{2b}\sqrt{10a^2b^2-3a^4-3b^4} \qquad (5)$$

From (4) and (5), $x=\frac{a^2+b^2\pm\sqrt{10a^2b^2-3a^4-3b^4}}{4b}$

$$y=\frac{a^2+b^2\mp\sqrt{10a^2b^2-3a^4-3b^4}}{4b}$$

47. Solve

$$\begin{cases} \dfrac{3\sqrt{x}+2\sqrt{y}}{4\sqrt{x}-2\sqrt{y}}=6 & (1) \\[2ex] \dfrac{x^2+1}{16}=\dfrac{y^2-64}{x^2} & (2) \end{cases}$$

Simplify (1),

$$3\sqrt{x}+2\sqrt{y}=24\sqrt{x}-12\sqrt{y}$$

$$14\sqrt{y}=21\sqrt{x}$$

$$2\sqrt{y}=3\sqrt{x}$$

$$4y=9x$$

$$y=\tfrac{9}{4}x \qquad (3)$$

Substitute value of y in (2),

$$\frac{x^2+1}{16}=\frac{\frac{81}{16}x^2-64}{x^2}.$$

$$\frac{x^2}{16}+\frac{1}{16}=\frac{81}{16}-\frac{64}{x^2}$$

$$\frac{x^2}{16}=5-\frac{64}{x^2}$$

$$x^4-80x^2=-1024$$

$$x^4-80x^2+1600=576$$

$$x^2-40=\pm 24$$

$$x^2=64, \text{ or } 16$$

$$\therefore\ x=\pm\ 8, \text{ or } \pm 4$$

From (3),

$$y=\pm 18, \text{ or } \pm 9$$

48. Solve

$$\begin{cases} \sqrt{x}-\sqrt{y}=x^{\frac{1}{2}}(\sqrt{x}+\sqrt{y}) & (1) \\ (x+y)^2=2(x-y)^2 & (2) \end{cases}$$

Expand (2),

$$x^2+2xy+y^2=2x^2-4xy+2y^2$$

Hence

$$6xy=x^2+y^2 \qquad (3)$$

But

$$2xy=\qquad 2xy$$

Therefore

$$4xy=x^2-2xy+y^2$$

and

$$2\sqrt{xy}=\pm(x-y) \qquad (4)$$

Square (1),

$$x-2\sqrt{xy}+y=x(x+2\sqrt{xy}+y) \qquad (5)$$

Substitute $+(x-y)$ for $2\sqrt{xy}$ in (5), and we have

$$x-x+y+y=x(x+x-y+y)$$

$$\therefore\ y=x^2$$

Substitute the value of y in (3), and we have

$$x^2(6x) = x^2(1 + x^2)$$

$$\therefore\ x = 0, \text{ and } \therefore\ y = 0$$

or $$6x = 1 + x^2$$

That is, $$x^2 - 6x = -1$$

$$x^2 - (\ ) + 9 = 8$$

$$x - 3 = \pm 2\sqrt{2}$$

$$x = 3 \pm 2\sqrt{2}$$

and $$y = (3 \pm 2\sqrt{2})^2$$

49. Solve

$$\begin{cases} \sqrt{\dfrac{3x}{x+y}} + \sqrt{\dfrac{x+y}{3x}} = 2 & (1) \\ x + y = xy - 54 & (2) \end{cases}$$

Let $$u = \sqrt{\frac{3x}{x+y}}$$

$$\therefore\ \frac{1}{u} = \sqrt{\frac{x+y}{3x}}$$

From (1), $$u + \frac{1}{u} = 2$$

$$u^2 - 2u + 1 = 0$$

$$u = 1$$

$$\sqrt{\frac{3x}{x+y}} = 1$$

$$\frac{3x}{x+y} = 1$$

$$3x = x + y$$

$$\therefore\ y = 2x$$

Substitute value of y in (2), $$3x = 2x^2 - 54$$

$$2x^2 - 3x - 54 = 0$$

$$(2x + 9)(x - 6) = 0$$

$$\therefore\ x = 6, \text{ or } -\tfrac{9}{2}$$

$$y = 12, \text{ or } -9$$

Exercise 28.

Form the equations of which the roots are:

1. $3, 2.$

$$(x-3)(x-2)=0$$
$$x^2-5x+6=0$$

2. $4, -5.$

$$(x-4)(x+5)=0$$
$$x^2+x-20=0$$

3. $-6, -8.$

$$(x+6)(x+8)=0$$
$$x^2+14x+48=0$$

4. $\frac{2}{3}, \frac{1}{2}.$

$$(x-\tfrac{2}{3})(x-\tfrac{1}{2})=0$$
$$(3x-2)(2x-1)=0$$
$$6x^2-7x+2=0$$

5. $-\frac{1}{3}, -\frac{3}{4}.$

$$(x+\tfrac{1}{3})(x+\tfrac{3}{4})=0$$
$$x^2+\tfrac{13}{12}x+\tfrac{1}{4}=0$$
$$12x^2+13x+3=0$$

6. $a+3b, a-3b.$

$$(x-a-3b)(x-a+3b)=0$$
$$x^2-2ax+a^2-9b^2=0$$

7. $\frac{a+2b}{3}, \frac{2a+b}{3}.$

$$\left(x-\frac{a+2b}{3}\right)\left(x-\frac{2a+b}{3}\right)=0$$
$$x^2-(a+b)x+\frac{2a^2+5ab+2b^2}{9}=0$$
$$9x^2-9(a+b)x+2a^2+5ab+2b^2=0$$

8. $2+\sqrt{3}, 2-\sqrt{3}.$

$$(x-2-\sqrt{3})(x-2+\sqrt{3})=0$$
$$x^2-4x+1=0$$

9. $-1+\sqrt{5}, -1-\sqrt{5}.$

$$(x+1-\sqrt{5})(x+1+\sqrt{5})=0$$
$$x^2+2x-4=0$$

10. $1+\sqrt{\frac{2}{3}}, 1-\sqrt{\frac{2}{3}}.$

$$(x-1-\sqrt{\tfrac{2}{3}})(x-1+\sqrt{\tfrac{2}{3}})=0$$
$$x^2-2x+\tfrac{1}{3}=0$$
$$3x^2-6x+1=0$$

Resolve into factors, real or imaginary:

11. $3x^2-15x-42=3(x^2-5x-14)=3(x-7)(x+2).$

12. $9x^2-27x-70=(3x-14)(3x+5).$

13. $49x^2+49x+6=(7x+6)(7x+1).$

14. $169x^2-52x+4=(13x-2)^2.$

15. $x^2 - 3x + 4.$

Solve
$$x^2 - 3x + 4 = 0$$
$$\therefore\ x = \frac{3 \pm \sqrt{-7}}{2}$$
$$\therefore\ x^2 - 3x + 4 = \left(x - \frac{3 + \sqrt{-7}}{2}\right)\left(x - \frac{3 - \sqrt{-7}}{2}\right)$$

16. $x^2 + x + 1.$

Solve
$$x^2 + x + 1 = 0$$
$$\therefore\ x = \frac{-1 \pm \sqrt{-3}}{2}$$
$$\therefore\ x^2 + x + 1 = \left(x + \frac{1 - \sqrt{-3}}{2}\right)\left(x + \frac{1 + \sqrt{-3}}{2}\right)$$

17. $4x^2 - 28x + 49 = (2x - 7)^2.$

18. $4x^2 + 12x + 13.$

Solve
$$4x^2 + 12x + 13 = 0$$
$$\therefore\ x = \frac{-3 \pm 2\sqrt{-1}}{2}$$
$$\therefore\ 4x^2 + 12x + 13 = (2x + 3 - 2\sqrt{-1})(2x + 3 + 2\sqrt{-1})$$

In examples 19–27, α and β are to be taken as the roots of the equation $x^2 - 7x + 8 = 0$.

Find the values of:

19. $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = 49 - 32 = 17.$

20. $\alpha^2\beta + \alpha\beta^2 = \alpha\beta(\alpha + \beta) = 8 \times 7 = 56.$

21. $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} = \dfrac{\alpha^2 + \beta^2}{\alpha^2\beta^2} = \dfrac{(\alpha + \beta)^2 - 2\alpha\beta}{(\alpha\beta)^2} = \dfrac{49 - 16}{64} = \dfrac{33}{64}.$

22. $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha} = \dfrac{\alpha^2 + \beta^2}{\alpha\beta} = \dfrac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta} = \dfrac{49 - 16}{8} = \dfrac{33}{8}.$

23. $\dfrac{\alpha}{\beta^2} + \dfrac{\beta}{\alpha^2} = \dfrac{\alpha^3 + \beta^3}{\alpha^2\beta^2} = \dfrac{(\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2)}{(\alpha\beta)^2} = \dfrac{(\alpha + \beta)[(\alpha + \beta)^2 - 3\alpha\beta]}{(\alpha\beta)^2}$

$$= \frac{7 \times (49 - 24)}{64} = \frac{175}{64}.$$

24. $\dfrac{\alpha^2 + \beta^2}{\alpha + \beta} = \dfrac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha + \beta} = \dfrac{49 - 16}{7} = \dfrac{33}{7}.$

25. $\frac{1}{\alpha^3}+\frac{1}{\beta^3}=\frac{\alpha^3+\beta^3}{\alpha^3\beta^3}=\frac{(\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)}{\alpha^3\beta^3}=\frac{(\alpha+\beta)[(\alpha+\beta)^2-3\alpha\beta]}{(\alpha\beta)^3}$

$$=\frac{7\times(49-24)}{512}=\frac{175}{512}.$$

26. $(\alpha^2-\beta^2)^2=(\alpha+\beta)^2(\alpha-\beta)^2=(\alpha+\beta)^2[(\alpha+\beta)^2-4\alpha\beta]$

$$=49(49-32)=833.$$

27. $\frac{\alpha^2}{\beta^2}+\frac{\beta^2}{\alpha^2}=\frac{\alpha^4+\beta^4}{\alpha^2\beta^2}=\frac{(\alpha^2+\beta^2)^2-2\alpha^2\beta^2}{\alpha^2\beta^2}=\frac{[(\alpha+\beta)^2-2\alpha\beta]^2-2\alpha^2\beta^2}{\alpha^2\beta^2}$

$$=\frac{(49-16)^2-128}{64}=\frac{961}{64}.$$

In examples 28–33, α and β are to be taken as the roots of the equation $x^2+px+q=0$. The results are to be found in terms of p and q.

Find the values of:

28. $\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}=-\frac{p}{q}.$

29. $\alpha^2\beta+\alpha\beta^2=\alpha\beta(\alpha+\beta)=-pq.$

30. $\alpha^3+\beta^3=(\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)=(\alpha+\beta)[(\alpha+\beta)^2-3\alpha\beta]$

$$=-p(p^2-3q).$$

31. $\alpha^3\beta+\alpha\beta^3=\alpha\beta(\alpha^2+\beta^2)=\alpha\beta[(\alpha+\beta)^2-2\alpha\beta]=q(p^2-2q).$

32. $\alpha^4+\beta^4=(\alpha^2+\beta^2)^2-2\alpha^2\beta^2=[(\alpha+\beta)^2-2\alpha\beta]^2-2\alpha^2\beta^2$

$$=(p^2-2q)^2-2q^2=p^4-4p^2q+2q^2.$$

33. $\frac{\alpha^2}{\beta^2}+\frac{\beta^2}{\alpha^2}=\frac{\alpha^4+\beta^4}{\alpha^2\beta^2}=\frac{(\alpha^2+\beta^2)^2-2\alpha^2\beta^2}{\alpha^2\beta^2}=\frac{[(\alpha+\beta)^2-2\alpha\beta]^2-2\alpha^2\beta^2}{\alpha^2\beta^2}$

$$=\frac{(p^2-2q)^2-2q^2}{q^2}=\frac{p^4-4p^2q+2q^2}{q^2}.$$

34. When will the roots of the equation $ax^2+bx+c=0$ be both positive? Both negative? One positive and one negative?

If α and β represent the roots we have:

$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a},\quad \beta=\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

In order that the roots may be real b^2-4ac must be positive.

This condition being fulfilled, a and c may have either the same or opposite signs. If they have the same sign, $4ac$ is positive and $b^2-4ac<b^2$, so that $\sqrt{b^2-4ac}<b$. Then, if b is positive, $-b+\sqrt{b^2-4ac}$ and $-b-\sqrt{b^2-4ac}$ are both negative, and, if a is positive, α and β are both negative, while if a is negative, α and β are both positive.

If a and c have the same sign, but b is negative, $-b+\sqrt{b^2-4ac}$ and $-b-\sqrt{b^2-4ac}$ are both positive, and α and β are both positive if a is positive, and both negative if a is negative.

Hence, if a and c have the same signs, α and β are both positive if a and b have opposite signs, and both negative if a and b have the same sign.

If a and c have opposite signs, $4ac$ is negative, and $b^2-4ac<b^2$, so that $\sqrt{b^2-4ac}>b$. In this case $-b+\sqrt{b^2-4ac}$ is always positive, and $-b-\sqrt{b^2-4ac}$ is always negative, so that α and β have opposite signs.

Arranging the results we have:

If $b^2-4ac>0$ and if a, b, c have same sign, both roots are negative.

If $b^2-4ac>0$ and if $\left\{\begin{array}{l}a \text{ and } c \text{ have same sign} \\ b \text{ has opposite sign}\end{array}\right\}$, both roots are positive.

If a and c have opposite signs one root is positive and one negative.

35. When will one root be the square of the other?

Let
$$\alpha=\beta^2$$

$$\frac{-b+\sqrt{b^2-4ac}}{2a}=\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)^2$$

$$\frac{-b+\sqrt{b^2-4ac}}{2a}=\frac{b^2+2b\sqrt{b^2-4ac}+b^2-4ac}{4a^2}$$

$$-ab+a\sqrt{b^2-4ac}=b^2-2ac+b\sqrt{b^2-4ac}$$

$$(a-b)\sqrt{b^2-4ac}=b^2-2ac+ab$$

$$(a^2-2ab+b^2)(b^2-4ac)=(b^2-2ac+ab)^2$$

$$a^2b^2-4a^3c-2ab^3+8a^2bc+b^4-4ab^2c$$
$$=b^4-4ab^2c+4a^2c^2+2ab^3-4a^2bc+a^2b^2$$

$$12a^2bc-4a^3c-4ab^3-4a^2c^2=0$$

$$a(3abc-a^2c-b^3-ac^2)=0$$

$a=0$ evidently does not satisfy the requirements.

$$\therefore\ 3abc-b^3-a^2c-ac^2=0$$

$$b^3+a^2c+ac^2=3abc$$

The same result is obtained if we put $\beta=\alpha^2$.

36. When will the sum of the reciprocals of the roots be unity.

$$\frac{1}{\alpha}+\frac{1}{\beta}=1$$

$$\frac{\alpha+\beta}{\alpha\beta}=1$$

$$\alpha+\beta=-b$$

$$\alpha\beta=c$$

$$\therefore\ -\frac{b}{c}=1$$

$$b=-c$$

$$b+c=0$$

37. Show that the roots of the equation

$$x^2+2(a+b)x+2(a^2+b^2)=0$$

are imaginary if a and b are real and unequal.

Here, instead of $\sqrt{b^2-4ac}$, we have

$$\sqrt{4(a+b)^2-8(a^2+b^2)}=\sqrt{8ab-4a^2-4b^2}=\sqrt{-4(a-b)^2}.$$

But if a and b are real and unequal $(a-b)^2$ is positive and not equal to 0.

$\therefore\ \sqrt{-4(a-b)^2}$ is imaginary, and consequently the roots of the given equation are imaginary.

38. Show that the roots of the equation

$$-x^2+(x-b)(x-c)+(x-c)(x-a)+(x-a)(x-b)=0$$

are real if a, b, and c are real.

Simplify, $\quad -4x^2-2(a+b+c)x+ab+bc+ac=0.$

Here, instead of $\sqrt{b^2-4ac}$, we have

$$\sqrt{4(a+b+c)^2+16(ab+bc+ac)}=\sqrt{4(a^2+b^2+c^2+2ab+2ac+2bc)}$$
$$=\pm 2(a+b+c),$$

which is real if a, b, and c are real.

39. Show that the equations $ax^2+bx+c=0$, $a'x+c'=0$, will have a common root if $\frac{a}{a'^2}+\frac{c}{c'^2}=\frac{b}{a'c'}$.

The equation $a'x+c'=0$ has only the one root,

$$x=-\frac{c'}{a'}$$

If this is also a root of the equation $ax^2+bx+c=0$,

then,
$$a\left(-\frac{c'}{a'}\right)^2+b\left(-\frac{c'}{a'}\right)+c=0$$

$$\frac{ac'^2}{a'^2}-\frac{bc'}{a'}+c=0$$

$$\therefore\ \frac{a}{a'^2}-\frac{b}{a'c'}+\frac{c}{c'^2}=0$$

$$\frac{a}{a'^2}+\frac{c}{c'^2}=\frac{b}{a'c'}$$

40. Show that the equations $ax^2+bx+c=0$, $a'x^2+b'x+c'=0$, will have a common root if $(a'c-ac')^2=(b'c-bc')(a'b-ab')$.

Since the two equations have a common root, they are true simultaneously.

$$\therefore\ ax^2+bx+c=0 \qquad (1)$$

$$a'x^2+b'x+c'=0 \qquad (2)$$

Multiply (1) by a', $\quad aa'x^2+a'bx+a'c=0$

Multiply (2) by a, $\quad aa'x^2+ab'x+ac'=0$

Subtract, $\quad (a'b-ab')x+(a'c-ac')=0$

$$\therefore\ x=\frac{ac'-a'c}{ab'-a'b}$$

This is the common root. Substitute this value of x in either equation, say (1).

$$a\left(\frac{ac'-a'c}{ab'-a'b}\right)^2+b\left(\frac{a'c-ac'}{ab'-a'b}\right)+c=0$$

$$a(ac'-a'c)^2+b(a'c-ac')(ab'-a'b)+c(ab'-a'b)^2=0$$

$$a(ac'-a'c)^2+(ab'-a'b)(a'bc-abc'+ab'c-a'bc)=0$$

$$a(ac'-a'c)^2+(ab'-a'b)(ab'c-abc')=0$$

$$\therefore\ (ac'-a'c)^2=(a'b-ab')(b'c-bc')$$

Exercise 29.

For what values of m are the two roots of each of the following equations (1) equal, (2) real and unequal, (3) imaginary?

1. $(3m+1)x^2+2(m+1)x+m=0.$

$$b^2-4ac=4(m+1)^2-4m(3m+1)$$

$$=4+4m-8m^2$$

$$=8(1-m)(\tfrac{1}{2}+m)$$

$\therefore$ The roots are equal if $m = 1$, or $-\frac{1}{2}$,
real and unequal if m lies between 1 and $-\frac{1}{2}$,
imaginary if $m > 1$, or $< -\frac{1}{2}$.

2. $(m-2)x^2 + (m-5)x + 2m - 5 = 0$.

$$\begin{aligned} b^2 - 4ac &= (m-5)^2 - 4(m-2)(2m-5) \\ &= -7m^2 + 26m - 15 \\ &= 7(\tfrac{5}{7} - m)(m-3) \end{aligned}$$

$\therefore$ The roots are equal if $m = \frac{5}{7}$, or 3,
real and unequal if m lies between $\frac{5}{7}$ and 3,
imaginary if $m > 3$, or $< \frac{5}{7}$.

3. $2mx^2 + x^2 - 6mx - 6x + 6m + 1 = 0$.

$$\begin{aligned} (2m+1)x^2 - 6(m+1)x + 6m + 1 &= 0 \\ b^2 - 4ac &= 36(m+1)^2 - 4(2m+1)(6m+1) \\ &= -12m^2 + 40m + 32 \\ &= 12(m + \tfrac{2}{3})(4 - m) \end{aligned}$$

$\therefore$ The roots are equal if $m = 4$, or $-\frac{2}{3}$,
real and unequal if m lies between 4 and $-\frac{2}{3}$,
imaginary if $m > 4$, or $< -\frac{2}{3}$.

4. $mx^2 + 2x^2 + 2m - 3mx + 9x - 10 = 0$.

$$\begin{aligned} (m+2)x^2 - 3(m-3)x + 2m - 10 &= 0 \\ b^2 - 4ac &= 9(m-3)^2 - 4(m+2)(2m-10) \\ &= m^2 - 30m + 161 \\ &= (m-7)(m-23) \end{aligned}$$

$\therefore$ The roots are equal if $m = 7$, or 23,
real and unequal if $m > 23$, or < 7,
imaginary if m lies between 7 and 23.

5. $6mx^2 + 8mx + 2m = 2x - x^2 - 1$.

$$\begin{aligned} (6m+1)x^2 + 2(4m-1)x + 2m + 1 &= 0 \\ b^2 - 4ac &= 4(4m-1)^2 - 4(6m+1)(2m+1) \\ &= 4(4m^2 - 16m) \\ &= 16(m^2 - 4m) \\ &= 16m(m-4) \end{aligned}$$

$\therefore$ The roots are equal if $m = +4$, or 0,
real and unequal if m is > 4, or < 0,
imaginary if m lies between 0 and 4.

Find the maximum or minimum value of each of the following expressions, and determine which:

6. $x^2 - 6x + 13$.

Let $x^2 - 6x + 13 = m$
Then $x^2 - 6x + 9 = m - 4$
$x - 3 = \pm\sqrt{m-4}$
$x = \pm 3\sqrt{m-4}$

$\therefore$ m is not less than 4.
The minimum value is 4.

7. $4x^2 - 12x + 16$.

Let $4x^2 - 12x + 16 = m$
Then $4x^2 - 12x + 9 = m - 7$
$2x - 3 = \pm\sqrt{m-7}$
$x = \dfrac{3 \pm \sqrt{m-7}}{2}$

$\therefore$ m is not less than 7.
The minimum value is 7.

8. $3 + 12x - 9x^2$.

Let $3 + 12x - 9x^2 = m$
Then $9x^2 - 12x = 3 - m$
$9x^2 - 12x + 4 = 7 - m$
$3x - 2 = \pm\sqrt{7-m}$
$x = \dfrac{2 \pm \sqrt{7-m}}{3}$

$\therefore$ m is not greater than 7.
The maximum value is 7.

9. $x^2 + 8x + 20$.

Let $x^2 + 8x + 20 = m$
Then $x^2 + 8x + 16 = m - 4$
$x + 4 = \pm\sqrt{m-4}$
$x = -4 \pm \sqrt{m-4}$

$\therefore$ m is not less than 4.
The minimum value is 4.

10. $4x^2 - 12x + 25$.

Let $4x^2 - 12x + 25 = m$
Then $4x^2 - 12x + 9 = m - 16$
$2x - 3 = \pm\sqrt{m-16}$
$x = \dfrac{3 \pm \sqrt{m-16}}{2}$

$\therefore$ m is not less than 16.
The minimum value is 16.

11. $25x^2 - 40x - 16$.

Let $25x^2 - 40x - 16 = m$
Then $25x^2 - 40x + 16 = m + 32$
$5x - 4 = \pm\sqrt{m+32}$
$x = \dfrac{4 \pm \sqrt{m+32}}{5}$

$\therefore$ m is not less than -32.
The minimum value is -32.

12. $\dfrac{x-6}{x^2}$.

Let $\dfrac{x-6}{x^2} = m$
Then $x - 6 = mx^2$
$mx^2 - x = -6$
$4m^2x^2 - 4mx + 1 = 1 - 24m$
$2mx - 1 = \pm\sqrt{1-24m}$
$x = \dfrac{1 \pm \sqrt{1-24m}}{2m}$

$\therefore$ m is not greater than $\frac{1}{24}$. The maximum value is $\frac{1}{24}$.

13. $\dfrac{(x+12)(x-3)}{x^2}$.

Let $$\frac{(x+12)(x-3)}{x^2}=m$$

Then $$(x+12)(x-3)=mx^2$$
$$(1-m)x^2+9x-36=0$$
$$(1-m)^2x^2+9(1-m)=36(1-m)$$
$$(1-m)^2x^2+9(1-m)+\tfrac{81}{4}=\tfrac{225}{4}-36m$$
$$(1-m)x+\tfrac{9}{2}=\pm\tfrac{1}{2}\sqrt{225-144m}$$
$$x=\frac{-9\pm\sqrt{225-144m}}{2(1-m)}$$

$\therefore$ m is not greater than $\frac{225}{144}$. The maximum value is $\frac{225}{144}$.

14. $\dfrac{4x}{(x+2)^2}$.

Let $$\frac{4x}{(x+2)^2}=m$$

Then $$4x=mx^2+4mx+4m$$
$$mx^2+(4m-4)x+4m=0$$
$$m^2x^2+4m(m-1)x+4(m-1)^2=4-8m$$
$$mx+2(m-1)=\pm2\sqrt{1-2m}$$
$$x=\frac{2(1-m)\pm2\sqrt{1-2m}}{m}$$

$\therefore$ m is not greater than $\frac{1}{2}$. The maximum value is $\frac{1}{2}$.

15. $\dfrac{x^2-x-1}{x^2-x+1}$.

Let $$\frac{x^2-x-1}{x^2-x+1}=m$$

Then $$(1-m)x^2+(m-1)x-1-m=0$$
$$x^2-x=\frac{m+1}{1-m}$$
$$x^2-x+\tfrac{1}{4}=\frac{3m+5}{4(1-m)}$$
$$x-\tfrac{1}{2}=\pm\tfrac{1}{2}\sqrt{\frac{3m+5}{1-m}}$$
$$x=\frac{1\pm\sqrt{\dfrac{3m+5}{1-m}}}{2}$$

$\therefore$ m is not greater than 1, nor less than $-\frac{5}{3}$.
The maximum value is 1, and the minimum value is $-\frac{5}{3}$.

16. $\dfrac{x^2+2x-3}{x^2-2x+3}.$

Let
$$\frac{x^2+2x-3}{x^2-2x+3}=m$$

Then
$$(1-m)x^2+2(1+m)x-3(1+m)=0$$
$$(1-m)^2x^2+2(1-m^2)=3(1-m^2)$$
$$(1-m)^2x^2+2(1-m^2)+(1+m)^2=4+2m-2m^2$$
$$(1-m)x+1+m=\pm\sqrt{4+2m-2m^2}$$
$$x=\frac{-1-m\pm\sqrt{4+2m-2m^2}}{1-m}$$
$$=\frac{-1-m\pm\sqrt{(2-m)(2+2m)}}{1-m}$$

∴ m is not greater than 2 nor less than -1.
The maximum value is 2, the minimum value is -1.

17. $\dfrac{1}{2+x}-\dfrac{1}{2-x}.$

Let
$$\frac{1}{2+x}-\frac{1}{2-x}=m$$

Then
$$-2x=m(4-x^2)$$
$$mx^2-2x=4m$$
$$m^2x^2-2mx+1=4m^2+1$$
$$mx-1=\pm\sqrt{m^2+1}$$
$$x=\frac{1\pm\sqrt{4m^2+1}}{m}$$

There is no maximum or minimum value.

18. $\dfrac{x^2+3x+5}{x^2+1}.$

Let
$$\frac{x^2+3x+5}{x^2+1}=m$$

Then
$$(1-m)x^2+3x+5-m=0$$
$$(1-m)^2x^2+3(1-m)x+\tfrac{9}{4}=\frac{-11+24m-4m^2}{4}$$
$$(1-m)x+\tfrac{3}{2}=\pm\tfrac{1}{2}\sqrt{-11+24m-4m^2}$$
$$x=\frac{-3\pm\sqrt{(11-2m)(2m-1)}}{2(1-m)}$$

∴ m is not greater than $\frac{11}{2}$ nor less than $\frac{1}{2}$.
The maximum value is $\frac{11}{2}$, the minimum value is $\frac{1}{2}$.

19. $\dfrac{(x+1)^2}{x^2-x+1}$.

Let
$$\frac{(x+1)^2}{x^2-x+1}=m$$

Then
$$(1-m)x^2+(2+m)x+(1-m)=0$$
$$(1-m)^2x^2+(1-m)(2+m)x+\left(\frac{2+m}{2}\right)^2=\frac{12m-3m^2}{4}$$
$$(1-m)x+\frac{2+m}{2}=\pm\tfrac{1}{2}\sqrt{12m-3m^2}$$
$$x=\frac{2+m\pm\sqrt{12m-3m^2}}{2(m-1)}$$
$$=\frac{2+m\pm\sqrt{3m(4-m)}}{2(m-1)}$$

∴ m is not greater than 4, nor less than 0.

The maximum value is 4, the minimum value is 0.

20. $\dfrac{2x^2-2x+5}{x^2-2x+3}$.

Let
$$\frac{2x^2-2x+5}{x^2-2x+3}=m$$

Then
$$(2-m)x^2+2(m-1)x+5-3m=0$$
$$(2-m)^2x^2+2(2-m)(m-1)x+(m-1)^2=-2m^2+9m-9$$
$$(2-m)x+m-1=\pm\sqrt{-2m^2+9m-9}$$
$$x=\frac{1-m\pm\sqrt{-2m^2+9m-9}}{2-m}$$
$$=\frac{1-m\pm\sqrt{(m-3)(3-2m)}}{2-m}$$

∴ m must lie between $\frac{3}{2}$ and 3.

The maximum value is 3, the minimum value is $\frac{3}{2}$.

21. Divide a line $2a$ inches long into two parts such that the rectangle of these parts shall be the greatest possible.

Let x = number of inches in the one part.

Then $2a-x$ = number of inches in the other part,

and $x(2a-x)$ = number of square inches in the area of the rectangle.

Let
$$x(2a-x)=m^2$$
$$x^2-2ax=-m^2$$
$$x^2-2ax+a^2=a^2-m^2$$
$$x-a=\pm\sqrt{a^2-m^2}$$
$$x=a\pm\sqrt{a^2-m^2}$$

$\therefore m^2$ is not greater than a^2. The maximum area is a^2.

If $$x(2a - x) = a^2$$
$$x^2 - 2ax + a^2 = 0$$
$$\therefore x = a$$

$\therefore$ The line must be bisected, and the rectangle is a square.

22. Divide a line 20 inches long into two parts such that the hypotenuse of the right triangle of which the two parts are the legs shall be the least possible.

Let $x =$ number of inches in the one part.

Then $20 - x =$ number of inches in the other part,

and $x^2 + (20 - x)^2 =$ square of the number of inches in the hypotenuse.

Then $x^2 + (20 - x)^2$ is to be a minimum.

Let $$x^2 + (20 - x)^2 = m^2$$
$$2x^2 - 40x + 400 = m^2$$
$$x^2 - 20x + 100 = \frac{m^2 - 200}{2}$$
$$x - 10 = \pm\sqrt{\frac{m^2 - 200}{2}}$$
$$x = 10 \pm \sqrt{\frac{m^2 - 200}{2}}$$

$\therefore m^2$ is not less than 200.

If $$m^2 = 200$$
then $$x = 10$$

$\therefore$ The line must be bisected, and the right triangle is isosceles.

23. Divide $2a$ into two parts such that the sum of their square roots shall be a maximum.

Let $x =$ the one part.

Then $2a - x =$ the other part,

and $\sqrt{x} + \sqrt{2a - x}$ is to be a maximum.

Let $$\sqrt{x} + \sqrt{2a - x} = m$$
$$\sqrt{(2a - x)} = m - \sqrt{x}$$
$$2a - x = m^2 - 2m\sqrt{x} + x$$
$$2(a - x) - m^2 = -2m\sqrt{x}$$
$$4a^2 - 8ax + 4x^2 - 4am^2 + 4m^2x + m^4 = 4m^2x$$
$$4x^2 - 8ax + 4a^2 = m^2(4a - m^2)$$

$$2x - 2a = \pm m\sqrt{4a - m^2}$$

$$x = a \pm \frac{m}{2}\sqrt{4a - m^2}$$

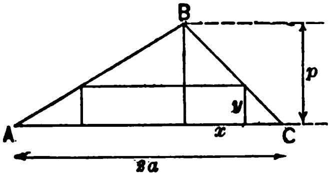
$\therefore$ m is not greater than $2\sqrt{a}$.

If $m = 2\sqrt{a}$

then $x = a$

$\therefore$ The line must be bisected.

24. Find the greatest rectangle that can be inscribed in a given triangle.



Let $2a =$ the base of the triangle,
$p =$ its altitude,
and $2x =$ the base of the rectangle,
$y =$ its altitude,
$2xy =$ its area.

Then, by similar triangles, $\frac{p - y}{2x} = \frac{p}{2a}$

$$p - y = \frac{px}{a}$$

$$y = \frac{p(a - x)}{a}$$

$$2xy = \frac{2px(a - x)}{a}$$

$x(a - x)$ is to be a maximum.

Let $x(a - x) = m$

$$x^2 - ax + \frac{a^2}{4} = \frac{a^2 - 4m}{4}$$

$$x - \frac{a}{2} = \pm \tfrac{1}{2}\sqrt{a^2 - 4m}$$

$$x = \frac{a \pm \sqrt{a^2 - 4m}}{2}$$

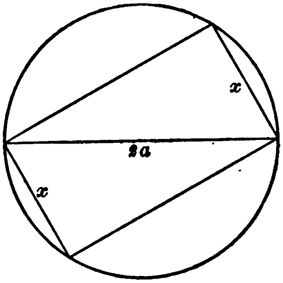
$\therefore$ m is not greater than $\frac{a^2}{4}$.

If $m=\frac{a^2}{4}$, $x=\frac{a}{2}$, $2x=a$, $y=\frac{p}{2}$.

That is, the altitude of the rectangle is half the altitude of the triangle.

The area of the rectangle is $2xy$, or $\frac{ap}{2}$, which is $\frac{1}{2}$ the area of the triangle. It makes no difference therefore which side of the triangle is taken for the base of the rectangle, since the areas of the three maximum rectangles are all equal.

25. Find the greatest rectangle that can be inscribed in a given circle.



Let $2a=$ diameter of the circle,

and $x=$ the length of one side of the rectangle.

Since the angles of the rectangle are right angles, the diagonals are diameters of the circle.

Then, $\sqrt{4a^2-x^2}=$ the length of the other side of the rectangle.

$x\sqrt{4a^2-x^2}=$ the area of the rectangle.

$\therefore$ $x\sqrt{4a^2-x^2}$ is to be a maximum.

Let

$$x\sqrt{4a^2-x^2}=m$$

$$4a^2x^2-x^4=m^2$$

$$x^4-4a^2x^2+4a^4=4a^4-m^2$$

$$x^2-2a^2=\pm\sqrt{4a^4-m^2}$$

$$x^2=2a^2\pm\sqrt{4a^4-m^2}$$

$\therefore$ m is not greater than $2a^2$.

If

$$m=2a^2$$

$$x^2=2a^2$$

$$x=a\sqrt{2}$$

and

$$\sqrt{4a^2-x^2}=a\sqrt{2}$$

The rectangle is the inscribed square.

26. Find the rectangle of greatest perimeter that can be inscribed in a given circle.

See figure for example 25.

Let $2a =$ diameter of the circle.

$x =$ length of one side of rectangle.

Then $\sqrt{4a^2 - x^2} =$ length of the other side of rectangle.

Then $x + \sqrt{4a^2 - x^2}$ is to be a maximum.

Let
$$x + \sqrt{4a^2 - x^2} = 2m$$
$$\sqrt{4a^2 - x^2} = 2m - x$$
$$4a^2 - x^2 = 4m^2 - 4mx + x^2$$
$$x^2 - 2mx + m^2 = 2a^2 - m^2$$
$$x - m = \pm\sqrt{2a^2 - m^2}$$
$$x = m \pm \sqrt{2a^2 - m^2}$$

$\therefore$ m is not greater than $a\sqrt{2}$.

If
$$m = a\sqrt{2}$$
$$x = m = a\sqrt{2}$$
and
$$\sqrt{4a^2 - x^2} = a\sqrt{2}$$

$\therefore$ The rectangle is the inscribed square.

Exercise 30.

Extract the square roots of:

1. $14 + 6\sqrt{5}$.

Let
$$\sqrt{x} + \sqrt{y} = \sqrt{14 + 6\sqrt{5}}$$
Then
$$\sqrt{x} - \sqrt{y} = \sqrt{14 - 6\sqrt{5}}$$
By multiplying,
$$x - y = \sqrt{16}$$
$$x - y = 4$$
But
$$x + y = 14$$
$$\therefore x = 9$$
$$y = 5$$
$$\sqrt{x} + \sqrt{y} = 3 + \sqrt{5}$$

2. $17 + 4\sqrt{15}$.

Let
$$\sqrt{x} + \sqrt{y} = \sqrt{17 + 4\sqrt{15}}$$
Then
$$\sqrt{x} - \sqrt{y} = \sqrt{17 - 4\sqrt{15}}$$
By multiplying,
$$x - y = \sqrt{49}$$
$$x - y = 7$$

But $x + y = 17$
$\therefore x = 12$
$y = 5$
$\sqrt{x} + \sqrt{y} = \sqrt{12} + \sqrt{5}$
$= 2\sqrt{3} + \sqrt{5}$

3. $10 + 2\sqrt{21}$.

Let $\sqrt{x} + \sqrt{y} = \sqrt{10 + 2\sqrt{21}}$
Then $\sqrt{x} - \sqrt{y} = \sqrt{10 - 2\sqrt{21}}$
By multiplying, $x - y = \sqrt{16}$
$x - y = 4$
But $x + y = 10$
$\therefore x = 7$
$y = 3$
$\sqrt{x} + \sqrt{y} = \sqrt{7} + \sqrt{3}$

4. $16 + 2\sqrt{55}$.
Let $\sqrt{x} + \sqrt{y} = \sqrt{16 + 2\sqrt{55}}$
Then $\sqrt{x} - \sqrt{y} = \sqrt{16 - 2\sqrt{55}}$
By multiplying, $x - y = \sqrt{36}$
$x - y = 6$
But $x + y = 16$
$\therefore x = 11$
$y = 5$
$\sqrt{x} + \sqrt{y} = \sqrt{11} + \sqrt{5}$

5. $9 - 2\sqrt{14} = 7 - 2\sqrt{14} + 2.$
$\therefore \sqrt{9 - 2\sqrt{14}} = \sqrt{7} - \sqrt{2}.$

6. $20 - 8\sqrt{6} = 20 - 2\sqrt{96} = 12 - 2\sqrt{96} + 8.$
$\therefore \sqrt{20 - 8\sqrt{6}} = \sqrt{12} - \sqrt{8} = 2\sqrt{3} - 2\sqrt{2}.$

7. $9 - 6\sqrt{2} = 9 - 2\sqrt{18} = 6 - 2\sqrt{18} + 3.$
$\therefore \sqrt{9 - 6\sqrt{2}} = \sqrt{6} - \sqrt{3}.$

8. $94 - 42\sqrt{5} = 94 - 2\sqrt{2205} = 49 - 2\sqrt{2205} + 45.$
$\therefore \sqrt{94 - 42\sqrt{5}} = 7 - \sqrt{45} = 7 - 3\sqrt{5}.$

9. $13 - 2\sqrt{30} = 10 - 2\sqrt{30} + 3.$

$\therefore \sqrt{13 - 2\sqrt{30}} = \sqrt{10} - \sqrt{3}.$

10. $11 - 6\sqrt{2} = 11 - 2\sqrt{18} = 9 - 2\sqrt{18} + 2.$

$\therefore \sqrt{11 - 6\sqrt{2}} = 3 - \sqrt{2}.$

11. $14 - 4\sqrt{6} = 14 - 2\sqrt{24} = 12 - 2\sqrt{24} + 2.$

$\therefore \sqrt{14 - 4\sqrt{6}} = \sqrt{12} - \sqrt{2} = 2\sqrt{3} - \sqrt{2}.$

12. $38 - 12\sqrt{10} = 38 - 2\sqrt{360} = 20 - 2\sqrt{360} + 18.$

$\therefore \sqrt{38 - 12\sqrt{10}} = \sqrt{20} - \sqrt{18} = 2\sqrt{5} - 3\sqrt{2}.$

13. $103 - 12\sqrt{11} = 103 - 2\sqrt{396} = 99 - 2\sqrt{396} + 4.$

$\therefore \sqrt{103 - 12\sqrt{11}} = \sqrt{99} - 2 = 3\sqrt{11} - 2.$

14. $57 - 12\sqrt{15} = 57 - 2\sqrt{540} = 45 - 2\sqrt{540} + 12.$

$\therefore \sqrt{57 - 12\sqrt{15}} = \sqrt{45} - \sqrt{12} = 3\sqrt{5} - 2\sqrt{3}.$

15. $3\tfrac{1}{2} - \sqrt{10} = \dfrac{7 - 2\sqrt{10}}{2} = \dfrac{14 - 4\sqrt{10}}{4} = \dfrac{14 - 2\sqrt{40}}{4}$

$= \dfrac{10 - 2\sqrt{40} + 4}{4}.$

$\therefore \sqrt{3\tfrac{1}{2} - \sqrt{10}} = \dfrac{\sqrt{10} - 2}{2} = \tfrac{1}{2}\sqrt{10} - 1.$

16. $2a + 2\sqrt{a^2 - b^2} = a + b + 2\sqrt{a^2 - b^2} + a - b.$

$\therefore \sqrt{2a + 2\sqrt{a^2 - b^2}} = \sqrt{a + b} + \sqrt{a - b}.$

17. $a^2 - 2b\sqrt{a^2 - b^2} = a^2 - b^2 - 2b\sqrt{a^2 - b^2} + b^2.$

$\therefore \sqrt{a^2 - 2b\sqrt{a^2 - b^2}} = \sqrt{a^2 - b^2} - b.$

18. $87 - 12\sqrt{42} = 87 - 2\sqrt{1512} = 63 - 2\sqrt{1512} + 24.$

$\therefore \sqrt{87 - 12\sqrt{42}} = \sqrt{63} - \sqrt{24} = 3\sqrt{7} - 2\sqrt{6}.$

19. $(a + b)^2 - 4(a - b)\sqrt{ab} = (a - b)^2 - 4(a - b)\sqrt{ab} + 4ab.$

$\therefore \sqrt{(a + b)^2 - 4(a - b)\sqrt{ab}} = a - b - 2\sqrt{ab}.$

Exercise 31.

1. Multiply:

$\sqrt{-8}$ by $\sqrt{-2}$; $2\sqrt{-3}$ by $4\sqrt{-27}$; $3\sqrt{-5}$ by $\dfrac{3}{\sqrt{27}}$.

$$8^{\frac{1}{2}}\sqrt{-1}\times 2^{\frac{1}{2}}\sqrt{-1} = -\sqrt{16} = -4.$$

$$(2\times 3^{\frac{1}{2}}\sqrt{-1})\times(4\times 27^{\frac{1}{2}}\sqrt{-1}) = -8\sqrt{81} = -72.$$

$$3\sqrt{-5}\times\frac{3}{\sqrt{27}} = \frac{9\sqrt{-5}}{3\sqrt{3}} = 3^{\frac{1}{2}}\sqrt{-5} = \sqrt{-15}.$$

2. Divide:

$\sqrt{7}$ by $\sqrt{-3}$; $\sqrt{-8}$ by $\sqrt{-2}$; $3\sqrt{-6}$ by $\sqrt{2}\sqrt{-3}$.

$$\sqrt{7}\div\sqrt{-3} = \frac{\sqrt{7}}{\sqrt{-3}} = \frac{\sqrt{-21}}{-3} = -\tfrac{1}{3}\sqrt{-21}.$$

$$\sqrt{-8}\div\sqrt{-2} = \sqrt{4} = 2.$$

$$3\sqrt{-6}\div\sqrt{2}\sqrt{-3} = 3\sqrt{-6}\div\sqrt{-6} = 3.$$

3. Reduce to the typical form:

$4+\sqrt{-81}$; $5+2\sqrt{-6}$; $(3+\sqrt{-27})^2$.

$$4+\sqrt{-81} = 4+81^{\frac{1}{2}}\sqrt{-1} = 4+9\sqrt{-1}.$$

$$5+2\sqrt{-6} = 5+2\times 6^{\frac{1}{2}}\sqrt{-1}.$$

$$(3+\sqrt{-27})^2 = -18+6\sqrt{-27} = -18+6\times 27^{\frac{1}{2}}\sqrt{-1}$$
$$= -18+18\times 3^{\frac{1}{2}}\sqrt{-1}.$$

4. Multiply $4+\sqrt{-3}$ by $4-\sqrt{-3}$.

$$(4+\sqrt{-3})(4-\sqrt{-3}) = 16+3 = 19.$$

5. Multiply $\sqrt{3}-2\sqrt{-2}$ by $\sqrt{3}+2\sqrt{-2}$.

$$(\sqrt{3}-2\sqrt{-2})(\sqrt{3}+2\sqrt{-2}) = 3+8 = 11.$$

6. Multiply

$7+\sqrt{-27}$ by $4+\sqrt{-3}$.

$7+\sqrt{-27}=7+3\sqrt{-3}$.

$$\begin{array}{r} 7+3\sqrt{-3} \\ 4+\sqrt{-3} \\ \hline 28+7\sqrt{-3} \\ 12\sqrt{-3}-9 \\ \hline 19+19\sqrt{-3} \end{array}$$

7. Multiply

$5+2\sqrt{-8}$ by $3-5\sqrt{-2}$.

$5+2\sqrt{-8}=5+4\sqrt{-2}$.

$$\begin{array}{r} 5+4\sqrt{-2} \\ 3-5\sqrt{-2} \\ \hline 15+12\sqrt{-2} \\ -25\sqrt{-2}+40 \\ \hline 55-13\sqrt{-2} \end{array}$$

8. Multiply $2\sqrt{3}-6\sqrt{-5}$ by $4\sqrt{3}-\sqrt{-5}$.

$$\begin{array}{r} 2\sqrt{3}-6\sqrt{-5} \\ 4\sqrt{3}-\sqrt{-5} \\ \hline 24-24\sqrt{-15} \\ -2\sqrt{-15}-30 \\ \hline -6-26\sqrt{-15} \end{array}$$

9. Multiply $\sqrt{a}+b\sqrt{-c}$ by $\sqrt{c}+a\sqrt{-b}$.

$$\begin{array}{r} \sqrt{a}+b\sqrt{-c}\phantom{+a\sqrt{-ab}-ab\sqrt{bc}} \\ \sqrt{c}+a\sqrt{-b}\phantom{+a\sqrt{-ab}-ab\sqrt{bc}} \\ \hline \sqrt{ac}+bc\sqrt{-1}\phantom{+a\sqrt{-ab}-ab\sqrt{bc}} \\ +a\sqrt{-ab}-ab\sqrt{bc} \\ \hline \sqrt{ac}+bc\sqrt{-1}+a\sqrt{ab}\sqrt{-1}-ab\sqrt{bc} \\ =\sqrt{ac}-ab\sqrt{bc}+(bc+a\sqrt{ab})\sqrt{-1} \end{array}$$

10. Divide 26 by $3+\sqrt{-4}$; 86 by $6\sqrt{-7}$.

$$26\div(3+\sqrt{-4})=\frac{26}{3+\sqrt{-4}}=\frac{26}{3+2\sqrt{-1}}=\frac{26(3-2\sqrt{-1})}{(3+2\sqrt{-1})(3-2\sqrt{-1})}$$

$$=\frac{26(3-2\sqrt{-1})}{13}=6-4\sqrt{-1}.$$

$$86\div(6-\sqrt{-7})=\frac{86}{6-\sqrt{-7}}=\frac{86(6+\sqrt{-7})}{(6-\sqrt{-7})(6+\sqrt{-7})}=\frac{86(6+\sqrt{-7})}{43}$$

$$=12+2\sqrt{-7}.$$

11. Divide $3+\sqrt{-1}$ by $4+3\sqrt{-1}$.

$$(3+\sqrt{-1})\div(4+3\sqrt{-1})=\frac{3+\sqrt{-1}}{4+3\sqrt{-1}}=\frac{(3+\sqrt{-1})(4-3\sqrt{-1})}{(4+3\sqrt{-1})(4-3\sqrt{-1})}$$

$$=\frac{15-5\sqrt{-1}}{25}=\frac{3}{5}-\frac{1}{5}\sqrt{-1}.$$

12. Divide $-9+19\sqrt{-2}$ by $3+\sqrt{-2}$.

$$(-9+19\sqrt{-2})\div(3+\sqrt{-2})=\frac{-9+19\sqrt{-2}}{3+\sqrt{-2}}$$

$$=\frac{(-9+19\sqrt{-2})(3-\sqrt{-2})}{(3+\sqrt{-2})(3-\sqrt{-2})}=\frac{11+66\sqrt{-2}}{11}=1+6\sqrt{-2}.$$

13. Extract the square root of $1+4\sqrt{-3}$.

$$1+4\sqrt{-3}=1+2\sqrt{-12}=4+2\sqrt{-12}-3.$$

$$\therefore\ \sqrt{1+4\sqrt{-3}}=2+\sqrt{-3}.$$

14. Extract the square root of $10-8\sqrt{-6}$.

$$10-8\sqrt{-6}=10-2\sqrt{-96}=16-2\sqrt{-96}-6.$$

$$\therefore\ \sqrt{10-8\sqrt{-6}}=4-\sqrt{-6}.$$

15. Extract the square root of $-17+4\sqrt{-15}$.

$$-17+4\sqrt{-15}=-17+2\sqrt{-60}=3+2\sqrt{-60}-20.$$

$$\therefore\ \sqrt{-17+4\sqrt{-15}}=\sqrt{3}+\sqrt{-20}=\sqrt{3}+2\sqrt{-5}.$$

16. Extract the square root of $-38-15\sqrt{-28}$.

$$-38-15\sqrt{-28}=-38-30\sqrt{-7}=-38-2\sqrt{-1575}$$

$$=25-2\sqrt{-1575}-63.$$

$$\therefore\ \sqrt{-38-15\sqrt{-28}}=5-\sqrt{-63}=5-3\sqrt{-7}.$$

17. Show that $4x^2-12x+25$ is positive for all real values of x, and find its minimum value.

$$4x^2-12x+25=4x^2-12x+9+16=(2x-3)^2+4^2$$

This is positive for all real values of x.

Its minimum value occurs when $2x=3$, or $x=\frac{3}{2}$, and is equal to 16.

18. Show that $6x - 4 - 9x^2$ is negative for all real values of x, and find its maximum value.

$$6x - 4 - 9x^2 = -3 - 1 + 6x - 9x^2 = -3 - (1 - 3x)^2$$

This is negative for all real values of x.

Its maximum value occurs when $1 - 3x = 0$, or $x = \frac{1}{3}$, and is -3.

19. If ω be one of two imaginary roots of the equation $x^3 = 1$, show that the other is ω^2.

Solve $$x = \frac{-1 \pm \sqrt{-3}}{2}$$

Let $$\omega = \frac{-1 + \sqrt{-3}}{2}$$

Then $$\omega^2 = \frac{1 - 2\sqrt{-3} - 3}{4} = \frac{-2 - 2\sqrt{-3}}{4} = \frac{-1 - \sqrt{-3}}{2}$$

Again, let $$\omega = \frac{-1 - \sqrt{-3}}{2}$$

Then $$\omega^2 = \frac{1 + 2\sqrt{-3} - 3}{4} = \frac{-2 + 2\sqrt{-3}}{4} = \frac{-1 + \sqrt{-3}}{2}$$

∴ Each of the imaginary roots is the square of the other.

20. Show that $\omega^n + (\omega^2)^n = -1$, if n is any integer which is not a multiple of 3.

$$\omega^n + (\omega^2)^n = \omega^n + \omega^{2n} = \omega^n(1 + \omega^n)$$

Now if n is not a multiple of 3 it must be of one of the forms

$$\left.\begin{array}{l} n = 3k + 1, \\ \text{or} \quad n = 3k + 2, \end{array}\right\} \text{where } k \text{ is an integer.}$$

Suppose $n = 3k + 1$

Then $\omega^n = \omega^{3k+1} = \omega^{3k}\omega$

Now $\omega^3 = 1$ since ω is a root of the equation $x^3 = 1$.

$\therefore \omega^{3k} = 1$

$\therefore \omega^n = \omega$

$$\omega^n(1 + \omega^n) = \omega(1 + \omega) = \omega + \omega^2$$

But $$\omega = \frac{-1 + \sqrt{-3}}{2}$$

and $$\omega^2 = \frac{-1 - \sqrt{-3}}{2}$$

$$\therefore\ \omega + \omega^2 = -\tfrac{2}{2} = -1$$

Again suppose $n = 3k + 2$

Then $\omega^n = \omega^{3k+2} = \omega^3\omega^2 = \omega^2$

$$\therefore\ \omega^n(1 + \omega^n) = \omega^2(1 + \omega^2) = \omega^2 + \omega^4 = \omega^2 + \omega^3 \times \omega$$
$$= \omega^2 + \omega = -1$$

If n is a multiple of 3, we may write $n = 3k$

Then $\omega^n = \omega^{3k} = 1$

$$\therefore\ \omega^n(1 + \omega^n) = 2$$

21. Show that

$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z).$

$$\begin{array}{llllll}
x & +\,\omega^2 y & +\,\omega z & & & \\
x & +\,\omega y & +\,\omega^2 z & & & \\
\hline
x^2 & +\,\omega^2 xy & +\,\omega xz & & & \\
 & +\,\omega xy & & +\,\omega^3 y^2 & +\,\omega^2 yz & \\
 & & +\,\omega^2 xz & & +\,\omega^4 yz & +\,\omega^3 z^2 \\
\hline
x^2 & +\,(\omega^2 + \omega)\,xy & +\,(\omega + \omega^2)\,xz & +\,\omega^3 y^2 & +\,(\omega^2 + \omega^4)\,yz & +\,\omega^3 z^2
\end{array}$$

But from Example 20, when $n = 1$, $\omega^2 + \omega = -1$

$$\omega^3 = 1$$
$$\omega^4 = \omega^3 \times \omega = \omega$$

$\therefore$ The product above $= x^2 - xy - xz + y^2 - yz + z^2$

$$\begin{array}{llllllllll}
x^2 & -\,xy & -\,xz & +\,y^2 & -\,yz & +\,z^2 & & & & \\
x & +\,y & +\,z & & & & & & & \\
\hline
x^3 & -\,x^2y & -\,x^2z & +\,xy^2 & -\,xyz & +\,xz^2 & & & & \\
 & +\,x^2y & & -\,xy^2 & -\,xyz & & +\,y^3 & -\,y^2z & +\,yz^2 & \\
 & & +\,x^2z & & -\,xyz & -\,xz^2 & & +\,y^2z & -\,yz^2 & +\,z^3 \\
\hline
x^3 & & & & -\,3xyz & & +\,y^3 & & & +\,z^3
\end{array}$$

22. Find all the fourth roots of -1.

Let $x^4 = -1$

Then $x^4 + 1 = 0$

Divide by x^2, $x^2 + \dfrac{1}{x^2} = 0$

Add 2 to both sides, $x^2 + 2 + \dfrac{1}{x^2} = 2$

Extract square root, $x + \dfrac{1}{x} = \pm\sqrt{2}$

$$x^2+1=\pm\sqrt{2}\,x$$
$$x^2\mp\sqrt{2}\,x=-1$$
$$x^2\mp\sqrt{2}\,x+\tfrac{1}{2}=-\tfrac{1}{2}$$
$$x\mp\tfrac{1}{2}\sqrt{2}=\pm\tfrac{1}{2}\sqrt{-2}$$
$$\therefore\ x=\pm\tfrac{1}{2}\sqrt{2}\pm\tfrac{1}{2}\sqrt{-2}$$

The four roots are:

$$\frac{\sqrt{2}+\sqrt{-2}}{2},\ \frac{\sqrt{2}-\sqrt{-2}}{2},\ \frac{-\sqrt{2}+\sqrt{-2}}{2},\ \frac{-\sqrt{2}-\sqrt{-2}}{2}.$$

23. Find all the sixth roots of + 1.

Let $x^6=1$

Then $x^6-1=0$

$$(x^3+1)(x^3-1)=0$$

If $x^3+1=0$

$$(x+1)(x^2-x+1)=0$$
$$\therefore\ x=-1$$

or $x^2-x+1=0$

$$x^2-x+\tfrac{1}{4}=-\tfrac{3}{4}$$
$$x-\tfrac{1}{2}=\pm\tfrac{1}{2}\sqrt{-3}$$
$$x=\frac{1\pm\sqrt{-3}}{2}$$

If $x^3-1=0$

$$(x-1)(x^2+x+1)=0$$
$$\therefore\ x=1$$

or $x^2+x+1=0$

$$x^2+x+\tfrac{1}{4}=-\tfrac{3}{4}$$
$$x+\tfrac{1}{2}=\pm\tfrac{1}{2}\sqrt{-3}$$
$$x=\frac{-1\pm\sqrt{-3}}{2}$$

∴ The six roots are:

$$1,\ -1,\ \frac{1+\sqrt{-3}}{2},\ \frac{1-\sqrt{-3}}{2},\ \frac{-1+\sqrt{-3}}{2},\ \frac{-1-\sqrt{-3}}{2}.$$

24. Find all the eighth roots of + 1.

Let $x^8=1$

Then $x^8-1=0$

$$(x^4+1)(x^4-1)=0$$

If $x^4+1=0$

the four roots are by Example 22:

$$\frac{\sqrt{2}+\sqrt{-2}}{2},\ \frac{\sqrt{2}-\sqrt{-2}}{2},\ \frac{-\sqrt{2}+\sqrt{-2}}{2},\ \frac{-\sqrt{2}-\sqrt{-2}}{2}.$$

If $$x^4 - 1 = 0$$

$$(x^2 + 1)(x^2 - 1) = 0$$

$$\therefore\ x^2 + 1 = 0$$

$$x = \pm\sqrt{-1}$$

or $$x^2 - 1 = 0$$

$$x = \pm 1$$

$\therefore$ The eight roots are: $1,\ -1,\ \sqrt{-1},\ -\sqrt{-1},$

$$\frac{\sqrt{2}+\sqrt{-2}}{2},\ \frac{\sqrt{2}-\sqrt{-2}}{2},\ \frac{-\sqrt{2}+\sqrt{-2}}{2},\ \frac{-\sqrt{2}-\sqrt{-2}}{2}.$$

25. Reduce to the typical form

$$\frac{(2-3\sqrt{-1})(3+4\sqrt{-1})}{(6+4\sqrt{-1})(15-8\sqrt{-1})}.$$

$$\frac{(2-3\sqrt{-1})(3+4\sqrt{-1})}{(6+4\sqrt{-1})(15-8\sqrt{-1})} = \frac{18-\sqrt{-1}}{122+12\sqrt{-1}}$$

$$= \frac{(18-\sqrt{-1})(122-12\sqrt{-1})}{(122+12\sqrt{-1})(122-12\sqrt{-1})}$$

$$= \frac{2184-338\sqrt{-1}}{15028} = \frac{84-13\sqrt{-1}}{578}$$

$$= \tfrac{42}{289} - \tfrac{13}{578}\sqrt{-1}$$

Exercise 32.

Show that, the letters being unequal and positive:

1. $$a^2 + 3b^2 > 2b(a+b)$$

That is, $$a^2 + 3b^2 > 2ab + 2b^2$$

$$a^2 + b^2 > 2ab$$

But $$a^2 + b^2 > 2ab$$

$$\therefore\ a^2 + 3b^2 > 2b(a+b)$$

2. $$a^3b + ab^3 > 2a^2b^2$$

That is, $$ab(a^2 + b^2) > 2a^2b^2$$

$$a^2 + b^2 > 2ab$$

But $$a^2 + b^2 > 2ab$$

$$\therefore\ a^3b + ab^3 > 2a^2b^2$$

3. $$(a^2 + b^2)(a^4 + b^4) > (a^3 + b^3)^2$$

That is, $$a^6 + a^4b^2 + a^2b^4 + b^6 > a^6 + 2a^3b^3 + b^6$$

$$a^4b^2 + a^2b^4 > 2a^3b^3$$

$$a^2 + b^2 > 2ab$$

But $$a^2 + b^2 > 2ab$$

$$\therefore\ (a^2 + b^2)(a^4 + b^4) > (a^3 + b^3)^2$$

4. $a^2b + a^2c + ab^2 + b^2c + ac^2 + bc^2 > 6abc.$

For $$a^2 + c^2 > 2ac$$

$$\therefore\ (a^2 + c^2)b > 2abc$$

or $$a^2b + bc^2 > 2abc \qquad (1)$$

Also, $a^2 + b^2 > 2ab$

$\therefore (a^2 + b^2)c > 2abc$

or $a^2c + b^2c > 2abc$ (2)

And $b^2 + c^2 > 2bc$

$\therefore (b^2 + c^2)a > 2abc$

or $ab^2 + ac^2 > 2abc$ (3)

Add (1), (2), and (3),

$$a^2b + a^2c + ab^2 + b^2c + ac^2 + bc^2 > 6abc$$

5. The sum of any fraction and its reciprocal > 2.

Let $\frac{a}{b}$ be any fraction.

Then $\frac{b}{a}$ is its reciprocal,

and $\frac{a}{b} + \frac{b}{a}$ is their sum.

Now $\frac{a}{b} + \frac{b}{a} > 2$

if $a^2 + b^2 > 2ab$

But $a^2 + b^2 > 2ab$

$$\therefore \frac{a}{b} + \frac{b}{a} > 2$$

6. If $x^2 = a^2 + b^2$, and $y^2 = c^2 + d^2$, $xy > ac + bd$, or $ad + bc$, unless $\frac{a}{c}$ equals $\frac{b}{d}$.

$xy > ac + bd$

if $x^2y^2 > a^2c^2 + 2abcd + b^2d^2$.

But $x^2 = a^2 + b^2$

and $y^2 = c^2 + d^2$

Hence $xy > ac + bd$

if $(a^2 + b^2)(c^2 + d^2) > a^2c^2 + 2abcd + b^2d^2$

if $a^2c^2 + b^2c^2 + a^2d^2 + b^2d^2 > a^2c^2 + 2abcd + b^2d^2$

if $b^2c^2 + a^2d^2 > 2abcd$

But $b^2c^2 + a^2d^2 > 2abcd$

Also $xy > ad + bc$

if $x^2y^2 > a^2d^2 + 2abcd + b^2c^2$

if $a^2c^2 + b^2c^2 + a^2d^2 + b^2d^2 > a^2d^2 + 2abcd + b^2c^2$

if $a^2c^2 + b^2d^2 > 2abcd$

But $a^2c^2 + b^2d^2 > 2abcd$

$$\therefore xy > ad + bc$$

7. $ab+ac+bc<(a+b-c)^2+(a+c-b)^2+(b+c-a)^2$

if $$ab+ac+bc<3a^2+3b^2+3c^2-2ab-2ac-2bc$$

if $$3(ab+ac+bc)<3(a^2+b^2+c^2)$$

if $$ab+ac+bc<a^2+b^2+c^2$$

But $$ab+ac+bc<a^2+b^2+c^2$$

$$\therefore\ ab+ac+bc<(a+b-c)^2+(a+c-b)^2+(b+c-a)^2$$

8. Which is the greater, $(a^2+b^2)(c^2+d^2)$ or $(ac+bd)^2$?

$$(a^2+b^2)(c^2+d^2)=a^2c^2+a^2d^2+b^2c^2+b^2d^2$$
$$(ac+bd)^2=a^2c^2+2abcd+b^2d^2$$

Subtract the second equation from the first,

$$(a^2+b^2)(c^2+d^2)-(ac+bd)^2=a^2d^2-2abcd+b^2c^2$$
$$=(ad-bc)^2$$
$$\therefore\ (a^2+b^2)(c^2+d^2)=(ac+bd)^2+(ad-bc)^2$$
$$\therefore\ (a^2+b^2)(c^2+d^2)>(ac+bd)^2$$

9. Which is the greater, a^4-b^4 or $4a^3(a-b)$ when $a>b$?

$$a^4-b^4=(a^3+a^2b+ab^2+b^3)(a-b)$$

And since $$b<a,$$
$$a^2b<a^3$$
$$ab^2<a^3$$
$$b^3<a^3$$
$$\therefore\ a^3+a^2b+ab^2+b^3<4a^3$$

Since $a-b$ is positive,

$$(a^3+a^2b+ab^2+b^3)(a-b)<4a^3(a-b)$$
$$\therefore\ a^4-b^4<4a^3(a-b)\text{ if }a>b$$

10. Which is the greater, $\sqrt{\frac{a^2}{b}}+\sqrt{\frac{b^2}{a}}$ or $\sqrt{a}+\sqrt{b}$?

$$\sqrt{\frac{a^2}{b}}+\sqrt{\frac{b^2}{a}}=\sqrt{b\frac{a^2}{b^2}}+\sqrt{a\frac{b^2}{a^2}}=\frac{a}{b}\sqrt{b}+\frac{b}{a}\sqrt{a}=\frac{a^2\sqrt{b}+b^2\sqrt{a}}{ab}.$$

$$\frac{a^2\sqrt{b}+b^2\sqrt{a}}{ab}>\text{ or }<\sqrt{a}+\sqrt{b}$$

as $$a^2\sqrt{b}+b^2\sqrt{a}>\text{ or }<ab\sqrt{a}+ab\sqrt{b}$$

as $$(a^2-ab)\sqrt{b}+(b^2-ab)\sqrt{a}>\text{ or }<0$$

as $$(a-b)(a\sqrt{b}-b\sqrt{a})>\text{ or }<0$$

But, if $a>b$, $a-b$ is positive,

and $a\sqrt{b}-b\sqrt{a}$ is positive.

For $\frac{a\sqrt{b}-b\sqrt{a}}{ab}=\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}$

which is positive since $\sqrt{b}<\sqrt{a}$

and, therefore $\frac{1}{\sqrt{b}}>\frac{1}{\sqrt{a}}$

And, if $a<b$, $a-b$ is negative,

and $a\sqrt{b}-b\sqrt{a}$ is negative.

For $\frac{a\sqrt{b}-b\sqrt{a}}{ab}=\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}$

which is negative since $\sqrt{b}>\sqrt{a}$

and, therefore $\frac{1}{\sqrt{b}}<\frac{1}{\sqrt{a}}$

$\therefore$ $(a-b)(a\sqrt{b}-b\sqrt{a})$ is always positive.

$$\therefore \sqrt{\frac{a^2}{b}}+\sqrt{\frac{b^2}{a}}>\sqrt{a}+\sqrt{b}.$$

11. Which is the greater, $\frac{a+b}{2}$ or $\frac{2ab}{a+b}$?

$$\frac{a+b}{2}>\text{ or }<\frac{2ab}{a+b}$$

as $(a+b)^2>\text{ or }<4ab$

as $a^2+b^2>\text{ or }<2ab$

But $a^2+b^2>2ab$

$$\therefore \frac{a+b}{2}>\frac{2ab}{a+b}$$

12. Which is the greater, $\frac{a}{b^2}+\frac{b}{a^2}$ or $\frac{1}{b}+\frac{1}{a}$?

$$\frac{a}{b^2}+\frac{b}{a^2}=\frac{a^3+b^3}{a^2b^2}$$

$$\frac{1}{b}+\frac{1}{a}=\frac{a+b}{ab}$$

$$\frac{a^3+b^3}{a^2b^2}>\text{ or }<\frac{a+b}{ab}$$

as $(a^3+b^3)>\text{ or }<ab(a+b)$

as $a^2-ab+b^2>\text{ or }<ab$

as $a^2+b^2>\text{ or }<2ab$

But $a^2+b^2>2ab$

$$\therefore \frac{a}{b^2}+\frac{b}{a^2}>\frac{1}{b}+\frac{1}{a}$$

Exercise 33.

1. **Write down the ratio compounded of 3 : 5 and 8 : 7. Which of these ratios is increased, and which is diminished by the composition?**

$$\frac{3}{5}\times\frac{8}{7}=\frac{24}{35}.$$

$\therefore$ The compound ratio is 24 : 35.

As $3:5=21:35$

and $8:7=40:35$

$\therefore$ 3 : 5 is increased,

$\therefore$ 8 : 7 diminished.

2. **Compound the duplicate ratio of 4 : 15 with the triplicate of 5 : 2.**

$$\frac{4^2}{15^2}\times\frac{5^3}{2^3}=\frac{10}{9}$$

$\therefore$ The compound ratio is 10 : 9.

3. **Show that a duplicate ratio is greater or less than its simple ratio according as it is a ratio of greater inequality or a ratio of less inequality.**

Let $m:n=$ a ratio of greater inequality.

Then
$$m>n$$
$$\frac{m}{n}>1$$
$$\therefore \frac{m}{n}\frac{m}{n}>\frac{m}{n}$$
$$\frac{m^2}{n^2}>\frac{m}{n}$$

Let $m:n=$ a ratio of less inequality.

Then
$$m<n$$
$$\frac{m}{n}<1$$
$$\therefore \frac{m}{n}\frac{m}{n}<\frac{m}{n}$$
$$\frac{m^2}{n^2}<\frac{m}{n}$$

4. **Arrange in order of magnitude the ratios 3 : 4, 23 : 25, 10 : 11.**

$$3:4=825:1100$$
$$23:25=1012:1100$$
$$10:11=1000:1100$$

$\therefore$ The descending order of magnitude is 23 : 25, 10 : 11, 3 : 4.

5. If $a > b$, which is the greater ratio,

$$a + b : a - b \text{ or } a^2 + b^2 : a^2 - b^2?$$

$$a + b : a - b = (a + b)(a + b) : (a + b)(a - b)$$
$$= a^2 + 2ab + b^2 : a^2 - b^2$$

But $\quad a^2 + 2ab + b^2 > a^2 + b^2$

$$\therefore\ a + b : a - b\ > a^2 + b^2 : a^2 - b^2$$

Find the ratios compounded of:

6. $3 : 5,\ 10 : 21,\ 14 : 15.$

$$\tfrac{3}{5} \times \tfrac{10}{21} \times \tfrac{14}{15} = \tfrac{4}{15}.$$

$\therefore$ The compound ratio is $4 : 15$.

7. $7 : 9,\ 102 : 105,\ 15 : 17.$

$$\tfrac{7}{9} \times \tfrac{102}{105} \times \tfrac{15}{17} = \tfrac{2}{3}.$$

$\therefore$ The compound ratio is $2 : 3$.

8. $a^2 - x^2 : a^2 + 3ax + 2x^2$ and $a + x : a - x$.

$$\frac{a^2 - x^2}{a^2 + 3ax + 2x^2} \times \frac{a + x}{a - x} = \frac{(a + x)(a - x)}{(a + x)(a + 2x)} \times \frac{a + x}{a - x} = \frac{a + x}{a + 2x}$$

$\therefore$ The compound ratio is $a + x : a + 2x$.

9. $x^2 - 4 : 2x^2 - 5x + 3$ and $x - 1 : x - 2$.

$$\frac{x^2 - 4}{2x^2 - 5x + 3} \times \frac{x - 1}{x - 2} = \frac{(x + 2)(x - 2)}{(2x - 3)(x - 1)} \times \frac{x - 1}{x - 2} = \frac{x + 2}{2x - 3}$$

$\therefore$ The compound ratio is $x + 2 : 2x - 3$.

10. Prove that a ratio of greater inequality is diminished, and a ratio of less inequality increased, by adding the same number to both its terms.

Let $\quad m : n$ be the given ratio,

and $\quad a$ the quantity added.

Then $\quad \dfrac{m + a}{n + a}$ the second ratio.

Then $$\frac{m + a}{n + a} > \text{ or } < \frac{m}{n}$$

as $$mn + an > \text{ or } < mn + am$$

as $$an > \text{ or } < am$$

as $$n > \text{ or } < m$$

as $$m < \text{ or } > n$$

11. Prove that a ratio of greater inequality is increased, and a ratio of less inequality diminished, by subtracting the same number from both its terms.

Let $m:n$ be the given ratio,

and a the quantity subtracted.

The $$\frac{m-a}{n-a} > \text{ or } < \frac{m}{n}$$

as $$mn-an > \text{ or } < mn-am$$

as $$am > \text{ or } < an$$

as $$m > \text{ or } < n$$

12. Show that the ratio $a:b$ is the duplicate of the ratio $a+c:b+c$, if $c^2=ab$.

$$\left(\frac{a+c}{b+c}\right)^2=\frac{a^2+2ac+c^2}{b^2+2bc+c^2}$$

But $$c^2=ab$$

$$\therefore \left(\frac{a+c}{b+c}\right)^2=\frac{a^2+2a\sqrt{ab}+ab}{b^2+2b\sqrt{ab}+ab}$$

$$=\frac{a(a+2\sqrt{ab}+b)}{b(b+2\sqrt{ab}+a)}=\frac{a}{b}$$

13. Two numbers are in the ratio $2:5$, and if 6 be added to each, they are in the ratio $4:7$. Find the numbers.

Let $x=$ one number,

and $y=$ the other.

Then $$\frac{x}{y}=\frac{2}{5}$$

$$\frac{x+6}{y+6}=\frac{4}{7}$$

$$\therefore 5x=2y$$

$$-4y=-18$$

$$\therefore x=6$$

$$y=15$$

14. What must be added to each of the terms of the ratio $m:n$, that it may become equal to the ratio $p:q$?

Let $x=$ number to be added.

$$\therefore \frac{m+x}{n+x}=\frac{p}{q}$$

$$qm+qx=pn+px$$

$$(p-q)x=qm-pn$$

$$\therefore x=\frac{qm-pn}{p-q}$$

15. If x and y be such that, when they are added to the antecedent and consequent respectively of the ratio $a:b$, its value is unaltered, show that $x:y=a:b$.

$$a+x:b+y=a:b$$

By alternation, $a+x:a=b+y:b$

By division, $x:a=y:b$

By alternation, $x:y=a:b$

Find x from the proportions:

16.
$$27:90=45:x$$
$$\therefore\ 27x=45\times 90$$
$$x=150$$

17.
$$11\tfrac{1}{4}:4\tfrac{1}{2}=3\tfrac{3}{4}:x$$
$$\therefore\ 11\tfrac{1}{4}x=4\tfrac{1}{2}\times 3\tfrac{3}{4}$$
$$\tfrac{45}{4}x=\tfrac{9}{2}\times\tfrac{15}{4}$$
$$x=\tfrac{3}{2}=1\tfrac{1}{2}$$

18.
$$\frac{3a}{5b}:\frac{12a}{7c}=\frac{14c}{15b}:x$$
$$\therefore\frac{3a}{5b}x=\frac{12\times 14ac}{7\times 15bc}=\frac{8a}{5b}$$
$$\therefore\ x=\tfrac{8}{3}=2\tfrac{2}{3}$$

Find a third proportional to:

19. $\tfrac{27}{10}$ and $\tfrac{5}{12}$.

Let $x=$ the third proportional.

Then,
$$\tfrac{27}{10}:\tfrac{5}{12}=\tfrac{5}{12}:x$$
$$\therefore\ \tfrac{27}{10}x=\tfrac{25}{144}$$
$$x=\frac{10\times 25}{27\times 144}$$
$$x=\tfrac{125}{1944}$$

20. $\dfrac{a^2-b^2}{c}$ and $\dfrac{a-b}{c}$.

Let $x=$ the third proportional.

Then,
$$\frac{a^2-b^2}{c}:\frac{a-b}{c}=\frac{a-b}{c}:x$$
$$\therefore\ \frac{a^2-b^2}{c}x=\frac{(a-b)^2}{c^2}$$
$$(a+b)x=\frac{a-b}{c}$$
$$x=\frac{a-b}{c(a+b)}$$

Find a mean proportional between:

21. **3 and $16\frac{1}{3}$.**

Let $x =$ the mean proportional.

Then $x = \sqrt{3 \times 16\frac{1}{3}} = \sqrt{49} = 7.$

22. $\frac{(m-5)^2}{m+5}$ **and** $\frac{(m+5)^2}{m-5}$.

Let $x =$ the mean proportional.

Then $x = \sqrt{\frac{(m-5)^2}{m+5} \times \frac{(m+5)^2}{m-5}} = \sqrt{m^2 - 25}.$

If $a : b = c : d$, prove that:

23. $2a + b : b = 2c + d : d.$

Since

$$a : b = c : d$$

$$\frac{a}{b} = \frac{c}{d}$$

$$\frac{2a}{b} = \frac{2c}{d}$$

$$\frac{2a}{b} + 1 = \frac{2c}{d} + 1$$

$$\therefore \frac{2a+b}{b} = \frac{2c+d}{d}$$

$$2a + b : b = 2c + d : d$$

24. $3a - b : a = 3c - d : c$

Since

$$a : b = c : d$$

$$\frac{a}{b} = \frac{c}{d}$$

$$\frac{b}{a} = \frac{d}{c}$$

$$-\frac{b}{a} = -\frac{d}{c}$$

$$3 - \frac{b}{a} = 3 - \frac{d}{c}$$

$$\frac{3a-b}{a} = \frac{3c-d}{c}$$

$$\therefore 3a - b : a = 3c - d : c$$

25. $4a+3b:4a-3b=4c+3d:4c-3d.$

Since $$a:b=c:d$$

$$\frac{a}{b}=\frac{c}{d}$$

$$\frac{4a}{b}=\frac{4c}{d}$$

$$\frac{4a}{b}+3=\frac{4c}{d}+3$$

$$\therefore\ \frac{4a+3b}{b}=\frac{4c+3d}{d} \qquad (1)$$

Also $$\frac{4a}{b}-3=\frac{4c}{d}-3$$

$$\frac{4a-3b}{b}=\frac{4c-3d}{d} \qquad (2)$$

Divide (1) by (2), $$\frac{4a+3b}{4a-3b}=\frac{4c+3d}{4c-3d}$$

$$\therefore\ 4a+3b:4a-3b=4c+3d:4c-3d$$

26. $2a^3+3b^3:2a^3-3b^3=2c^3+3d^3:2c^3-3d^3.$

Since $$a:b=c:d$$

$$\frac{a}{b}=\frac{c}{d}$$

$$\frac{a^3}{b^3}=\frac{c^3}{d^3}$$

$$\frac{2a^3}{b^3}=\frac{2c^3}{d^3}$$

$$\frac{2a^3}{b^3}+3=\frac{2c^3}{d^3}+3$$

$$\frac{2a^3+3b^3}{b^3}=\frac{2c^3+3d^3}{d^3} \qquad (1)$$

Also $$\frac{2a^3}{b^3}-3=\frac{2c^3}{d^3}-3$$

$$\frac{2a^3-3b^3}{b^3}=\frac{2c^3-3d^3}{d^3} \qquad (2)$$

Divide (1) by (2), $$\frac{2a^3+3b^3}{2a^3-3b^3}=\frac{2c^3+3d^3}{2c^3-3d^3}$$

$$\therefore\ 2a^3+3b^3:2a^3-3b^3=2c^3+3d^3:2c^3-3d^3$$

If $a : b = b : c$, prove that:

27. $a^2 + ab : b^2 + bc :: a : c$.

If $$a : b = b : c$$

By composition and alternation,

$$a + b : b + c = b : c$$

$$\frac{a+b}{b+c} = \frac{b}{c}$$

Also $$\frac{a}{b} = \frac{b}{c}$$

Multiply together, $$\frac{a(a+b)}{b(b+c)} = \frac{b^2}{c^2}$$

But $$b^2 = ac$$

$$\therefore \frac{a(a+b)}{b(b+c)} = \frac{a}{c}$$

$$a^2 + ab : b^2 + bc = a : c$$

28. $a : c :: (a+b)^2 : (b+c)^2$.

If $$a : b = b : c$$

By composition and alternation,

$$a : b = a + b : b + c$$

$$\frac{a}{b} = \frac{a+b}{b+c}$$

$$\therefore \frac{a^2}{b^2} = \frac{(a+b)^2}{(b+c)^2}$$

But $$b^2 = ac$$

$$\therefore \frac{a}{c} = \frac{(a+b)^2}{(b+c)^2}$$

$$\therefore a : c = (a+b)^2 : (b+c)^2$$

29. If $a : b = b : c$, and a is the greatest of the three numbers, show that $a + c > 2b$.

$$a : b = b : c$$

$$\therefore b^2 = ac$$

Now $$a^2 + c^2 > 2ac$$

$$\therefore a^2 + 2ac + c^2 > 4ac$$

$$a^2 + 2ac + c^2 > 4b^2$$

$$\therefore a + c > 2b$$

30. If $\frac{x-y}{l}=\frac{y-z}{m}=\frac{z-x}{n}$, and x, y, z be unequal, show that $l+m+n=0$.

$$\frac{x-y}{l}=\frac{y-z}{m}=\frac{z-x}{n}$$

Add the numerators together and add the denominators together.

Then
$$\frac{x-y+y-z+z-x}{l+m+n}=\frac{z-x}{n}$$

or
$$\frac{0}{l+m+n}=\frac{z-x}{n}$$
$$0=(z-x)(l+m+n)$$

But z is *not* equal to x,

$$\therefore\ l+m+n=0$$

Find x from the proportions:

31. $$x+1:x-1=x+2:x-2$$

By composition and division,

$$2x:2=2x:4$$
$$8x=4x$$
$$\therefore\ x=0$$

32. $$x+a:2x-b=3x+b:4x-a$$

By division, $$x+a:x-a-b=3x+b:x-a-b$$
$$(x+a)(x-a-b)=(x-a-b)(3x+b)$$
$$\therefore\ x+a=3x+b$$
$$2x=a-b$$
$$x=\frac{a-b}{2}$$

or
$$x-a-b=0$$
$$\therefore\ x=a+b$$

33. $$x^2-4x+2:x^2-2x-1=x^2-4x:x^2-2x-2$$

By alternation,

$$x^2-4x+2:x^2-4x=x^2-2x-1:x^2-2x-2$$

By division, $$2:x^2-4x=1:x^2-2x-2$$
$$\therefore\ 2(x^2-2x-2)=x^2-4x$$
$$2x^2-4x-4=x^2-4x$$
$$x^2=4$$
$$\therefore\ x=\pm 2$$

34. $3+x:4+x=9+x:13+x$

By division, $3+x:1=9+x:4$

$$4(3+x)=9+x$$
$$12+4x=9+x$$
$$3x=-3$$
$$\therefore\ x=-1$$

35. $a+x:b+x=c+x:d+x$

By division, $a-b:b+x=c-d:d+x$

$$(a-b)(d+x)=(c-d)(b+x)$$
$$(a-b-c+d)\,x=bc-ad$$
$$\therefore\ x=\frac{bc-ad}{a-b-c+d}$$

36. If $a:b=c:d$, show that

$$a^2+b^2:\frac{a^3}{a+b}=c^2+d^2:\frac{c^3}{c+d}$$

$$a:b=c:d$$

Then $a^3:b^3=c^3:d^3$

$$b^3:a^3=d^3:c^3$$

By composition, $a^3+b^3:a^3=c^3+d^3:c^3$ (1)

Also, $b:a=d:c$

By composition, $a+b:a=c+d:c$

and $ab:a^2=cd:c^2$

Multiply together, $ab(a+b):a^3=cd(c+d):c^3$ (2)

Add (1) and (2),

$$a^3+ab(a+b)+b^3:a^3=c^3+cd(c+d)+d^3:c^3$$
$$(a^2+b^2)(a+b):a^3=(c^2+d^2)(c+d):c^3$$
$$a^2+b^2:\frac{a^3}{a+b}=c^2+d^2:\frac{c^3}{c+d}$$

37. When a, b, c, d are proportional and all unequal, show that no number x can be found such that $a+x$, $b+x$, $c+x$, $d+x$ shall be proportionals.

$$a:b=c:d$$
$$a+x:b+x=c+x:d+x$$

By division, $a-b:b-x=c-d:d-x$

$$(a-b)\,d-(a-b)\,x=(c-d)\,b-(c-d)\,x$$
$$(c-d-a+b)\,x=bc-ad$$

But $bc-ad=0$

$$\therefore\ (c-d-a+b)x=0$$

$c-d-a+b$ is not 0.

For, if $c-d-a+b=0$

$$a-b=c-d$$

But $a-b:b=c-d:d$

$$\therefore \text{ if } a-b=c-d$$

$b=d$, which is not allowed.

$$\therefore\ x=0$$

Exercise 34.

1. A rectangular field contains 5270 acres, and its length is to its breadth in the ratio of 31 : 17. Find its dimensions.

Let x = number of rods in length.
y = number of rods in width.
160 sq. rds. = 1 acre.

Then $x:y=31:17$ (1)

$xy=5270\times 160$ (2)

From (1), $17x=31y$

$$x=\frac{31y}{17}$$

Substitute value of x in (2),

$$\frac{31y^2}{17}=5270\times 160$$

$$y^2=17\times 170\times 160$$

$$\therefore\ y=17\times 40=680$$

$$x=31\times 40=1240$$

The field is 1240 rods long and 680 rods wide.

2. If five gold coins and four silver ones be worth as much as three gold coins and twelve silver ones, find the ratio of the value of a gold coin to that of a silver one.

Let x = number of units of value in 1 gold coin.
y = number of units of value in 1 silver coin.

Then $5x+4y=3x+12y$

$$2x=8y$$

$$x=4y$$

$$\therefore\ x:y=4:1$$

The ratio is 4 : 1.

3. The lengths of two rectangular fields are in the ratio of 2 : 3, and the breadths in the ratio of 5 : 6. Find the ratio of their areas.

Let $x =$ number of units in length of one field.
$y =$ number of units in breadth of this field.
$z =$ number of units in length of second field.
$w =$ number of units in breadth of second field.
Then $xy =$ number of units in area of first field.
$zw =$ number of units in area of second field.

$$\therefore\ x : z = 2 : 3$$
$$y : w = 5 : 6$$
$$\frac{x}{z} = \frac{2}{3}$$
$$\frac{y}{w} = \frac{5}{6}$$
$$\therefore\ \frac{xy}{zw} = \frac{5}{9}$$
$$xy : zw = 5 : 9$$

The areas of the fields are in the ratio 5 : 9.

4. Two workmen are paid in proportion to the work they do. A can do in 20 days the work that it takes B 24 days to do. Compare their wages.

Let $x =$ amount of work A can do in 20 days.
Then $x =$ amount of work B can do in 24 days.
$\frac{x}{20} =$ amount of work A can do in 1 day.
$\frac{x}{24} =$ amount of work B can do in 1 day.

$$\therefore \text{A's wages : B's wages} = \frac{x}{20} : \frac{x}{24} = \frac{1}{20} : \frac{1}{24} = \frac{120}{20} : \frac{120}{24} = 6 : 5$$

5. In a mile race between a bicycle and a tricycle their rates were as 5 : 4. The tricycle had half a minute start, but was beaten by 176 yards. Find the rate of each.

Let $x =$ number of yards the bicycle goes in 1 minute,
and $y =$ number of yards the tricycle goes in 1 minute.

$$\therefore\ x : y = 5 : 4$$
$$1760 \text{ yards} = 1 \text{ mile.}$$

$\frac{1760}{x} =$ number of minutes in which the bicycle goes 1 mile.

$\frac{y}{2}$ = number of yards the tricycle goes in $\frac{1}{2}$ minute.

$1760 - 176$ = number of yards the tricycle goes in all.

$\frac{1760 - 176}{y}$ = number of minutes in which tricycle goes $1760 - 176$ yards.

$$\therefore\ 1760 - \frac{176}{y} - \frac{1}{2} = \frac{1760}{x} \qquad (2)$$

$$3168\,x - xy = 3520\,y \qquad (3)$$

From (1), $4\,x = 5\,x$

$$x = \tfrac{5}{4}\,y$$

Substitute value of x in (3),

$$3960\,y - \tfrac{5}{4}\,y^2 = 3520\,y$$

$$\tfrac{5}{4}\,y^2 = 440\,y$$

$$y^2 = 352\,y$$

$$\therefore\ y = 0, \text{ or } 352$$

$$x = 0, \text{ or } 440$$

∴ The rate of the bicycle is 440 yards a minute, or 15 miles an hour; that of the tricycle is 352 yards a minute, or 12 miles an hour.

6. A railway passenger observes that a train passes him, moving in the opposite direction, in 2 seconds; but moving in the same direction with him, it passes him in 30 seconds. Compare the rates of the two trains.

Let x = number of units the first train moves per second,

and y = number of units the second train moves per second.

Then $y + x$ = number of units the second train moves past the first in 1 second when moving in the opposite direction.

$y - x$ = number of units the second train moves past the first in 1 second when moving in the same direction.

But the rates of motion of the two trains are inversely proportional to the time of passing each other.

$$\therefore\ y + x : y - x = 30 : 2$$

By composition and division,

$$2\,y : 2\,x = 32 : 28$$

$$\therefore\ y : x = 8 : 7$$

The rate of the second train is to the rate of the first as 8 : 7.

7. A vessel is half full of a mixture of wine and water. If filled up with wine, the ratio of the quantity of wine to that of water is ten times what it would be if the vessel were filled up with water. Find the ratio of the original quantity of wine to that of water.

Let $x =$ number of gallons of wine in the vessel at first.

$y =$ number of gallons of water in the vessel at first.

Then $x + y =$ number of gallons the vessel contains when half full.

$x + y =$ number of gallons which must be put in to fill it.

$2x + y =$ number of gallons of wine in the vessel if it is filled up with wine.

$x + 2y =$ number of gallons of water in the vessel if it is filled up with water.

$$\therefore\ 2x + y : y = 10x : x + 2y$$

$$\therefore\ (2x + y)(x + 2y) = 10xy$$

$$2x^2 + 5xy + 2y^2 = 10xy$$

$$2x^2 - 5xy + 2y^2 = 0$$

Factoring,

$$(2x - y)(x - 2y) = 0$$

$$\therefore\ x = \frac{y}{2}, \text{ or } 2y$$

$$x : y = 1 : 2,$$

or

$$x : y = 2 : 1$$

The ratio of the original quantity of wine to that of water was 1 : 2, or 2 : 1.

8. A quantity of milk is increased by watering in the ratio 4 : 5, and then 3 gallons are sold; the remainder is increased in the ratio 6 : 7 by mixing it with 3 quarts of water. How many gallons of milk were there at first?

Let $4x =$ number of quarts of milk originally.

Then $5x =$ number of quarts of milk and water after first watering.

$5x - 12 =$ number of quarts remaining after 3 gallons are sold.

$\frac{7}{6}(5x - 12) =$ number of quarts after second watering.

$5x - 9 =$ number of quarts after second watering.

$$\therefore\ \tfrac{7}{6}(5x - 12) = (5x - 9)$$

$$35x - 84 = 30x - 54$$

$$5x = 30$$

$$x = 6$$

$$4x = 24$$

There were 24 quarts or 6 gallons of milk at first.

9. Each of two vessels, A and B, contains a mixture of wine and water; A in the ratio of $7:3$, and B in the ratio of $3:1$. How many gallons from B must be put with 5 gallons from A to give a mixture of wine and water in the ratio of $11:4$?

Let $7x =$ number of gallons of wine in A.
Then $3x =$ number of gallons of water in A.
Let $3y =$ number of gallons of wine in B.
Then $y =$ number of gallons of water in B.
Let $z =$ number of gallons to be taken from B.

A contains in all $10x$ gallons, $\frac{7}{10}$ of which is wine and $\frac{3}{10}$ water.

Hence 5 gallons of the mixture in A will contain $\frac{35}{10}$ gallons of wine and $\frac{15}{10}$ gallons of water.

B contains in all $4y$ gallons, $\frac{3}{4}$ of which is wine and $\frac{1}{4}$ water.

Hence z gallons of the mixture from B will contain $\frac{3z}{4}$ gallons of wine and $\frac{z}{4}$ gallons of water.

$$\therefore \frac{35}{10}+\frac{3z}{4}:\frac{15}{10}+\frac{z}{4}=11:4$$

$$4\left(3\tfrac{1}{2}+\tfrac{3}{4}z\right)=11\left(1\tfrac{1}{2}+\tfrac{1}{4}z\right)$$

$$14+3z=\tfrac{33}{2}+\tfrac{11}{4}z$$

$$\tfrac{1}{4}z=\tfrac{5}{2}$$

$$z=10$$

$\therefore$ 10 gallons must be taken from B.

10. The time which an express train takes to travel 180 miles is to that taken by an ordinary train as $9:14$. The ordinary train loses as much time from stopping as it would take to travel 30 miles; the express train loses only half as much time as the other by stopping, and travels 15 miles an hour faster. What are their respective rates?

Let $x =$ number of miles the ordinary train travels per hour.
Then $x + 15 =$ number of miles the express train travels per hour.

$\frac{180+30}{x} =$ number of hours required by ordinary train to travel 180 miles.

$\frac{30}{2x} =$ number of hours lost by express train in stopping.

$\frac{180}{x+15}+\frac{30}{2x} =$ number of hours required by express train to travel 180 miles.

$$\therefore\ \frac{180}{x+15}+\frac{30}{2x}:\frac{180+30}{x}=9:14$$

$$\begin{aligned}
14\left(\frac{180}{x+15}+\frac{15}{x}\right)&=9\,\frac{210}{x}\\
\frac{2520}{x+15}+\frac{210}{x}&=\frac{1890}{x}\\
\frac{2520}{x+15}&=\frac{1680}{x}\\
\frac{3}{x+15}&=\frac{2}{x}\\
3x&=2x+30\\
x&=30\\
x+15&=45
\end{aligned}$$

∴ The ordinary train travels 30 miles an hour, the express train travels 45 miles an hour.

11. A and B trade with different sums. A gains $200 and B loses $50, and now A's stock is to B's as 2 : ½. But if A had gained $100 and B lost $85, their stocks would have been as 15 : 3¼. Find the original stock of each.

Let x = number of dollars in A's stock at first,

and y = number of dollars in B's stock at first.

$$\therefore\ x+200:y-50=2:\tfrac{1}{2} \qquad (1)$$

$$x+100:y-85=15:3\tfrac{1}{4} \qquad (2)$$

From (1),

$$\begin{aligned}
\frac{x+200}{2}&=2y-100\\
x-4y&=-400
\end{aligned}$$

From (2),

$$\begin{aligned}
\frac{13(x+100)}{4}&=15(y-85)\\
13x-60y&=-6400\\
13x-52y&=-5200\\
\hline
8y&=\quad 1200\\
y&=150\\
x&=200
\end{aligned}$$

∴ A's original stock was $200, B's was $150.

12. A line is divided into two parts in the ratio 2 : 3, and into two parts in the ratio 3 : 4; the distance between the points of section is 2. Find the length of the line.

Let $35x =$ length of the line.
Then $14x =$ length of one part after first division.
$21x =$ length of second part after first division.
$15x =$ length of one part after second division.
$20x =$ length of second part after second division.

$$\therefore\ 15x - 14x = 21x - 20x = 2$$
$$\therefore\ x = 2$$
$$35x = 70$$

The line is 70 units long.

13. A railway consists of two sections; the annual expenditure on one is increased this year 5%, and on the other 4%, producing on the whole an increase of $4\frac{3}{10}$%. Compare the amounts expended on the two sections last year, and also the amounts expended this year.

Let $x =$ amount expended on the first section last year.
$y =$ amount expended on second section last year.
Then $x + y =$ amount expended on both sections last year.
$\frac{105}{100}x =$ amount expended on first section this year.
$\frac{104}{100}y =$ amount expended on second section this year.
$\frac{105x + 104y}{100} =$ amount expended on both sections this year.
But $\frac{1043(x+y)}{1000} =$ amount expended on both sections this year.

$$\therefore\ \frac{1043(x+y)}{1000} = \frac{105x + 104y}{100}$$
$$7x = 3y$$
$$x = \tfrac{3}{7}y$$
$$105x = 45y$$
$$\frac{105x}{104y} = \frac{45}{104}$$
$$x : y = 3 : 7$$
$$\tfrac{105}{100}x : \tfrac{104}{100}y = 45 : 104$$

∴ The amounts expended last year are to each other as 3 : 7; this year the ratio is 45 : 104.

Exercise 35.

1. If $y \propto x$, and $y = 4$ when $x = 5$, find y when $x = 12$.

Here $y = mx$
When $y = 4,\ x = 5$

$\therefore\ 4 = 5m$

$m = \frac{4}{5}$

$y = \frac{4}{5}x$

When $x = 12,\ y = \frac{48}{5} = 9\frac{3}{5}$

2. If $y \propto x$, and when $x = \frac{1}{2}$, $y = \frac{1}{3}$, find y when $x = \frac{1}{3}$.

Here $y = mx.$

When $x = \frac{1}{2},\ y = \frac{1}{3}$

$\therefore\ \frac{1}{3} = \frac{1}{2}m$

$m = \frac{2}{3}$

$y = \frac{2}{3}x$

When $x = \frac{1}{3},\ y = \frac{2}{9}$

3. If z vary jointly as x and y, and 3, 4, 5, be simultaneous values of x, y, z, find z when $x = y = 10$.

Here $z = mxy.$

When $x = 3,\ y = 4,\ z = 5$

$\therefore\ 5 = 12m$

$m = \frac{5}{12}$

$z = \frac{5}{12}xy$

When $x = y = 10$

$z = \frac{500}{12} = 41\frac{2}{3}$

4. If $y \propto \frac{1}{x}$, and when $y = 10$, $x = 2$, find the value of x when $y = 4$.

Here $y = \frac{m}{x}.$

When $y = 10,\ x = 2$

$\therefore\ 10 = \frac{m}{2}$

$m = 20$

$y = \frac{20}{x}$

When $y = 4$, $4 = \frac{20}{x}$

$x = 5$

5. If $z \propto \frac{x}{y}$, and when $z = 6$, $x = 4$, and $y = 3$, find the value of z when $x = 5$ and $y = 7$.

Here $z = \frac{mx}{y}.$

When $z = 6,\ x = 4,\ y = 3$

$$\therefore\ 6 = \frac{4m}{3}$$

$$m = \tfrac{9}{2}$$

$$z = \frac{9x}{2y}$$

When $x = 5,\ y = 7$

$$z = \tfrac{45}{14} = 3\tfrac{3}{14}$$

6. If the square of x vary as the cube of y, and $x = 3$ when $y = 4$, find the equation between x and y.

Here $x^2 = my^3$

When $x = 3,\ y = 4$

$$\therefore\ 9 = 64m$$

$$m = \tfrac{9}{64}$$

$$x^2 = \tfrac{9}{64}y^3$$

$$64x^2 = 9y^3$$

7. If the square of x vary inversely as the cube of y, and $x = 2$ when $y = 3$, find the equation between x and y.

Here $x^2 = \dfrac{m}{y^3}$

When $x = 2,\ y = 3$

$$\therefore\ 4 = \frac{m}{27}$$

$$m = 108$$

$$x^2 = \frac{108}{y^3}$$

$$x^2y^3 = 108$$

8. If x vary as x directly and y inversely, and if when $z = 2,\ x = 3$, and $y = 4$, find the value of z when $x = 15$ and $y = 8$.

Here $z = \dfrac{mx}{y}$

When $z = 2,\ x = 3,\ y = 4$

$$\therefore\ 2 = \frac{3m}{4}$$

$$m = \tfrac{8}{3}$$

$$z = \frac{8x}{3y}$$

When $x = 15,\ y = 8$

$$z = \tfrac{120}{24} = 5$$

9. If $y \propto x + c$ where c is constant, and if $y = 2$ when $x = 1$, and if $y = 5$ when $x = 2$, find y when $x = 3$.

Here $\quad y = m(x + c) \quad (1)$
When $\quad x = 1,\ y = 2$
$\therefore\ 2 = m + mc \quad (2)$
When $\quad x = 2,\ y = 5$
$\therefore\ 5 = 2m + mc \quad (3)$
From (2) and (3), $\quad m = 3,\ mc = -1$
$\therefore\ y = 3x - 1$
When $\quad x = 3,\ y = 8$

10. The velocity acquired by a stone falling from rest varies as the time of falling; and the distance fallen varies as the square of the time. If it be found that in 3 seconds a stone has fallen 145 feet, and acquired a velocity of $96\frac{2}{3}$ feet per second, find the velocity and distance fallen at the end of 5 seconds.

Let $\quad v =$ velocity in feet per second,
$t =$ number of seconds,
$x =$ distance in feet.
Then $v \propto t$, and $x \propto t^2$.
Put $\quad v = mt$
When $\quad v = 96\frac{2}{3},\ t = 3$
$\therefore\ 96\frac{2}{3} = 3m$
$m = \frac{290}{9}$
$v = \frac{290}{9}t$
When $\quad t = 5,\ v = 169\frac{1}{9}$
Put $\quad x = nt^2$
When $\quad x = 145,\ t = 3$
$145 = 9n$
$n = \frac{145}{9}$
$x = \frac{145}{9}t^2$
When $\quad t = 5,\ x = 402\frac{7}{9}$

11. If a heavier weight draw up a lighter one by means of a string passing over a fixed wheel, the space described in a given time will vary directly as the difference between the weights, and inversely as their sum. If 9 ounces draw 7 ounces through 8 feet in 2 seconds, how high will 12 ounces draw 9 ounces in the same time?

Let $\quad x =$ heavy weight,
$y =$ light weight,
$z =$ space.

$$z \propto \frac{x-y}{x+y}$$

$$z = \frac{(x-y)m}{x+y}$$

$$m = \frac{z(x+y)}{x-y}$$

Substitute values, $$m = \frac{(7+9)8}{9-7}$$

$$\therefore\ m = 64$$

$$64 = \frac{(12+9)z}{12-9}$$

$$64 = \frac{21z}{3}$$

$$7z = 64$$

$$\therefore\ z = 9\tfrac{1}{7}$$

12. The space will vary also as the square of the time. Find the space in Example 11, if the time in the latter case be 3 seconds.

We have from last example, $9\frac{1}{7}$ feet for 2 seconds.

Since space varies as square of time, we have

$$9\tfrac{1}{7} : x :: 2^2 : 3^2$$

$$\therefore\ 4x = 9 \times \tfrac{64}{7}$$

$$x = 9 \times \tfrac{16}{7} = \tfrac{144}{7} = 20\tfrac{4}{7} \text{ feet.}$$

13. Equal volumes of iron and copper are found to weigh 77 and 89 ounces respectively. Find the weight of $10\frac{1}{2}$ feet of round copper rod when 9 inches of iron rod of the same diameter weigh $31\frac{9}{10}$ ounces.

Let $x =$ required weight.

9 inches $= \frac{3}{4}$ of a foot.

If $\frac{3}{4}$ of a foot weigh 31.9 ounces, $\frac{1}{4}$ of a foot would weigh $10.03\frac{1}{3}$ ounces, and $10\frac{1}{2}$ feet would weigh 446.60 ounces.

And, as equal volumes of iron and copper weigh 77 and 89 ounces respectively,

$$77 : 89 :: 446\tfrac{3}{5} : x$$

$$\therefore\ x = 516\tfrac{1}{5} \text{ ounces.}$$

14. The square of the time of a planet's revolution varies as the cube of its distance from the sun. The distances of the Earth and Mercury from the sun being 91 and 35 millions of miles, find in days the time of Mercury's revolution.

Let $x =$ time of Mercury's revolution.

$$91^3 : 35^3 :: 1^2 : x^2$$

$$13^3 : 5^3 :: 1 : x^2$$

Whence, $x^2 = .056895$

$$\therefore\ x = .238, \text{ time in years},$$

$$= 87.1, \text{ time in days}.$$

15. A spherical iron shell 1 foot in diameter weighs $\frac{91}{216}$ of what it would weigh if solid. Find the thickness of the metal, it being known that the volume of a sphere varies as the cube of its diameter.

Let $D =$ diameter of shell,

$d =$ diameter of sphere required to fill the shell,

and 1 represent the weight of iron sphere having diameter $= D$.

Then $1 - \frac{91}{216}$ will represent the weight of iron sphere having diameter $= d$.

Now the weights vary as the cubes of their diameters,

$$\therefore\ D^3 : d^3 :: 1 : 1 - \tfrac{91}{216}$$

That is, $D^3 : d^3 :: 1 : \frac{125}{216}$

or, by extracting the cube root of each term,

$$D : d :: 1 : \tfrac{5}{6}$$

or $d = \frac{5}{6} D$

Since the thickness of the shell $= \frac{1}{2}(D - d)$

the thickness of the shell $= \frac{1}{2}(1 - \frac{5}{6}) = \frac{1}{12}$

Hence, the thickness of the shell is $\frac{1}{12}$ of a foot, $= 1$ inch.

16. The volume of a sphere varies as the cube of its diameter. Compare the volume of a sphere 6 inches in diameter with the sum of the volumes of three spheres whose diameters are 3, 4, 5 inches respectively.

Let $x =$ volume of first sphere in cubic inches,

and $y =$ sum of volumes of other three in cubic inches.

Then $x : y = 6^3 : 3^3 + 4^3 + 5^3$

$x : y = 216 : 216.$

The volumes in question are equal.

17. Two circular gold plates, each an inch thick, the diameters of which are 6 inches and 8 inches respectively, are melted and formed into a single circular plate 1 inch thick. Find its diameter, having given that the area of a circle varies as the square of its diameter.

Let $x =$ required diameter of plate,

$y =$ volume of first plate,

$z =$ volume of second plate.

Then $y + z =$ volume of required plate.

Now $y + z : x^2 = y : 36 = z : 64$

In the last two ratios $y + z : 100 = y : 36$

But $y + z : x^2 = y : 36$

$\therefore x^2 = 100$

$x = 10$ in.

Table on Page 179.

1.

The series is $a, a + d, a + 2d, \ldots\ldots$

$\therefore$ *nth* term is $a + (n - 1)d$

Let l denote last or *nth* term.

Then $l = a + (n - 1)d$

2.

$$l = a + (n - 1)d$$
$$l = a + nd - d$$
$$nd = l - a + d$$
$$n = \frac{l - a + d}{d}$$
$$s = \frac{n}{2}(a + l)$$
$$2s = n(a + l)$$
$$n = \frac{2s}{a + l}$$
$$\frac{2s}{a + l} = \frac{l - a + d}{d}$$

Simplify, $2sd = l^2 - a^2 + ad + ld$

$$l^2 + dl = 2sd + a^2 - ad$$

Complete the square,

$$4l^2 + (\quad) + d^2 = 8sd + 4a^2 - 4ad + d^2$$

Extract the root, $2l + d = \pm\sqrt{8sd + 4a^2 - 4ad + d^2}$

$$l = -\tfrac{1}{2}d \pm \sqrt{2ds + a^2 - ad + \frac{d^2}{4}}$$
$$l = -\tfrac{1}{2}d \pm \sqrt{2ds + (a - \tfrac{1}{2}d)^2}$$

3.
$$s=\frac{n}{2}(a+l)$$
$$2s=na+nl$$
Transpose,
$$2s-na=nl$$
$$\therefore\ l=\frac{2s-na}{n}$$
$$\therefore\ l=\frac{2s}{n}-a$$

4.
$$l=a+(n-1)d \qquad (1)$$
$$s=\frac{n}{2}(a+l) \qquad (2)$$
$$2s=na+nl$$
$$\therefore\ a=\frac{2s}{n}-l$$
Substitute value of a in (1),
$$l=\frac{s}{n}+\frac{(n-1)d}{2}$$

5.
$$l=a+(n-1)d \qquad (1)$$
$$s=\frac{n}{2}(a+l) \qquad (2)$$
Substitute in (2) $a+(n-1)d$ for l,
$$s=\frac{n}{2}[2a+(n-1)d]$$

6.
$$l=a+(n-1)d \qquad (1)$$
$$s=\frac{n}{2}(a+l) \qquad (2)$$
From (1), $l=a+nd-d$

From (2), $2s=na+nl$
$$\therefore\ n=\frac{l-a+d}{d}$$
and
$$n=\frac{2s}{a+l}$$
$$\therefore\ \frac{l-a+d}{d}=\frac{2s}{a+l}$$
$$l^2-a^2+ad+dl=2sd$$
$$\therefore\ s=\frac{l^2-a^2+ad+dl}{2d}$$
or
$$s=\frac{l^2-a^2}{2d}+\frac{l+a}{2}$$

7.
$$\begin{array}{rl}
s= & a+(a+d)+(a+2d)+\cdots\cdots+(l-d)+l\\
\text{or}\quad s= & l+(l-d)+(l-2d)+\cdots\cdots+(a+d)+a\\
\hline
\therefore\ 2s= & (a+l)+(a+l)+(a+l)+\cdots\cdots+(a+l)+(a+l)
\end{array}$$
$$2s=n(a+l)$$
$$\therefore\ s=\frac{n(a+l)}{2}$$

8.

$$l = a + (n-1)d \qquad (1)$$

$$s = \tfrac{1}{2}n(a + l) \qquad (2)$$

From (1), $a = l - (n-1)d$

From (2), $a = \dfrac{2s - ln}{n}$

Then $\dfrac{2s - ln}{n} = l - (n-1)d$

$$2s = 2ln - (n-1)dn$$

$$\therefore\ s = \tfrac{1}{2}n[2l - (n-1)d]$$

9.

$$l = a + (n-1)d$$

Transposing, $a = l - (n-1)d$

10.

$$l = a + (n-1)d$$

$$s = \tfrac{1}{2}n(a + l)$$

From (2), $2s = na + nl$

$$\therefore\ l = \frac{2s - na}{n}$$

$$\therefore\ a + (n-1)d = \frac{2s - na}{n}$$

$$na + dn^2 - dn = 2s - na$$

$$2na = 2s - dn^2 + dn$$

$$\therefore\ a = \frac{s}{n} - \frac{(n-1)d}{2}$$

11.

$$l = a + (n-1)d$$

$$dn = l - a + d$$

$$n = \frac{l - a + d}{d}$$

$$s = \tfrac{1}{2}n(a + l)$$

$$2s = na + nl$$

$$n = \frac{2s}{a + l}$$

$$\therefore\ \frac{l - a + d}{d} = \frac{2s}{a + l}$$

$$\therefore\ l^2 - a^2 + ad + ld = 2ds.$$
$$a^2 - ad = l^2 + ld - 2ds$$
$$a^2 - (\ \ ) + (\tfrac{1}{2}d)^2 = l^2 + ld - 2ds + (\tfrac{1}{2}d)^2$$
$$a - \tfrac{1}{2}d = \pm\sqrt{l^2 + ld - 2ds + (\tfrac{1}{2}d)^2}$$
$$\therefore\ a = \tfrac{1}{2}d \pm \sqrt{(l + \tfrac{1}{2}d)^2 - 2ds}$$

12.

$$s = \tfrac{1}{2}n(a + l)$$

Multiply by $\frac{2}{n}$,

$$\frac{2s}{n} = a + l$$
$$\therefore\ a = \frac{2s}{n} - l$$

13.

$$a = l - (n - 1)d$$
$$d(n - 1) = l - a$$
$$\therefore\ d = \frac{l - a}{n - 1}$$

14.

$$l = a + (n - 1)d$$
$$s = \tfrac{1}{2}n(a + l),$$
$$2s = an + nl$$
$$\therefore\ l = \frac{2s - an}{n}$$
$$\therefore\ a + nd - d = \frac{2s - an}{n}$$
$$an + n^2d - nd = 2s - an$$
$$n^2d - nd = 2s - 2an$$
$$d(n^2 - n) = 2(s - an)$$
$$\therefore\ d = \frac{2(s - an)}{n(n - 1)}$$

15.

$$l = a + (n - 1)d$$
$$\therefore\ n = \frac{l - a}{d} + 1$$
$$s = \tfrac{1}{2}n(a + l)$$
$$\therefore\ n = \frac{2s}{a + l}$$
$$\therefore\ \frac{l - a}{d} + 1 = \frac{2s}{a + l}$$
$$\therefore\ l^2 - a^2 + ad + bl = 2ds$$
$$\therefore\ 2ds - ad - dl = l^2 - a^2$$
$$\therefore\ d(2s - l - a) = l^2 - a^2$$
$$\therefore\ d = \frac{l^2 - a^2}{2s - l - a}$$

16.

$$l = a + (n - 1)d$$
$$\therefore\ a = l - (n - 1)d$$
$$s = \tfrac{1}{2}n(a + l)$$
$$\therefore\ a = \frac{2s - ln}{n}$$
$$\therefore\ l - (n - 1)d = \frac{2s - ln}{n}$$
$$ln - dn^2 + dn = 2s - ln$$
$$-dn^2 + dn = 2s - 2ln$$
$$d(n - 1)n = 2(ln - s)$$
$$\therefore\ d = \frac{2(ln - s)}{n(n - 1)}$$

17.

$$l = a + (n - 1)d$$
$$l = a + dn - d$$
$$dn = l - a + d$$
$$\therefore\ n = \frac{l - a}{d} + 1$$

18. (1)

$$l = a + (n-1)d \qquad (2)$$

$$s = \tfrac{1}{2}n(a + l)$$

From (2),

$$l = \frac{2s}{n} - a.$$

$$\therefore\ a + (n-1)d = \frac{2s}{n} - a.$$

$$an + d^2n - dn = 2s - an$$

$$dn^2 + n(2a - d) = 2s.$$

Complete the square,

$$4d^2n^2 + (\ \ ) + (2a-d)^2 = (2a-d)^2 + 8ds.$$

Extract the root,

$$2dn + (2a - d) = \pm\sqrt{(2a-d)^2 + 8ds}$$

$$2dn = d - 2a \pm \sqrt{(2a-d)^2 + 8ds}$$

$$\therefore\ n = \frac{d - 2a \pm \sqrt{(2a-d)^2 + 8ds}}{2d}$$

19.

$$s = \tfrac{1}{2}n(a + l)$$

$$2s = an + ln$$

$$\therefore\ n = \frac{2s}{l + a}$$

20.

$$l = a + (n-1)d \qquad (1)$$

$$s = \tfrac{1}{2}n(a + l) \qquad (2)$$

From (1),

$$a = l - (n-1)d$$

From (2),

$$a = \frac{2s - ln}{n}$$

$$\therefore\ l - (n-1)d = \frac{2s - ln}{n}$$

$$\therefore\ ln - dn^2 + dn = 2s - ln$$

$$dn^2 - (2l + d)n = -2s$$

$$4d^2n^2 - (\ \ ) + (2l + d)^2 = (2l + d)^2 - 8ds$$

$$2dn - (2l + d) = \sqrt{(2l + d)^2 - 8ds}$$

$$\therefore\ n = \frac{2l + d \pm \sqrt{(2l + d)^2 - 8ds}}{2d}$$

Exercise 36.

1. **Find the 10th term of 3, 8, 13 .....**

$$l = a + (n-1)d = 3 + 9 \times 5 = 48.$$

2. **Find the 8th term of 12, 9, 6 .....**

$$l = a + (n-1)d = 22 + 7(-3) = -9.$$

3. **Find the 12th term of -4, -9, -14 .....**

$$l = a + (n-1)d = -4 + 11(-5) = -59.$$

4. **Find the 11th term of $2\frac{1}{2}$, $1\frac{5}{6}$, $1\frac{1}{6}$ .....**

$$l = a + (n-1)d = 2\tfrac{1}{2} + 10(-\tfrac{2}{3}) = -4\tfrac{1}{6}.$$

5. **Find the 14th term of $1\frac{1}{3}$, $\frac{1}{4}$, $-\frac{5}{6}$ .....**

$$l = a + (n-1)d = 1\tfrac{1}{3} + 13(-1\tfrac{1}{12}) = -\tfrac{51}{4} = -12\tfrac{3}{4}.$$

6. **Find the sum of 8 terms of 4, 7, 10 .....**

$$l = a + (n-1)d = 4 + 7 \times 3 = 25.$$

$$s = \frac{n}{2}(a + l) = 4\,(4 + 25) = 116.$$

7. **Find the sum of 10 terms of 8, 5, 2 .....**

$$l = a + (n-1)d = 8 + 9(-3) = -19.$$

$$s = \frac{n}{2}(a + l) = 5\,(8 - 19) = -55.$$

8. **Find the sum of 12 terms of -3, 1, 5 .....**

$$l = a + (n-1)d = -3 + 11 \times 4 = 41.$$

$$s = \frac{n}{2}(a + l) = 6\,(-3 + 41) = 228.$$

9. **Find the sum of n terms of 2, $1\frac{1}{6}$, $\frac{1}{3}$ .....**

$$l = a + (n-1)d = 2 + (n-1)(-\tfrac{5}{6}) = 2 - \tfrac{5}{6}n + \tfrac{6}{5} = \frac{17 - 5n}{6}.$$

$$s = \frac{n}{2}(a + l) = \frac{n}{2}\left(2 + \frac{17 - 5n}{6}\right) = \frac{29n - 5n^2}{12}.$$

10. Find the sum of n terms of $2\frac{1}{4}, 1\frac{5}{6}, 1\frac{5}{12}\cdots\cdots$

$$l = a + (n-1)d = 2\tfrac{1}{4} + (n-1)(-\tfrac{5}{12}) = \frac{9}{4} - \frac{5n}{12} + \frac{5}{12} = \frac{32-5n}{12}.$$

$$s = \frac{n}{2}(a+l) = \frac{n}{2}\left(\frac{9}{4} + \frac{32-5n}{12}\right) = \frac{59n - 5n^2}{24}.$$

11. Given $a = 3, l = 55, n = 13$. Find d and s.

$$l = a + (n-1)d. \qquad 55 = 3 + 12d. \qquad d = \tfrac{13}{3}.$$

$$s = \frac{n}{2}(a+l) = \tfrac{13}{2}(3 + 55) = 377.$$

12. Given $a = 3\frac{1}{4}, l = 64, n = 82$. Find d and s.

$$l = a + (n-1)d. \qquad 64 = 3\tfrac{1}{4} + 81d. \qquad 81d = \tfrac{243}{4}. \qquad d = \tfrac{3}{4}.$$

$$s = \frac{n}{2}(a+l) = 41(3\tfrac{1}{4} + 64) = \frac{41 \times 269}{4} = \frac{11029}{4} = 2757\tfrac{1}{4}.$$

13. Given $a = 1, n = 20, 3 = 305$. Find d and l.

$$\begin{aligned} l &= a + (n-1)d \\ s &= \frac{n}{2}(a+l) \\ l &= 1 + 19d && (1) \\ 305 &= 10(1+l) && (2) \end{aligned}$$

$$\therefore\ l = \tfrac{59}{2}.$$

In (1),

$$\begin{aligned} \tfrac{59}{2} &= 1 + 19d \\ 57 &= 38d \\ d &= \tfrac{3}{2} = 1\tfrac{1}{2}. \end{aligned}$$

14. Given $l = 105, n = 16, s = 840$. Find a and d.

$$\begin{aligned} l &= a + (n-1)d \\ s &= \frac{n}{2}(a+l) \\ 105 &= a + 15d && (1) \\ 840 &= 8(a + 105) && (2) \end{aligned}$$

From (2),

$$\begin{aligned} a + 105 &= 105 \\ a &= 0 \end{aligned}$$

In (1),

$$\begin{aligned} 15d &= 105 \\ d &= 7 \end{aligned}$$

15. Given $d = 7, n = 12, s = 594$. Find a and l.

$$\begin{aligned} l &= a + (n-1)d \\ s &= \frac{n}{2}(a+l) \\ l &= a + 77 \\ 594 &= 6(a+l) \\ l - a &= 77 \end{aligned}$$

$$l + a = 99$$

Adding,

$$\begin{aligned} 2l &= 176 \\ l &= 88 \end{aligned}$$

Subtracting,

$$\begin{aligned} 2a &= 22 \\ a &= 11 \end{aligned}$$

16. Given $a = 9$, $d = 4$, $s = 624$. Find n and l.

$$l = a + (n - 1)d$$
$$s = \frac{n}{2}(a + l)$$
$$l = 9 + 4(n - 1)$$
$$l = 5 + 4n \qquad (1)$$
$$624 = \frac{n}{2}(9 + l)$$
$$9n + nl = 1248 \qquad (2)$$

Substitute in (2) the value of l obtained from (1),

$$9n + n(5 + 4n) = 1248$$
$$4n^2 + 14n = 1248$$
$$4n^2 + 14n + \tfrac{49}{4} = \tfrac{5041}{4}$$
$$2n + \tfrac{7}{2} = \pm\tfrac{71}{2}$$
$$n = 16, \text{ or } -\tfrac{39}{2}$$

If $n = 16$, $\quad l = 5 + 4n = 69.$

If $n = -\tfrac{39}{2}$, $l = 5 + 4n = -73.$

This last answer may be rejected. It may be retained, however, if we interpret the $\tfrac{39}{2}$th term to be $a + \tfrac{39}{2}d$, and interpret the $-$ sign to mean that the terms are to be counted *before* a.

17. Given $d = 5$, $l = 77$, $s = 623$. Find a and n.

$$l = a + (n - 1)d$$
$$s = \frac{n}{2}(a + l)$$
$$77 = a + 5(n - 1)$$
$$a + 5n = 82 \qquad (1)$$
$$623 = \frac{n}{2}(a + 77)$$
$$an + 77n = 1246 \qquad (2)$$

From (1), $\qquad a = 82 - 5n$

Substitute in (2), $\quad 82n - 5n^2 + 77n = 1246$

$$5n^2 - 159 = -1246$$
$$100n^2 - (\ ) + (159)^2 = 25281 - 24920 = 361$$
$$10n - 159 = \pm 19$$
$$n = 14, \text{ or } \tfrac{89}{5}$$
$$a = 82 - 5n$$

If $\qquad n = 14,\ a = 12$

If $\qquad n = \tfrac{89}{5},\ a = -7$

This last answer may be rejected or retained.

18. When a train arrives at the top of a long slope, the last car is detached and begins to descend, passing over 3 feet in the first second, three times 3 feet in the second second, five times 3 feet in the third second, etc. At the end of 12 minutes it reaches the bottom of the slope. What was its velocity in the last second?

$$l = a + (n - 1)d$$

Here $\qquad a = 3,\ d = 6,\ n = 120$

$$l = 3 + 119 \times 6 = 717 \text{ feet per second.}$$

19. Insert eleven arithmetical means between 1 and 12.

Counting 1 and 12 the number of terms is 13.

$$l = a + (n-1)\,d$$
$$12 = 1 + 12\,d$$
$$d = \tfrac{11}{12}$$

The means are $1\tfrac{11}{12}$, $2\tfrac{5}{6}$, $3\tfrac{3}{4}$, $4\tfrac{2}{3}$, $5\tfrac{7}{12}$, $6\tfrac{1}{2}$, $7\tfrac{5}{12}$, $8\tfrac{1}{3}$, $9\tfrac{1}{4}$, $10\tfrac{1}{6}$, $11\tfrac{1}{12}$.

20. The first term of an arithmetical series is 3, and the sum of six terms is 28. What term will be 9?

$$s = \frac{n}{2}(a + l)$$
$$28 = 3\,(3 + l)$$
$l = \tfrac{19}{3}$, 6th term.
$$l = a + (n-1)d$$
$$\tfrac{19}{3} = 3 + 5\,d$$
$$d = \tfrac{2}{3}$$

If 9 is nth term,
$$9 = 3 + \tfrac{2}{3}(n-1)$$
$$27 = 9 + 2\,(n-1)$$
$$n - 1 = 19$$
$$n = 10$$
The 10th term is 9.

21. How many terms of the series $-5 - 2 + 1 + \cdots\cdots$ must be taken in order that their sum may be 63?

Here $a = -5,\ d = 3,\ s = 63.$
$$l = a + (n-1)d$$
$$s = \frac{n}{2}(a + l)$$
$$l = -5 + 3\,(n-1) \quad (1)$$
$$63 = \frac{n}{2}(-5 + l) \quad (2)$$
$$l = 3n - 8 \quad (3)$$
$$ln - 5n = 126 \quad (4)$$
Nine terms must be taken.

Substitute in (4) the value of l obtained from (3),
$$3n^2 - 8n - 5n = 126$$
$$3n^2 - 13n = 126$$
$$n^2 - \frac{13n}{3} + \frac{169}{36} = 41 + \frac{169}{36} = \frac{1681}{36}$$
$$n = \frac{13 \pm 41}{36}$$
$$n = 9, \text{ or } -\tfrac{14}{3}$$

22. The arithmetical mean between two numbers is 10, and the mean between the double of the first and the triple of the second is 27. Find the numbers.

Let x = first number, and y = second number.
$$\frac{x+y}{2} = 10$$
$$\frac{2x+3y}{2} = 27$$
$$x + y = 20$$

$$2x + 3y = 54$$
$$2x + 2y = 40$$
$$y = 14$$
$$\therefore\ x = 6$$
The numbers are 6 and 14.

23. The first term of an arithmetical progression is 3, the third term is 11. Find the sum of seven terms.

$$\text{Third term is } a + 2d.$$
$$3 + 2d = 11$$
$$d = 4$$
$$l = a + (n - 1)d = 3 + 6 \times 4 = 27$$
$$s = \frac{n}{2}(a + l) = \tfrac{7}{2}(3 + 27) = 105.$$

24. Arithmetical means are inserted between 8 and 32, so that the sum of the first two is to the sum of the last two as 7 is to 25. How many means are inserted?

The first two means are $8 + d$ and $8 + 2d$.
The last two means are $32 - d$ and $32 - 2d$

$$\therefore\ 8 + d + 8 + 2d \cdot 32 - d + 32 - 2d = 7 : 25$$
$$\frac{16 + 3d}{64 + 3d} = \frac{7}{25}$$
$$400 + 75d = 448 - 21d$$
$$96d = 48$$
$$d = \tfrac{1}{2}$$
$$l = a + (n - 1)d$$
$$31\tfrac{1}{2} = 8\tfrac{1}{2} + \tfrac{1}{2}(n - 1)$$
$$n - 1 = 46$$
$$n = 47$$

47 means are inserted.

25. In an arithmetical series the common difference is 2, and the square roots of the first, third, and sixth terms form a new arithmetical series. Find the series.

Let $a =$ first term.
Then $a + 4 =$ third term.
$a + 10 =$ sixth term.

$$\therefore\ \sqrt{a + 10} - \sqrt{a + 4} = \sqrt{a + 4} - \sqrt{a}$$
$$\sqrt{a} + \sqrt{a + 10} = 2\sqrt{a + 4}$$
$$2a + 10 + 2\sqrt{a^2 + 10a} = 4a + 16$$
$$\sqrt{a^2 + 10a} = a + 3$$
$$a^2 + 10a = a^2 + 6a + 9$$
$$4a = 9$$
$$a = \tfrac{9}{4} = 2\tfrac{1}{4}$$

$\therefore$ The series is $2\tfrac{1}{4}, 4\tfrac{1}{4}, 6\tfrac{1}{4}, \cdots\cdots$

26. Find three numbers in arithmetical progression of which the sum is 21, and the sum of the first and second $\frac{3}{4}$ of the sum of the second and third.

Let $a - d =$ the first number.
$a =$ the second number.
Then $a + d =$ the third number.

$$\therefore\ a - d + a + a + d = 21 \qquad (1)$$
$$a - d + a = \tfrac{3}{4}(a + a + d) \qquad (2)$$

From (1), $3a = 21$
$a = 7$

From (2), $8a - 4d = 6a + 3d$
$2a = 7d$
$\therefore\ d = 2$
$a - d = 5$
$a + d = 7$

$\therefore$ The numbers are 5, 7, and 9.

27. The sum of three numbers in arithmetical progression is 33, and the sum of their squares is 461. Find the numbers.

Let $a - d$, a, $a + d$ be the three numbers.

Then $$a - d + a + a + d = 33 \qquad (1)$$
$$(a - d)^2 + a^2 + (a + d)^2 = 461 \qquad (2)$$

From (1), $3a = 33$
$$a = 11 \qquad (3)$$

From (2), $3a^2 + 2d^2 = 461$
Substitute value of a from (3), $363 + 2d^2 = 461$
$d^2 = 49$
$d = \pm 7$
$a - d = 4$ or 18
$a + d = 18$ or 4

$\therefore$ The three numbers are 4, 11, and 18.

28. The sum of four numbers in arithmetical progression is 12, and the sum of their squares 116. What are these numbers.

Let $a - 3d$, $a - d$, $a + d$, and $a + 3d$ be the numbers.

Then $$a - 3d + a - d + a + d + a + 3d = 12 \qquad (1)$$
$$(a - 3d)^2 + (a - d)^2 + (a + d)^2 + (a + 3d)^2 = 116 \qquad (2)$$

From (1), $4a = 12$
$$a = 3 \qquad (3)$$

From (2), $4a^2 + 20d^2 = 116$

Substitute value of a from (3), $36 + 20d^2 = 116$

$$d^2 = 4$$
$$d = \pm 2$$

From (3), $a = 3$

$$a - 3d = -3 \text{ or } 9$$
$$a - d = 1 \text{ or } 5$$
$$a + d = 5 \text{ or } 1$$
$$a + 3d = 9 \text{ or } -3$$

∴ The numbers are -3, 1, 5, and 9.

29. How many terms of the series 1, 4, 7 ····· must be taken, in order that the sum of the first half may bear to the sum of the second half the ratio 7 : 22?

Let $2n$ = number of terms,
s = sum of the whole series,
s' = sum of the first half.

For first half,

$$l = a + (n-1)d = 1 + 3(n-1) = 3n - 2$$
$$s' = \frac{n}{2}(1 + 3n - 2) = \frac{n}{2}(3n - 1)$$

For the whole series,

$$l = 1 + 3(2n - 1) = 6n - 2$$
$$s = n(6n - 1)$$

Now $s' : s :: 7 : 29$

$$\therefore \frac{\frac{n}{2}(3n-1)}{n(6n-1)} = \frac{7}{29}$$
$$87n - 29 = 84n - 14$$
$$3n = 15$$
$$n = 5$$
$$2n = 10$$

10 terms must be taken.

30. The sum of the squares of the extremes of four numbers in arithmetical progression is 200, and the sum of the squares of the means is 136. What are the numbers?

Let $a - 3d, a - d, a + d, a + 3d$ be the four numbers.

Then $(a - 3d)^2 + (a + 3d)^2 = 200$ (1)

$(a - d)^2 + (a + d)^2 = 136$ (2)

From (1), $2a^2 + 18d^2 = 200$

From (2), $2a^2 + 2d^2 = 136$

$$\therefore 16d^2 = 64$$
$$d^2 = 4$$
$$d = \pm 2$$
$$\therefore a = \pm 8$$
$$a - 3d = \pm 2, \text{ or } \pm 14$$
$$a - d = \pm 6, \text{ or } \pm 10$$
$$a + d = \pm 10, \text{ or } \pm 6.$$
$$a + 3d = \pm 14, \text{ or } \pm 2$$

∴ The numbers are 2, 6, 10, 14, or -2, -6, -10, -14.

31. A man wishes to have his horse shod. The blacksmith asks him $2 a shoe, or 1 cent for the first nail, 3 for the second, 5 for the third, etc. Each shoe has 8 nails. Ought the man to accept the second proposition?

Here $a = 1,\ d = 2,\ n = 32$

$$l = 1 + 31 \times 2 = 63$$
$$s = 16(1 + 63) = 1024$$

The shoeing would therefore cost $10.24 if the second proposition were accepted. At $2.00 a shoe it would only cost $8.00.

32. A number consists of three digits which are in arithmetical progression; and this number divided by the sum of its digits is equal to 26; if 198 be added to the number, the digits in the units' and hundreds' places will be interchanged. Required the number.

Let $a - b$, a, $a + b$ be the three digits.

Then

$100(a - b) + 10a + a + b =$ the number,

$100(a + b) + 10a + a - b =$ the number if the first and third digits are interchanged.

$$\therefore \frac{100(a - b) + 10a + a + b}{3a} = 26 \qquad (1)$$

$$100(a - b) + 10a + a + b + 198 = 100(a + b) + 10a + a - b \qquad (2)$$

From (1), $\dfrac{111a - 99b}{3a} = 26$

$$111a - 99b = 78a$$
$$33a = 99b$$
$$a = 3b$$

From (2), $111a - 99b + 198 = 111a + 99b$

$$198b = 198$$
$$b = 1$$
$$a = 3$$
$$a - b = 2$$
$$a + b = 4$$

∴ The number is 234.

33. There are placed in a straight line upon a lawn 50 eggs 3 feet distant from each other. A person is required to pick them up one by one and carry them to a basket in the line of the eggs and 3 feet from the first egg, while a runner, starting from the basket, touches a goal and returns. At what distance ought the goal to be placed that both men may have the same distance to pass over?

In picking up the eggs the man goes successively 6, 12, 18, ... feet.

Hence the distances form an arithmetical series for which $a = 6$, $d = 6$, $n = 50$.

$$l = 6 + 49 \times 6 = 300$$
$$s = 25(16 + 300) = 7650$$

∴ The first man travels 7650 feet. The goal must therefore be placed at a distance of 3825 feet.

34. Starting from a box, there are placed upon a straight line 40 stones, at the distances 1 foot, 3 feet, 5 feet, etc. A man placed at the box is required to take them and carry them back one by one. What is the total distance that he has to accomplish?

The man goes successively 2, 6, 10, ... feet.

Hence $a = 2$, $d = 4$, $n = 40$

$$l = 2 + 39 \times 4 = 158$$
$$s = 20(2 + 158) = 3200$$

∴ He has to travel 3200 feet.

35. The sum of five numbers in arithmetical progression is 45, and the product of the first and fifth is $\frac{5}{8}$ of the product of the second and fourth. Find the numbers.

Let the number be $a - 4b$, $a - 2b$, a, $a + 2b$, $a + 4b$.

Then,

$$a - 4b + a - 2b + a + a + 2b + a + 4b = 45 \qquad (1)$$
$$(a - 4b)(a + 4b) = \tfrac{5}{8}(a - 2b)(a + 2b) \qquad (2)$$

From (1),
$$5a = 45$$
$$a = 9 \qquad (3)$$

From (2),
$$8(a^2 - 16b^2) = 5(a^2 - 4b^2)$$
$$3a^2 = 108b^2$$
$$a^2 = 36b^2$$

Substitute value of a from (3),
$$81 = 36b^2$$
$$\therefore b = \pm \tfrac{3}{2}$$
$$a - 4b = 3, \text{ or } 15$$
$$a - 2b = 6, \text{ or } 12$$
$$n + 2b = 12, \text{ or } 6$$
$$a + 4b = 15, \text{ or } 3$$

$\therefore$ The numbers are 3, 6, 9, 12, and 15.

TABLE ON PAGE 186.

1.
$$\text{1st term} = a$$
$$\text{2d term} = ar$$
$$\text{3d term} = ar^2$$
$$\cdots\cdots\cdots$$
$$n\text{th term} = ar^{n-1}$$
$$\therefore l = ar^{n-1}$$

2.
$$s = \frac{rl - a}{r - 1}$$
$$rs - s = rl - a$$
$$rl = a + (r - 1)s$$
$$\therefore l = \frac{a + (r - 1)s}{r}$$

3.
$$l = ar^{n-1} \qquad (1)$$
$$\therefore rl = ar^n \qquad (2)$$

From (1),
$$r^{n-1} = \frac{l}{a}$$
$$\therefore r = \sqrt[n-1]{\frac{l}{a}}$$

Also,
$$s = \frac{a(r^n - 1)}{r - 1}$$
$$sr - s = ar^n - a$$
$$sr - ar^n = s - a$$

Substitute rl for ar^n,
$$sr - rl = s - a$$
$$r(s - l) = s - a$$
$$\therefore r = \frac{s - a}{r - l}$$
$$\therefore \sqrt[n-1]{\frac{l}{a}} = \frac{s - a}{s - l}$$
$$\frac{l}{a} = \left(\frac{s - a}{s - l}\right)^{n-1}$$
$$l(s - l)^{n-1} - a(s - a)^{n-1} = 0$$

4.
$$l = ar^{n-1}$$
$$\therefore a = \frac{l}{r^{n-1}}$$
$$s = \frac{a(r^n - 1)}{r - 1}$$
$$s(r - 1) = a(r^n - 1)$$
$$a = \frac{s(r - 1)}{r^n - 1}$$
$$\therefore \frac{l}{r^{n-1}} = \frac{s(r - 1)}{r^n - 1}$$
$$l(r^n - 1) = sr^{n-1}(r - 1)$$
$$l = \frac{sr^{n-1}(r - 1)}{r^n - 1}$$

5.

$$s = a + ar + ar^2 + ar^3 + \cdots\cdots + ar^{n-1}$$

Multiply by r, $$rs = \quad ar + ar^2 + ar^3 + \cdots\cdots + ar^{n-1} + ar^n$$

Subtract, $$rs - s = ar^n - a$$

$$s(r-1) = a(r^n - 1)$$

$$s = \frac{a(r^n - 1)}{r - 1}$$

6.

$$s = a + ar + ar^2 + ar^3 + \cdots\cdots + ar^{n-1}$$

Multiply by r, $$rs = \quad ar + ar^2 + ar^3 + \cdots\cdots + ar^{n-1} + ar^n$$

Subtract, $$rs - s = ar^n - a$$

$$(r-1)s = a(r^n - 1)$$

$$s = \frac{ar^n - a}{r - 1}$$

Since $$l = ar^{n-1}$$

$$rl = ar^n$$

Substitute rl for ar^n, $$s = \frac{rl - a}{r - 1}$$

7. $$l = ar^{n-1}$$

$$r^{n-1} = \frac{l}{a}$$

$$r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}}$$

$$s = \frac{a(r^n - 1)}{r - 1}$$

$$rs - s = a(r^n - 1)$$

Substitute value of r,

$$s\left(\frac{l}{a}\right)^{\frac{1}{n-1}} - s = a\left(\frac{l}{a}\right)^{\frac{n}{n-1}} - a$$

$$s\left\{\left(\frac{l}{a}\right)^{\frac{1}{n-1}} - 1\right\} = a\left(\frac{l}{a}\right)^{\frac{n}{n-1}} - a$$

$$s = \frac{a\left(\frac{l}{a}\right)^{\frac{n}{n-1}} - a}{\left(\frac{l}{a}\right)^{\frac{1}{n-1}} - 1}$$

$$= \frac{\sqrt[n-1]{l^n} - \sqrt[n-1]{a^n}}{\sqrt[n-1]{l} - \sqrt[n-1]{a}}$$

8. $$l = ar^{n-1} \qquad (1)$$

$$s = \frac{a(r^n - 1)}{r - 1} \qquad (2)$$

From (1), $$a = \frac{l}{r^{n-1}}$$

From (2), $$a = \frac{s(r-1)}{r^n - 1}$$

$$\therefore \frac{s(r-1)}{r^n - 1} = \frac{l}{r^{n-1}}$$

$$s(r^n - r^{n-1}) = lr^n - l$$

$$s = \frac{lr^n - l}{r^n - r^{n-1}}$$

9. $$l = ar^{n-1}$$

$$ar^{n-1} = l$$

$$\therefore a = \frac{l}{r^{n-1}}$$

10.
$$s=\frac{ar^n-a}{r-1}$$
$$(r-1)s=(r^n-1)a$$
$$\therefore\ a=\frac{(r-1)s}{r^n-1}$$

11.
$$s=\frac{rl-a}{r-1}$$
$$(r-1)s=rl-a$$
$$a=rl-(r-1)s$$

12.
$$l=ar^{n-1} \qquad (1)$$
$$s=\frac{rl-a}{r-1} \qquad (2)$$

From (1), $r=\left(\frac{l}{a}\right)^{\frac{1}{n-1}}$

From (2), $r=\frac{s-a}{s-l}$

$$\therefore\ \left(\frac{l}{a}\right)^{\frac{1}{n-1}}=\frac{s-a}{s-l}$$
$$\therefore\ \frac{l}{a}=\frac{(s-a)^{n-1}}{(s-l)^{n-1}}$$
$$\therefore\ a(s-a)^{n-1}-l(s-l)^{n-1}=0$$

13.
$$l=ar^{n-1}$$
$$r^{n-1}=\frac{l}{a}$$
$$\therefore\ r=\sqrt[n-1]{\frac{l}{a}}$$

14.
$$s=\frac{a(r^n-1)}{r-1}$$
$$rs-s=ar^n-a$$
$$rs-ar^n=s-a$$
$$ar^n=rs-(s-a)$$
$$\therefore\ r^n=\frac{s}{a}r-\frac{s-a}{a}$$
$$\therefore\ r^n-\frac{s}{a}r+\frac{s-a}{a}=0$$

15.
$$s=\frac{rl-a}{r-1}$$
$$rs-s=rl-a$$
$$rs-rl=s-a$$
$$r(s-l)=s-a$$
$$\therefore\ r=\frac{s-a}{s-l}$$

16.
$$l=ar^{n-1}$$
$$s=\frac{a(r^n-1)}{r-1}$$

Then $a=\frac{l}{r^{n-1}}$

$$a=\frac{s(r-1)}{r^n-1}$$
$$\therefore\ \frac{l}{r^{n-1}}=\frac{s(r-1)}{r^n-1}$$
$$lr^n-l=sr^n-sr^{n-1}$$
$$sr^n-sr^{n-1}-lr^n+l=0$$
$$r^n(s-l)-sr^{n-1}+l=0$$
$$r^n-\frac{s}{s-l}r^{n-1}+\frac{l}{s-l}=0$$

Exercise 37.

1. Find the eighth term of
3, 6, 12, ...

$$l=ar^{n-1}$$
$$l=3\times 2^7$$
$$l=384$$

2. Find the twelfth term of
2, −4, 8, ...

$$l=ar^{n-1}$$
$$l=2(-2)^{11}$$
$$l=-4096$$

3. Find the twentieth term of

$1, -\frac{1}{3}, \frac{1}{9}, \ldots\ldots$

$l = ar^{n-1}$

$l = 1(-\frac{1}{3})^{19}$

$l = -\frac{1}{3^{19}}$

4. Find the eighteenth term of

$3, 2, 1\frac{1}{3}, \ldots\ldots$

$l = ar^{n-1}$

$l = 3(\frac{2}{3})^{17}$

$l = \frac{2^{17}}{3^{16}}$

5. Find the *n*th term of

$1, -1\frac{1}{3}, 1\frac{7}{9}, \ldots\ldots$

$l = ar^{n-1}$

$l = 1(-\frac{4}{3})^{n-1}$

$l = (-\frac{4}{3})^{n-1}$

6. Find the sum of eleven terms of 4, 8, 16, ...

$s = \frac{a(r^n - 1)}{r - 1}$

$s = \frac{4(2^{11} - 1)}{2 - 1}$

$s = 8188$

7. Find the sum of nineteen terms of 9, 3, 1, ...

$s = \frac{(ar^n - 1)}{r - 1}$

$s = \frac{9[(\frac{1}{3})^{19} - 1]}{\frac{1}{3} - 1}$

$s = -\frac{27[(\frac{1}{3})^{19} - 1]}{2}$

$s = \frac{27}{2} - \frac{27}{2}(\frac{1}{3})^{19}$

$s = \frac{27}{2} - \frac{1}{23^{16}}$

8. Find the sum of twelve terms of $5, -3, 1\frac{4}{5}, \ldots\ldots$

$s = \frac{a(r^n - 1)}{r - 1}$

$s = \frac{5[(-\frac{3}{5})^{12} - 1]}{-\frac{3}{5} - 1}$

$s = \frac{25[(\frac{3}{5})^{12} - 1]}{-8}$

$s = \frac{\frac{3^{12}}{5^{10}} - 25}{-8}$

$s = \frac{25}{8} - \frac{3^{12}}{8 \times 5^{10}}$

9. Find the sum of *n* terms of $1\frac{7}{8}, \frac{3}{8}, \frac{3}{40}, \ldots\ldots$

$s = \frac{a(r^n - 1)}{r - 1}$

$s = \frac{1\frac{7}{8}[(\frac{1}{5})^n - 1]}{\frac{1}{5} - 1}$

$s = \frac{\frac{15}{8}[1 - (\frac{1}{5})^n]}{\frac{4}{5}}$

$s = \frac{75}{32} - \frac{3}{32 \times 5^{n-2}}$

10. Sum to infinity

$4 - 2 + 1 - \ldots\ldots$

$s = \frac{a}{1 - r}$

$s = \frac{4}{1 + \frac{1}{2}}$

$s = \frac{8}{3}$

11. Sum to infinity

$\frac{1}{2} + \frac{1}{3} + \frac{2}{9} + \ldots\ldots$

$s = \frac{a}{1 - r}$

$s = \frac{\frac{1}{2}}{1 - \frac{2}{3}}$

$s = \frac{3}{2}$

12. Sum to infinity

$$1-\tfrac{2}{5}+\tfrac{4}{25}-\cdots\cdots$$

$$s=\frac{a}{1-r}$$

$$s=\frac{1}{1+\frac{2}{5}}$$

$$s=\tfrac{5}{7}$$

13. Sum to infinity

$$\tfrac{1}{5}+\tfrac{1}{15}+\tfrac{1}{45}+\cdots\cdots$$

$$s=\frac{a}{1-r}$$

$$s=\frac{\frac{1}{5}}{1-\frac{1}{3}}$$

$$s=\tfrac{3}{10}$$

14. Find the value of the recurring decimal .153153 + ...

$$s=\frac{a}{1-r}$$

$$.153153+\cdots\cdots=\tfrac{153}{1000}+\tfrac{153}{10000000}+\text{etc.}$$

$$\therefore\ a=\tfrac{153}{1000}\qquad r=\tfrac{1}{1000}$$

$$s=\frac{\frac{153}{1000}}{\frac{999}{1000}}=\tfrac{153}{999}=\tfrac{17}{111}$$

15. Find the value of the recurring decimal .123535 + ...

$$s=\frac{a}{1-r}$$

$$.123535+\cdots\cdots=.12+.003535+\cdots\cdots$$

$$.003535+\cdots\cdots=\tfrac{35}{10000}+\tfrac{35}{1000000}+\cdots\cdots$$

$$\therefore\ a=\tfrac{35}{10000}\qquad r=\tfrac{1}{100}$$

$$s=\frac{\frac{35}{10000}}{\frac{99}{100}}\qquad s=\tfrac{35}{9900}\qquad .12=\tfrac{1188}{9900}$$

$$\therefore\ .123535+\cdots\cdots=\frac{1188+35}{9900}=\frac{1223}{9900}$$

16. Find the value of the recurring decimal 3.17272 + ...

$$3.17272+\cdots\cdots=3.1+.07272+\cdots\cdots$$

$$.07272+\cdots\cdots=\tfrac{72}{1000}+\tfrac{72}{100000}+\cdots\cdots$$

$$s=\frac{a}{1-r}$$

$$a=\tfrac{72}{1000},\ r=\tfrac{1}{100}\qquad s=\frac{\frac{72}{1000}}{\frac{99}{100}}$$

$$s=\tfrac{72}{990}=\tfrac{8}{110}\qquad 3.1=\tfrac{31}{10}=\tfrac{341}{110}$$

$$\therefore\ 3.17272+\cdots\cdots=\tfrac{341}{110}+\tfrac{8}{110}=3\tfrac{19}{110}$$

17. **Find the value of the recurring decimal $4.2561561 + \cdots\cdots$**

$$4.2561561 + \cdots\cdots = 4.2 + .0561561 + \cdots\cdots$$

$$.0561561 + \cdots\cdots = \tfrac{561}{10000} + \tfrac{561}{10000000} + \cdots\cdots$$

$$s = \frac{a}{1-r}$$

$$a = \tfrac{561}{10000} \qquad r = \tfrac{1}{1000}$$

$$s = \frac{\tfrac{561}{10000}}{\tfrac{999}{1000}} \qquad s = \tfrac{561}{9990}$$

$$s = \tfrac{187}{3330} \qquad 4.2 = \tfrac{42}{10} = \tfrac{21}{5}$$

$$4.2561561 + \cdots\cdots = \tfrac{187}{3330} + \tfrac{21}{5} = \tfrac{14173}{3330} = 4\tfrac{853}{3330}$$

18. **Given $a = 36$, $l = 2\tfrac{1}{4}$, $n = 5$. Find r and s.**

$$l = ar^{n-1} \qquad \tfrac{9}{4} = 36r^4$$

$$r^4 = \tfrac{1}{16} \qquad r = \pm\tfrac{1}{2}$$

$$s = \frac{rl-a}{r-1}$$

$$r = +\tfrac{1}{2} \qquad s = \frac{\tfrac{1}{2}(\tfrac{9}{4}) - 36}{\tfrac{1}{2} - 1} = \frac{-\tfrac{279}{8}}{-\tfrac{1}{2}} = \tfrac{279}{4} = 69\tfrac{3}{4}$$

$$r = -\tfrac{1}{2} \qquad s = \frac{-\tfrac{1}{2}(\tfrac{9}{4}) - 36}{-\tfrac{1}{2} - 1} = \frac{-\tfrac{297}{8}}{-\tfrac{3}{2}} = \tfrac{99}{4} = 24\tfrac{3}{4}$$

19. **Given $l = 128$, $r = 2$, $n = 7$. Find a and s.**

$$l = ar^{n-1} \qquad 128 = a \times 2^6 \qquad a = 2$$

$$s = \frac{rl-a}{r-1} = \frac{256-2}{2-1} = 254$$

20. **Given $r = 2$, $n = 7$, $s = 635$. Find a and l.**

$$s = \frac{a(r^n-1)}{r-1} \qquad 635 = \frac{a(2^7-1)}{2-1}$$

$$635 = 127a \qquad a = 5$$

$$l = ar^{n-1} \qquad l = 5 \times 2^6 = 320$$

21. **Given $l = 1296$, $r = 6$, $s = 1555$. Find a and s.**

$$s = \frac{rl-a}{r-1} \qquad l = ar^{n-1}$$

$$1555 = \frac{7776-a}{6-1} \qquad 1296 = 6^{n-1}$$

$$n - 1 = 4$$

$$n = 5$$

$$a = 7776 - 7775 = 1$$

22. Insert three geometrical means between 14 and 224.

The total number of terms is 5.

$$l = ar^{n-1}$$
$$224 = 14r^4$$
$$r^4 = 16$$
$$r = \pm 2$$

$$r = +2$$

The means are

28, 56, 112

$$r = -2$$

The means are

$-28, +56, -112$

23. Insert five geometrical means between 2 and 145.

The total number of terms is 7.

$$l = ar^{n-1}$$
$$1458 = 2 \times r^6$$
$$r^6 = 729$$
$$r = \pm 3$$

$$r = +3$$

The means are

6, 18, 54, 162, 486

$$r = -3$$

The means are

$-6, 18, -54, 162, -486$

24. If the first term is 2 and the ratio 3, what term will be 162?

$$l = ar^{n-1}$$
$$162 = 2 \times 3^{n-1}$$
$$3^{n-1} = 81$$

$$n - 1 = 4$$
$$n = 5$$

The 5th term will be 162.

25. The fifth term of a geometrical series is 48, and the ratio 2. Find the first and seventh terms.

Fifth term is ar^4, $r = 2$

$$\therefore\ a2^4 = 48$$
$$a = 3$$

$$\text{Seventh term} = ar^6$$
$$= 3 \times 2^6$$
$$= 192$$

26. Four numbers are in geometrical progression; the sum of the first and fourth is 195, and the sum of the second and third is 60. Find the numbers.

Let a, ar, ar^2, ar^3 be the numbers.

Then $$a + ar^3 = 195 \qquad (1)$$

$$ar + ar^2 = 60 \qquad (2)$$

From (1), $$a(1 + r^3) = 195 \qquad (3)$$

From (2), $$ar(1 + r) = 60 \qquad (4)$$

Divide $$\frac{1 - r + r^2}{r} = \frac{195}{60}$$

$$60-60r+60r^2=195r$$
$$60r^2-255r+60=0$$
$$4r^2-17r+4=0$$
$$(4r-1)(r-4)=0$$
$$\therefore\ r=4,\text{ or }\tfrac{1}{4}$$

Substitute value of r in (4),

$$20a=60,\text{ or }\tfrac{5}{16}a=60$$
$$\therefore\ a=3,\text{ or }192$$
$$ar=12,\text{ or }48$$
$$ar^2=48,\text{ or }12$$
$$ar^3=192,\text{ or }3$$

$\therefore$ The numbers are 3, 12, 48, and 192.

27. The sum of four numbers in geometrical progression is 105; the difference between the first and last is to the difference between the second and third in the ratio of 7 : 2. Find the numbers.

Let a, ar, ar^2, ar^3 be the numbers.

$$a+ar+ar^2+ar^3=105 \qquad (1)$$

Then
$$ar^3-a:ar^2-ar=7:2 \qquad (2)$$

From (2),
$$2(ar^3-a)=7(ar^2-ar)$$
$$2(r^2+r+1)=7r$$
$$2r^2-5r+2=0$$
$$(2r-1)(r-2)=0$$
$$\therefore\ r=2,\text{ or }\tfrac{1}{2}$$

Substitute value of r in (1),

$$a(1+2+4+8)=105$$

or
$$a(1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8})=105$$
$$a=7,\text{ or }56$$
$$ar=14,\text{ or }28$$
$$ar^2=28,\text{ or }14$$
$$ar^3=56,\text{ or }7$$

$\therefore$ The numbers are 7, 14, 28, 56.

28. The first term of an arithmetical progression is 2, and the first, second, and fifth terms are in geometrical progression. Find the sum of 11 terms of the arithmetical progression.

Let $d=$ the common difference of the arithmetical progression.

Then 2, $2+d$, and $2+4d$ are in geometrical progression.

$$\therefore\ \frac{2+d}{2}=\frac{2+4d}{2+d}$$
$$4+4d+d^2=4+8d$$

$$d^2 - 4d = 0$$
$$d = 4$$

Then $$a = 2,\ d = 4,\ n = 11$$
$$l = a + (n-1)d$$
$$= 2 + 10 \times 4 = 42$$
$$s = \frac{n}{2}(a + l)$$
$$= \tfrac{11}{2}(2 + 42) = 242$$

∴ The sum of 11 terms of the arithmetical progression is 242.

29. The sum of three numbers in arithmetical progression is 6. If 1, 2, 5 be added to the numbers, the three resulting numbers are in geometrical progression. Find the numbers.

Let $a-d,\ a,\ a+d$ be the three numbers.

Then $$3a = 6 \qquad (1)$$
$$\frac{a+2}{a-d+1} = \frac{a+d+5}{a+2} \qquad (2)$$

From (1), $$a = 2$$

Substitute value of a in (2),
$$\frac{4}{3-d} = \frac{7+d}{4}$$
$$16 = 21 - 4d - d^2$$
$$d^2 + 4d - 5 = 0$$
$$(d-1)(d+5) = 0$$
$$\therefore\ d = 1, \text{ or } -5$$
$$a - d = 1, \text{ or } 7$$
$$a + d = 3, \text{ or } -3$$

∴ The numbers are 1, 2, and 3, or 7, 2, and −3.

30. The sum of three numbers in arithmetical progression is 15; if 1, 4, 19 be added to the numbers, the results are in geometrical progression. Find the numbers.

Let $a-d,\ a,\ a+d$ be the numbers.

Then $$3a = 15 \qquad (1)$$
$$\frac{a+4}{a-d+1} = \frac{a+d+19}{a+4} \qquad (2)$$

From (1), $$a = 5$$

Substitute value of a in (2),
$$\frac{9}{6-d} = \frac{24+d}{9}$$

$$\begin{aligned} 81 &= 144 - 18d - d^2 \\ d^2 + 18d - 63 &= 0 \\ (d+21)(d-3) &= 0 \\ \therefore d &= 3, \text{ or } -21 \\ a - d &= 2, \text{ or } 26 \\ a + d &= 8, \text{ or } -16 \end{aligned}$$

$\therefore$ The numbers are 2, 5, and 8; or 26, 5, and -16.

31. There are four numbers of which the sum is 84; the first three are in geometrical progression and the last three in arithmetical progression; the sum of the second and third is 18. Find the numbers.

Let a, ar, ar^2 be the first three numbers.

Then $ar^2 + ar^2 - ar$ is the fourth number.

$$\therefore a + ar + ar^2 + 2ar^2 - ar = 84 \qquad (1)$$

$$ar + ar^2 = 18 \qquad (2)$$

From (1), $$a + 3ar^2 = 84 \qquad (3)$$

Divide (2) by (3), $$\frac{r+r^2}{1+3r^2} = \frac{18}{84}$$

$$\begin{aligned} \frac{r+r^2}{1+3r^2} &= \frac{3}{14} \\ 14r + 14r^2 &= 3 + 9r^2 \\ 5r^2 + 14r - 3 &= 0 \\ (5r-1)(r+3) &= 0 \\ \therefore r &= \tfrac{1}{5}, \text{ or } -3 \end{aligned}$$

Substitute value of r in (2),

$$\begin{aligned} a\left(\tfrac{1}{5} + \tfrac{1}{25}\right) &= 18, \text{ or } a(-3+9) = 18 \\ \therefore a &= 75, \text{ or } \ 3 \\ ar &= 15, \text{ or } -9 \\ ar^2 &= \ 3, \text{ or } 27 \\ 2ar^2 - ar &= -9, \text{ or } 63 \end{aligned}$$

$\therefore$ The numbers are 75, 15, 3, and -9, or 3, -9, 27, and 63.

32. There are four numbers of which the sum is 13, the fourth being 3 times the second; the first three are in geometrical progression and the last three in arithmetical progression. Find the numbers.

Let a, ar, ar^2 be the first three numbers.

Then $2ar^2 - ar$ is the fourth number.

$$\therefore a + ar + ar^2 + 2ar^2 - ar = 13 \qquad (1)$$

$$2ar^2 - ar = 3ar \qquad (2)$$

From (1), $$a + 3ar^2 = 13 \qquad (3)$$

From (2), $2ar^2 - 4ar = 0$

$$ar^2 - 2ar = 0 \qquad (4)$$

$$\therefore\ ar = 0$$

or $r = 2$

Substitute $ar = 0$ in (3), $a = 13$

$$\therefore\ r = 0$$

This pair of solutions is to be rejected.

Substitute $r = 2$ in (3), $a + 12a = 13$

$$\begin{aligned} \therefore\ a &= 1 \\ ar &= 2 \\ ar^2 &= 4 \\ 2ar^2 - ar &= 6 \end{aligned}$$

$\therefore$ The numbers are 1, 2, 4, and 6.

33. The sum of the squares of two numbers exceeds twice their product by 576; the arithmetical mean of the two numbers exceeds the geometrical by 6. Find the numbers.

Let $x + y$ and $x - y$ be the numbers.

Then $$(x + y)^2 + (x - y)^2 = 2(x + y)(x - y) + 576 \qquad (1)$$

$$\frac{x + y + x - y}{2} = \sqrt{(x + y)(x - y)} + 6 \qquad (2)$$

From (1), $$\begin{aligned} 2x^2 + 2y^2 &= 2x^2 - 2y^2 + 576 \\ 4y^2 &= 576 \\ y^2 &= 144 \\ y &= \pm 12 \end{aligned}$$

From (2), $x - 6 = \sqrt{x^2 - y^2}$

Substitute $y = \pm 12$, $x - 6 = \sqrt{x^2 - 144}$

$$\begin{aligned} x^2 - 12x + 36 &= x^2 - 144 \\ 12x &= 180 \\ x &= 15 \\ \therefore\ x + y &= 27, \text{ or } 3 \\ x - y &= 3, \text{ or } 27 \end{aligned}$$

$\therefore$ The numbers are 3 and 27.

34. A number consists of three digits in geometrical progression. The sum of the digits is 13; and if 792 be added to the number, the digits in the units' and hundreds' places will be interchanged. Find the number.

Let a, ar, ar^2 be the three digits in order.

Then $100a + 10ar + ar^2 =$ the number.

$$\therefore\ a + ar + ar^2 = 13 \qquad (1)$$

$$100a + 10ar + ar^2 + 792 = a + 10ar + 100ar^2 \qquad (2)$$

From (2), $99a - 99ar^2 + 792 = 0$

$$a - ar^2 + 8 = 0$$

$$ar^2 - a = 8 \qquad (3)$$

Divide (1) by (3), $\frac{1 + r + r^2}{r^2 - 1} = \frac{13}{8}$

$$8 + 8r + 8r^2 = 13r^2 - 13$$

$$5r^2 - 8r - 21 = 0$$

$$(5r + 7)(r - 3) = 0$$

$$\therefore\ r = 3, \text{ or } -\tfrac{7}{5}$$

Substitute value of r in (3),

$$9a - a = 8, \text{ or } \tfrac{49}{25}a - a = 8$$

$$\therefore\ a = 1, \text{ or } \tfrac{25}{3}$$

$$ar = 3$$

$$ar^2 = 9$$

∴ The number is 139.

35. **Find an infinite geometrical series in which each term is five times the sum of all the terms that follow it.**

Let a be any term of the series.

ar the following term.

Then $\frac{ar}{1 - r}$ is the sum of all terms, beginning with ar.

$$\therefore\ a = \frac{5ar}{1 - r}$$

$$1 = \frac{5r}{1 - r}$$

$$1 - r = 5r$$

$$\therefore\ r = \tfrac{1}{6}$$

∴ Any series for which $r = \frac{1}{6}$ will satisfy the requirement.

36. **If a, b, c, d are four numbers in geometrical progression, show that** $(a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (ab + bc + cd)^2$.

Let r be the ratio of the series.

Then $b = ar$

$$c = ar^2$$

$$d = ar^3$$

$$\begin{aligned} \therefore\ (a^2 + b^2 + c^2)(b^2 + c^2 + d^2) &= (a^2 + a^2r^2 + a^2r^4)(a^2r^2 + a^2r^4 + a^2r^6) \\ &= a^4r^2(1 + r^2 + r^4)(1 + r^2 + r^4) \\ &= a^4r^2(1 + r^2 + r^4)^2 \end{aligned}$$

And $$(ab + bc + cd)^2 = (a^2r + a^2r^3 + a^2r^5)^2$$
$$= a^4r^2(1 + r^2 + r^4)^2$$
$$\therefore (a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (ab + bc + cd)^2$$

Exercise 38.

1. Insert four harmonical means between 2 and 12.

$$d = \frac{l - a}{m + 1}$$
$$\therefore d = \frac{\frac{1}{2} - \frac{1}{12}}{5} = \frac{1}{12}$$
$$\tfrac{2}{12}, \tfrac{3}{12}, \tfrac{4}{12}, \tfrac{5}{12} = \text{arithmetical means,}$$
$$\therefore 6, 4, 3, 2\tfrac{2}{5} = \text{harmonical means.}$$

2. Find two numbers whose difference is 8, and the harmonical mean between them $1\frac{4}{5}$.

Let x = one number.
Then $x + 8$ = the other number.

$$H = \frac{2ab}{a + b}$$
$$\therefore \frac{9}{5} = \frac{2x^2 + 16x}{2x + 8}$$
$$18x + 72 = 10x^2 + 80x$$
$$10x^2 + 62x = 72$$
$$100x^2 + (\ ) + (31)^2 = 1681$$
$$10x + 31 = \pm 41$$
$$x = 1, \text{ or } -7\tfrac{1}{5}$$
$$x + 8 = 9, \text{ or } \tfrac{4}{5}$$

Hence, the numbers are 1 and 9.

3. Find the seventh term of the harmonical series 3, $3\frac{3}{7}$, 4, ...

$$l = a + (n - 1)d$$
Here $$a = \tfrac{1}{3}, n = 7, d = \tfrac{7}{24} - \tfrac{7}{21} = -\tfrac{1}{24}$$
$$\therefore l = \tfrac{1}{3} - \tfrac{1}{4}$$
$$l = \tfrac{1}{12}$$

Hence, the seventh term is 12.

4. Continue to two terms each way the harmonical series two consecutive terms of which are 15, 16.

Harmonical series, 15, 16.

Arithmetical series, $\frac{1}{15}$, $\frac{1}{16}$.

$$d = \frac{1}{16} - \frac{1}{15} = -\frac{1}{240}$$

Subtract d from first term, $\frac{1}{15} + \frac{1}{240} = \frac{17}{240}$

Subtract d from $\frac{17}{240}$, $\frac{17}{240} + \frac{1}{240} = \frac{18}{240}$

Add d to last term, $\frac{1}{16} - \frac{1}{240} = \frac{14}{240}$

Add d to $\frac{14}{240}$, $\frac{14}{240} - \frac{1}{240} = \frac{13}{240}$

Hence, arithmetical series is $\frac{18}{240}$, $\frac{17}{240}$ ….. $\frac{14}{240}$, $\frac{13}{240}$.

Hence, harmonical series is $13\frac{1}{3}$, $14\frac{2}{17}$ ….. $17\frac{1}{7}$, $18\frac{6}{13}$.

5. The first two terms of a harmonical series are 5 and 6. Which term will equal 30?

$$l = a + (n-1)d$$
$$n = \frac{l-a}{d} + 1$$
$$n = \frac{\frac{1}{5} - \frac{1}{30}}{\frac{1}{30}} + 1$$
$$n = \frac{5}{30} + \frac{30}{1} + 1$$
$$n = 6$$

6. The fifth and ninth terms of a harmonical series are 8 and 12. Find the first four terms.

$$\frac{l-a}{m+1} = d$$
$$\frac{\frac{1}{12} - \frac{1}{8}}{4} = d$$
$$-\frac{1}{96} = d$$
$$\frac{1}{8} + \frac{1}{96} = \frac{13}{96} = \frac{1}{7\frac{5}{13}}$$
$$\frac{13}{96} + \frac{1}{96} = \frac{14}{96} = \frac{1}{6\frac{6}{7}}$$
$$\frac{14}{96} + \frac{1}{96} = \frac{15}{96} = \frac{1}{6\frac{2}{5}}$$
$$\frac{15}{96} + \frac{1}{96} = \frac{16}{96} = \frac{1}{6}$$

Hence, the first four terms are 6, $6\frac{2}{5}$, $6\frac{6}{7}$, $7\frac{5}{13}$.

7. The difference between the arithmetical and harmonical means between two numbers is $1\frac{4}{5}$, and one of the numbers is four times the other. Find the numbers.

Let x and $y =$ the numbers.

Then $\frac{x+y}{2}, \frac{2xy}{x+y} =$ the arithmetical and harmonical means.

Hence, $$x = 4y \quad (1)$$

and $$\frac{x+y}{2} - \frac{2xy}{x+y} = \frac{9}{5} \quad (2)$$

Substitute $4y$ for x in (2),

$$\frac{5y}{2} - \frac{8y^2}{5y} = \frac{9}{5}$$

$$\therefore\ y = 2$$

and $$x = 8$$

8. The arithmetical mean between two numbers exceeds the geometrical by 13, and the geometrical exceeds the harmonical by 12. What are the numbers?

Let a and $b =$ the numbers.

Then $\frac{a+b}{2}, \sqrt{ab}, \frac{2ab}{a+b} =$ the arithmetical, geometrical, and harmonical means.

Hence $$\frac{a+b}{2} - \sqrt{ab} = 13 \quad (1)$$

$$\sqrt{ab} - \frac{2ab}{a+b} = 12 \quad (2)$$

Add (1) and (2), $$\frac{a+b}{2} - \frac{2ab}{a+b} = 25 \quad (3)$$

Transpose in (1), $$\frac{a+b}{2} - 13 = \sqrt{ab}$$

$$a + b - 26 = 2\sqrt{ab} \quad (4)$$

Square (4), $$a^2 + 2ab + b^2 - 52a - 52b + 676 = 4ab$$

Simplify (3), $$a^2 + 2ab + b^2 - 50a - 50b = 4ab$$

Subtract, $$2a + 2b = 676$$

$$a + b = 338 \quad (5)$$

Substitute value of $a + b$ in (1),

$$169 - \sqrt{ab} = 13$$

$$156 = \sqrt{ab}$$

From (5), $$a = 338 - b$$

$$\therefore\ 156 = \sqrt{(338 - b)b}$$

$$\therefore\ 156^2 = 338b - b^2$$

$$b^2 - 338b = -24336$$

$$b^2 - (\ ) + (169)^2 = 4225$$

$$b - 169 = \pm 65$$

$$\therefore b = 234, \text{ or } 104$$

and $$a = 104, \text{ or } 234$$

9. The sum of three terms of a harmonical series is 39, and the third is the product of the other two. Find the terms.

Let $x =$ the first term,

and $y =$ the last term.

Then $\dfrac{2xy}{x+y} =$ the middle term.

$$\therefore x + y + \frac{2xy}{x+y} = 39 \qquad (1)$$

$$y = \frac{2x^2y}{x+y} \qquad (2)$$

From (1), $$x^2 + 4xy + y^2 = 39(x+y) \qquad (3)$$

From (2), $$1 = \frac{2x^2}{x+y}$$

$$x + y = 2x^2$$

$$y = 2x^2 - x$$

Substitute value of y in (3),

$$x^2 + 8x^3 - 4x^2 + 4x^4 - 4x^3 + x^2 = 78x^2$$

$$4x^4 + 4x^3 - 80x^2 = 0$$

$$4x^2(x^2 + x - 20) = 0$$

$$4x^2(x+5)(x-4) = 0$$

$$\therefore x = 0,\ 4, \text{ or } -5$$

$$y = 0,\ 28, \text{ or } 55$$

$$\frac{2xy}{x+y} = 7, \text{ or } -11.$$

∴ The numbers are 4, 7, and 28, or -5, -11, and 55.

10. When a, b, c, are in harmonical progression, show that

$$a : c :: a - b : b - c.$$

If a, b, c, are a harmonical series,

$$\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$$

Multiply by abc, $$ac - bc = ab - ac$$

or $$c(a - b) = a(b - c)$$

or $$a : c :: a - b : b - c$$

11. If a and b are positive, which is the greater, A or B?

$$A = \frac{a+b}{2}$$

$$H = \frac{2ab}{a+b}$$

$$A > \text{ or } < H$$

as $$\frac{a+b}{2} > \text{ or } < \frac{2ab}{a+b}$$

as $$(a+b)^2 > \text{ or } < 4ab$$

as $$a^2+b^2 > \text{ or } < 2ab$$

But $$a^2+b^2 > 2ab$$

$$\therefore A > H$$

Exercise 39.

Solve in positive integers:

1. $x + y = 12$

Transpose, $x = 12 - y$

If $y = 1, x = 11$

If $y = 2, x = 10$

. . . . . .

If $y = 11, x = 1$

2. $2x + 11y = 83$

Transpose,

$$2x = 83 - 11y$$

$$x = 41 - 5y + \frac{1-y}{2} \quad (1)$$

Let $\frac{1-y}{2} = m$

$$\therefore 1 - y = 2m$$

$$y = 1 - 2m$$

Substitute value of y in (1).

$$\therefore x = 36 + 11m$$

$$y = 1 - 2m$$

If $m = 0, x = 36, y = 1$

If $m = -1, x = 25, y = 3$

If $m = -2, x = 14, y = 5$

If $m = -3, x = 3, y = 7$

3. $4x + 9y = 53$

Transpose,

$$4x = 53 - 9y$$

$$x = 13 - 2y + \frac{1-y}{4} \quad (1)$$

Let $\frac{1-y}{4} = m$

$$\therefore 1 - y = 4m$$

$$y = 1 - 4m$$

Substitute value of y in (1).

$$\therefore x = 11 + 9m$$

$$y = 1 - 4m$$

If $m = 0, x = 11, y = 1$

If $m = -1, x = 2, y = 5$

4. $8x + 5y = 74$

Transpose,

$$5y = 74 - 8x$$

$$y = 14 - x + \frac{4-3x}{5} \quad (1)$$

Multiply by 2,

$$2y = 28 - 2x + \frac{8-6x}{5}$$

$$2y = 29 - 3x + \frac{3-x}{5}$$

Let $\frac{3-x}{5}=m$

$\therefore\ 3-x=5m$

$x=3-5m$

Substitute value of x in (1).

$24-40m+5y=74$

$\therefore\ y=10+8m$

$x=3-5m$

If $m=0,\ x=3,\ y=10$

If $m=-1,\ x=8,\ y=2$

5. $5x+3y=105$

Transpose, $3y=105-5x$

$y=35-x-\frac{2x}{3}$

Multiply by 2,

$2y=70-3x-\frac{x}{3}$ (1)

Let $\frac{x}{3}=m$

$\therefore\ x=3m$

Substitute value of x in (1).

$\therefore\ y=35-5m$

$x=3m$

If $m=1,\ x=3,\ y=30$

If $m=2,\ x=6,\ y=25$

If $m=3,\ x=9,\ y=20$

. . . .

If $m=6,\ x=18,\ y=5$

6. $\frac{3}{4}x+5y=92$

$3x+20y=368$

Transpose, $3x=368-20y$

$x=122-6y+\frac{2-2y}{3}$ (1)

Multiply by 2, $2x=245-13y+\frac{1-y}{3}$

Let $\frac{1-y}{3}=m$

$\therefore\ 1-y=3m$

$y=1-3m$

Substitute value of y in (1).

$\therefore\ x=116+20m$

$y=1-3m$

If $m=0,\ x=116,\ y=1$

If $m=-1,\ x=96,\ y=4$

If $m=-2,\ x=76,\ y=7$

. . . .

If $m=-5,\ x=16,\ y=16.$

7. $\frac{3}{4}x+\frac{1}{4}y=27$

$3x+y=108$

Transpose, $y=108-3x$

If $x=1,\ y=105$

If $x = 2,\ y = 102$

. . . .

If $x = 35,\ y = 3$

8. $\frac{2}{5}x + \frac{3}{4}y = 53$

$$8x + 15y = 1060$$

Transpose, $8x = 1060 - 15y$

$$x = 132 - 2y + \frac{4 + y}{8} \qquad (1)$$

Let $\frac{4 + y}{8} = m$

$$\therefore 4 + y = 8m$$

$$y = 8m - 4$$

Substitute value of y in (1).

$$x = 140 - 15m$$

$$y = 8m - 4$$

If $m = 1,\ x = 125,\ y = 4$

If $m = 2,\ x = 110,\ y = 12$

. . . .

If $m = 9,\ x = 5,\ y = 68$

Solve in least possible integers:

9. $7x - 2y = 12$

Transpose, $2y = 7x - 12$

$$y = 3x - 6 + \frac{x}{2} \qquad (1)$$

Let $\frac{x}{2} = m$

$$\therefore x = 2m$$

Substitute value of x in (1).

$$\therefore y = 7m - 6$$

$$x = 2m$$

If $m = 1,\ x = 2,\ y = 1$

10. $9x - 5y = 21$

Transpose, $5y = 9x - 21$

$$y = 2x - 4 + \frac{x + 1}{5} \qquad (1)$$

Let $\frac{x + 1}{5} = m$

$$\therefore\ x = 5m - 1$$

Substitute value of x in (1).

$$\therefore\ y = 9m - 6$$

$$x = 5m - 1$$

If $m = 1,\ x = 4,\ y = 3$

11. $7x - 4y = 45$

Transpose, $4y = 7x - 45$

$$y = 2x - 11 - \frac{x+1}{4} \qquad (1)$$

Let $\dfrac{x+1}{4} = m$

$$\therefore\ x = 4m - 1$$

Substitute value of x in (1).

$$\therefore\ y = 7m - 13$$

$$x = 4m - 1$$

If $m = 2,\ x = 7,\ y = 1$

12. $11x - 5y = 73$

Transpose, $5y = 11x - 73$

$$y = 2x - 14 + \frac{x-3}{5} \qquad (1)$$

Let $\dfrac{x-3}{5} = m$

$$\therefore\ x = 5m + 3$$

Substitute value of x in (1).

$$\therefore\ y = 11m - 8$$

$$x = 5m + 3$$

If $m = 1,\ x = 8,\ y = 3$

13. $15x - 47y = 11$

Transpose, $15x = 11 + 47y$

$$x = 3y + \frac{11 + 2y}{15} \qquad (1)$$

Multiply by 8, $8x = 25y + 5 + \dfrac{13 + y}{15}$

Let $\dfrac{13 + y}{15} = m$

$$\therefore\ y = 15m - 13$$

Substitute value of y in (1).

$$\therefore\ x = 47m - 40$$

$$y = 15m - 13$$

$$\text{If } m = 1,\ x = 7,\ y = 2$$

14. $$23x - 14y = 99$$

Transpose, $$14y = 23x - 99$$

$$y = x - 7 + \frac{9x - 1}{14} \qquad (1)$$

Multiply by 11, $$11y = 14x - 77 + \frac{99x - 11}{14}$$

$$11y = 18x - 77 + \frac{x - 11}{14}$$

Let $$\frac{x - 11}{14} = m$$

$$\therefore\ x = 14m + 11$$

Substitute value of x in (1).

$$\therefore\ y = 23m + 11$$

$$x = 14m + 11$$

$$\text{If } m = 0,\ x = 11,\ y = 11$$

15. Find two numbers which, multiplied respectively by 7 and 17, have for the sum of their products 1135.

Let x and y be the numbers.

Then $$7x + 17y = 1135$$

Transpose, $$7x = 1135 - 17y$$

$$x = 162 - 2y + \frac{1 - 3y}{7} \qquad (1)$$

Multiply by 5, $$5x = 810 - 12y + \frac{5 - y}{7}$$

Let $$\frac{5 - y}{7} = m$$

$$\therefore\ y = 5 - 7m$$

Substitute value of y in (1).

$$x = 17m + 150$$

$$y = 5 - 7m$$

$$\begin{aligned}&\text{If } m = 0, x = 150, y = 5\\&\text{If } m = -1, x = 133, y = 12\\&\quad\cdots\cdots\cdots\\&\text{If } m = -8, x = 14, y = 61\end{aligned}$$

16. If two numbers are multiplied respectively by 8 and 17, the difference of their products is 10. What are the numbers?

Let x and y be the numbers.

Then $8x - 17y = 10$

Transpose, $8x = 10 + 17y$

$$x = 1 + 2y + \frac{2+y}{8} \qquad (1)$$

Let $\frac{2+y}{8} = m$

$\therefore y = 8m - 2$

Substitute value of y in (1).

$\therefore x = 17m - 3$

$y = 8m - 2$

If $m = 1, x = 14, y = 6$

If $m = 2, x = 31, y = 14$

. . . .

17. If two numbers are multiplied respectively by 7 and 15, the first product is greater by 12 than the second. Find the numbers.

Let x and y be the numbers.

Then $7x - 15y = 12$

Transpose, $7x = 12 + 15y$

$$x = 1 + 2y + \frac{5+y}{7} \qquad (1)$$

Let $\frac{5+y}{7} = m$

$\therefore y = 7m - 5$

Substitute value of y in (1).

$\therefore x = 15m - 9$

$y = 7m - 5$

If $m = 1, x = 6, y = 2$

If $m = 2, x = 21, y = 9$

. . . .

18. Divide 89 in two parts, one of which is divisible by 3, and the other by 8.

Let $3x$ and $8y$ be the two parts.

Then $$3x + 8y = 89$$

Transpose, $$3x = 89 - 8y$$

$$x = 29 - 2y + \frac{2 - 2y}{3} \qquad (1)$$

Multiply by 2, $$2x = 59 - 5y + \frac{1 - y}{3}$$

Let $$\frac{1 - y}{3} = m$$

$$\therefore\ y = 1 - 3m$$

Substitute value of y in (1).

$$\therefore\ x = 27 + 8m$$
$$y = 1 - 3m$$

If $m = 0,\ x = 27,\ y = 1$

If $m = -1,\ x = 19,\ y = 4$

If $m = -2,\ x = 11,\ y = 7$

If $m = -3,\ x = 3,\ y = 10$

$$\therefore\ 3x = 81,\ 57,\ 33, \text{ or } 9$$
$$8y = 8,\ 32,\ 56, \text{ or } 80$$

19. Divide 314 in two parts, one of which is a multiple of 11, and the other a multiple of 13.

Let $11x$ and $13y$ be the two parts.

Then $$11x + 13y = 314$$

Transpose, $$11x = 314 - 13y$$

$$x = 28 - y + \frac{6 - 2y}{11}$$

Multiply by 6, $$6x = 171 - 7y + \frac{3 - y}{11} \qquad (1)$$

Let $$\frac{3 - y}{11} = m$$

$$\therefore\ y = 3 - 11m$$

Substitute value of y in (1).

$$\therefore\ x = 13m + 25$$
$$y = 3 - 11m$$

If $m = 0,\ x = 25,\ y = 3$

If $m = -1,\ x = 12,\ y = 14$

$$\therefore\ 11x = 275, \text{ or } 132$$
$$13y = 39, \text{ or } 182$$

20. What is the smallest number which, divided by 5 and by 7, gives each time 4 for a remainder?

Let $x = $ the number.

Then $\frac{x-4}{5} = $ an integer,

$\frac{x-4}{7} = $ an integer.

Let $\frac{x-4}{5} = m$

and $\frac{x-4}{7} = n$

$$\therefore\ x = 5m + 4$$
$$x = 7n + 4$$
$$\therefore\ 5m = 7n$$
$$m = n + \frac{2n}{5}$$

Multiply by 3, $3m = 4n + \frac{n}{5}$

Let $\frac{n}{5} = p$

$$\therefore\ n = 5p$$

Substitute value of n in equation $5m = 7n$.

$$\therefore\ m = 7p$$
$$n = 5p$$
$$\therefore\ x = 35p + 4$$

If $p = 1,\ x = 39$

$\therefore$ 39 is the smallest number, except 4 itself.

21. The difference of two numbers is 151. The first divided by 8 has 5 for a remainder, and 4 must be added to the second to make it divisible by 11. What are the numbers?

Let x be one of the two numbers.
Then $151 + x$ is the other.

$$\therefore\ \frac{x-5}{8} = \text{an integer.}$$

$$\frac{151 + x + 4}{11} = \text{an integer.}$$

Let $$\frac{x - 5}{8} = m$$

and $$\frac{155 + x}{11} = n$$

$$\therefore\ x = 8m + 5$$

$$x = 11n - 155$$

$$\therefore\ 8m - 11n = -160$$

$$8m = 11n - 160$$

$$m = n - 20 + \frac{3n}{8}$$

Multiply by 3, $$3m = 4n - 60 + \frac{n}{8}$$

Let $$\frac{n}{8} = p$$

$$\therefore\ n = 8p$$

$$x = 88p - 155$$

$$151 + x = 88p - 4$$

If $p = 2,\ x = 21,\ 151 + x = 172$

If $p = 3,\ x = 109,\ 151 + x = 260$

. . . .

22. Find pairs of fractions whose denominators are 24 and 16, and whose sum is $\frac{19}{24}$.

Let x and y be the numerators.

Then $$\frac{x}{24} + \frac{y}{16} = \frac{19}{24}$$

$$2x + 3y = 38$$

$$2x = 38 - 3y$$

$$x = 19 - y - \frac{y}{2}$$

Let $$\frac{y}{2} = m$$

$$\therefore\ y = 2m$$

$$2x + 6m = 38$$

$$\therefore\ x = 19 - 3m$$

$$y = 2m$$

If $m = 1, x = 16, y = 2$

If $m = 2, x = 13, y = 4$

. . . .

If $m = 6, x = 1, y = 12$

$\therefore$ The fractions are $\frac{16}{24}$ and $\frac{2}{16}$, $\frac{13}{24}$ and $\frac{4}{16}$, ….. $\frac{1}{24}$ and $\frac{12}{16}$.

23. How can one pay a sum of \$87, giving only bills of \$5 and \$2?

Let $x =$ number of \$5 bills,

and $y =$ number of \$2 bills.

Then

$$5x + 2y = 87$$
$$2y = 87 - 5x$$
$$y = 43 - 2x + \frac{1-x}{2}$$

Let

$$\frac{1-x}{2} = m$$
$$\therefore x = 1 - 2m$$
$$\therefore y = 5m + 41$$
$$x = 1 - 2m$$

If $m = 0, x = 1, y = 41$

If $m = -1, x = 3, y = 36$

If $m = -2, x = 5, y = 31$

. . . .

If $m = -8, x = 17, y = 1$

24. A man buys calves at \$5 apiece, and pigs for \$3 apiece. He spends in all \$114. How many did he buy of each?

Let $x =$ number of calves,

and $y =$ number of pigs.

Then

$$5x + 3y = 114$$
$$3y = 114 - 5x$$
$$y = 38 - x - \frac{2x}{3}$$
$$2y = 76 - 3x - \frac{x}{3}$$

Let

$$\frac{x}{3} = m$$
$$x = 3m$$
$$y = 38 - 5m$$

$$x = 3m$$

If $m = 1,\ x = 3,\ y = 33$

$m = 2,\ x = 6,\ y = 28$

. . . .

If $m = 7,\ x = 21,\ y = 3$

25. A person bought 40 animals, consisting of pigs, geese, and chickens, for $40. The pigs cost $5 apiece, the geese $1, and the chickens 25 cents each. Find the number he bought of each.

Let $x =$ number of pigs.

$y =$ number of geese.

$z =$ number of chickens.

Then
$$x + y + z = 40 \qquad (1)$$
$$5x + y + \tfrac{1}{4}z = 40 \qquad (2)$$

Subtract,
$$4x - \tfrac{3}{4}z = 0$$
$$16x - 3z = 0$$
$$3z = 16x$$
$$z = 5x + \frac{x}{3}$$

Let
$$\frac{x}{3} = m$$
$$\therefore\ x = 3m$$
$$3z = 16x$$
$$\therefore\ z = 16m$$

(1) is
$$x + y + z = 40$$
$$3m + y + 16m = 40$$
$$\therefore\ y = 40 - 19m$$

Then,
$$x = 3m$$
$$y = 40 - 19m$$
$$z = 16m$$

If $m = 1,\ x = 3,\ y = 21,\ z = 16$

If $m = 2,\ x = 6,\ y = 2,\ z = 32$

∴ He bought 3 pigs, 21 geese, and 16 chickens, or 6 pigs, 2 geese, and 32 chickens.

26. Solve $18x - 5y = 70$ so that y may be a multiple of x, and both positive.

$$18x - 5y = 70$$
$$y = mx$$

$$18x - 5mx = 70$$
$$x = \frac{70}{18 - 5m}$$
$$y = \frac{70m}{18 - 5m}$$

If $m = 1, x = 5\frac{5}{13}, y = 5\frac{5}{13}$
$m = 2, x = 8\frac{3}{4}, y = 17\frac{1}{2}$
$m = 3, x = 23\frac{1}{3}, y = 70$

27. Solve $8x + 12y = 23$ so that x and y may be positive, and their sum an integer.

$$8x + 12y = 23$$
$$x + y = m$$
$$x = m - y$$
$$8m - 8y + 12y = 23$$
$$4y = 23 - 8m$$
$$y = \tfrac{23}{4} - 2m$$
$$\therefore x = 3m - \tfrac{23}{4}$$

If $m = 2, x = \frac{1}{4}, y = 1\frac{3}{4}$

28. Divide 70 into three parts which shall give integral quotients when divided by 6, 7, 8, respectively, and the sum of the quotients shall be 10.

Let $6x$, $7y$, and $8z$ be the three parts.

Then $6x + 7y + 8z = 70$ (1)
$x + y + z = 10$ (2)

Subtract $6 \times (2)$ from (1),

$$y + 2z = 10$$
$$y = 10 - 2z$$

Substitute value of y in (2),

$$x + 10 - z = 10$$
$$x = z$$
$$y = 10 - 2z$$

If $z = 1, x = 1, y = 8, 6x = 6, 7y = 56, 8z = 8$
$z = 2, x = 2, y = 6, 6x = 12, 7y = 42, 8z = 16$
$z = 3, x = 3, y = 4, 6x = 18, 7y = 28, 8z = 24$
$z = 4, x = 4, y = 2, 6x = 24, 7y = 14, 8z = 32$

$\therefore$ The parts are 6, 56, and 8; or 12, 42, and 16; or 18, 28, and 24; or 24, 14, and 32.

29. In how many ways can $3.60 be paid with dollars and twenty-cent pieces?

Let $x =$ number of dollars,

and $y =$ number of 20 cent pieces.

Then $100x + 20y = 360$

$$5x + y = 18$$

$$y = 18 - 5x$$

If $x = 0, y = 18$

$$x = 1, y = 13$$

$$x = 2, y = 8$$

$$x = 3, y = 3$$

Hence it can be done in 3 ways (or 4 ways including the case in which the amount is all paid in twenty-cent pieces).

30. In how many ways can 300 pounds be weighed with 7 and 9 pound weights?

Let $x =$ number of 7 pound weights,

and $y =$ number of 9 pound weights.

Then $7x + 9y = 300$

$$7x = 300 - 9y$$

$$x = 42 - y + \frac{6 - 2y}{7}$$

$$4x = 171 - 5y + \frac{3 - y}{7}$$

Let $\frac{3 - y}{7} = m$

$$\therefore\ y = 3 - 7m$$

$$x = 39 + 9m$$

$$y = 3 - 7m$$

If $m = 0, x = 39, y = 3$

$$m = -1, x = 30, y = 10$$

$$m = -2, x = 21, y = 17$$

$$m = -3, x = 12, y = 24$$

$$m = -4, x = 3, y = 31$$

$\therefore$ It can be done in 5 ways.

31. Find the general form of the numbers that, divided by 2, 3, 7, have for remainders 1, 2, 5, respectively.

Let $\quad x =$ a number of the required kind.

Then $\quad \dfrac{x-1}{2}, \dfrac{x-2}{3}, \dfrac{x-5}{7}$ are integers.

Let $\quad \dfrac{x-1}{2} = m, \quad \dfrac{x-2}{3} = n, \quad \dfrac{x-5}{7} = p$

Then
$$x = 2m + 1 \qquad (1)$$
$$x = 3n + 2 \qquad (2)$$
$$x = 7p + 5 \qquad (3)$$

From (1) and (2), $\quad m = n + \dfrac{1+n}{2}$

Let $\quad \dfrac{1+n}{2} = a, \qquad \therefore n = 2a - 1$

$$x = 3n + 2, \quad \therefore x = 6a - 1 \qquad (4)$$

From (1) and (3), $\quad m = 2 + 3p + \dfrac{p}{2}$

Let $\quad \dfrac{p}{2} = b, \qquad \therefore p = 2b$

$$x = 7p + 5, \quad \therefore x = 14b + 5 \qquad (5)$$

From (4) and (5), $\quad 6a - 14b = 6$

$$a = 1 + 2b + \frac{b}{3}$$

Let $\quad \dfrac{b}{3} = c$

$$\therefore b = 3c$$
$$x = 14b + 5$$
$$\therefore x = 42c + 5$$

$\therefore$ The numbers are of the form $42c + 5$, where c is any integer.

32. Find the general form of the numbers that, divided by 7, 8, 9, have for remainders, 6, 7, 8, respectively.

Let $\quad x =$ any number of the required kind.

Then $\quad \dfrac{x-6}{7}, \dfrac{x-7}{8}, \dfrac{x-8}{9}$ are integers.

Let $\quad \dfrac{x-6}{7} = m, \quad \dfrac{x-7}{8} = n, \quad \dfrac{x-8}{9} = p$

$$x = 7m + 6 \qquad (1)$$
$$x = 8n + 7 \qquad (2)$$
$$x = 9p + 8 \qquad (3)$$

From (1) and (2), $\quad m = n + \dfrac{n+1}{7}$

Let $\frac{n+1}{7}=a, \qquad \therefore\ n=7a-1$

$$x=8n+7,\ \therefore\ x=56a-1 \tag{4}$$

From (1) and (3), $m=p+\frac{2p+2}{7}$

$$4m=5p+1+\frac{p+1}{7}$$

Let $\frac{p+1}{7}=b, \qquad \therefore\ p=7b-1$

$$x=9p+8,\ \therefore\ x=63b-1 \tag{5}$$

From (4) and (5), $56a=63b$

$$a=b+\frac{b}{8}$$

Let $\frac{b}{8}=c,\ \therefore\ b=8c$

$$x=63b-1,\ \therefore\ x=504c-1$$

$\therefore$ The numbers are of the form, $504c-1$, where c is any integer.

33. A farmer buys oxen, sheep, and hens. The whole number bought is 100, and the total cost £100. If the oxen cost £5, the sheep £1, and the hens 1*s*. each, how many of each did he buy?

Let $x=$ number of oxen.
$y=$ number of sheep.
$z=$ number of hens.

Then
$$x+y+z=100 \tag{1}$$
$$5x+y+\tfrac{1}{20}z=100 \tag{2}$$

Subtract, $4x-\frac{19}{20}z=0$

$$80x-19z=0$$
$$z=4x+\tfrac{4}{19}x$$
$$5z=21x-\tfrac{1}{19}x$$

Let $\frac{1}{19}x=m$

$$\therefore\ x=19m$$
$$80x-19z=0$$
$$80\times 19m-19z=0$$
$$\therefore\ z=80m$$

Substitute values of x and z in (1),

$$19m+y+80m=100$$
$$\therefore\ y=100-99m$$

If $m=1$, $x=19$, $y=1$, $z=80$.

$\therefore$ He bought 19 oxen, 1 sheep, and 80 hens.

34. A farmer sells 15 calves, 14 lambs, and 13 pigs, and receives \$200. Some days after, at the same price, he sells 7 calves, 11 lambs, and 16 pigs, for which he receives \$141. What is the price of each?

Let $x =$ number of dollars paid for 1 calf.
$y =$ number of dollars paid for 1 lamb.
$z =$ number of dollars paid for 1 pig.

Then
$$15x + 14y + 13z = 200 \qquad (1)$$
$$7x + 11y + 16z = 141 \qquad (2)$$

$7 \times (1)$ is $105x + 98y + 91z = 1400$
$15 \times (2)$ is $105x + 165y + 240z = 2115$

Subtract, $67y + 149z = 715$

$$67y = 715 - 149z$$
$$y = 10 - 2z + \frac{45 - 15z}{67}$$
$$9y = 94 - 20z + \frac{3 - z}{67}$$

Let $\frac{3 - z}{67} = m$

$$\therefore z = 3 - 67m$$
$$67y + 447 - 149 \times 67m = 715$$
$$y = 4 + 149m$$

Substitute values of y and z in (1),

$$15x + 56 + 2086m + 39 - 871m = 200$$
$$x = 7 - 81m$$

If $m = 0$, $x = 7$, $y = 4$, $z = 3$.

$\therefore$ The price of each calf was \$7, of each lamb, \$4, of each pig, \$3.

Exercise 40.

Expand

1. $(1 + 3x)^5$

$$= 1 + 5(3x) + \frac{5 \cdot 4}{1 \cdot 2} 9x^2 + \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} 27x^3 + \frac{5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4} 81x^4$$
$$+ \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} 243x^5$$
$$= 1 + 15x + 90x^2 + 270x^3 + 405x^4 + 243x^5$$

2. $\left(1+\frac{2x}{3}\right)^4$

$$=1+4\frac{2x}{3}+6\cdot\frac{4x^2}{9}+4\frac{8x^3}{27}+\frac{16x^4}{81}$$

$$=1+\frac{8x}{3}+\frac{8x^2}{3}+\frac{32x^3}{27}+\frac{16x^4}{81}$$

3. $\left(1-\frac{\sqrt{x^3}}{3}\right)^4$

$$=1-4\frac{\sqrt{x^3}}{3}+6\left(\frac{\sqrt{x^3}}{3}\right)^2-4\left(\frac{\sqrt{x^3}}{3}\right)^3+\left(\frac{\sqrt{x^3}}{3}\right)^4$$

$$=1-\tfrac{4}{3}x^{\frac{3}{2}}+\tfrac{2}{3}x^3-\tfrac{4}{27}x^{\frac{9}{2}}+\tfrac{1}{81}x^6$$

4. $(2+x^2)^6$

$$=(2)^6+6(2)^5(x^2)+\frac{6\cdot5}{1\cdot2}(2)^4(x^2)^2+\frac{6\cdot5\cdot4}{1\cdot2\cdot3}(2)^3(x^2)^3$$

$$+\frac{6\cdot5\cdot4\cdot3}{1\cdot2\cdot3\cdot4}(2)^2(x^2)^4+\frac{6\cdot5\cdot4\cdot3\cdot2}{1\cdot2\cdot3\cdot4\cdot5}(2)(x^2)^5$$

$$+\frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{1\cdot2\cdot3\cdot4\cdot5\cdot6}(x^2)^6$$

$$=64+192x^2+240x^4+160x^6+60x^8+12x^{10}+x^{12}.$$

5. $\left(\frac{2}{x}-\frac{x^2}{4}\right)^5$

$$=\left(\frac{2}{x}\right)^5-5\left(\frac{2}{x}\right)^4\left(\frac{x^2}{4}\right)+\frac{5\cdot4}{1\cdot2}\left(\frac{2}{x}\right)^3\left(\frac{x^2}{4}\right)^2-\frac{5\cdot4\cdot3}{1\cdot2\cdot3}\left(\frac{2}{x}\right)^2\left(\frac{x^2}{4}\right)^3$$

$$+\frac{5\cdot4\cdot3\cdot2}{1\cdot2\cdot3\cdot4}\left(\frac{2}{x}\right)\left(\frac{x^2}{4}\right)^4-\frac{5\cdot4\cdot3\cdot2\cdot1}{1\cdot2\cdot3\cdot4\cdot5}\left(\frac{x^2}{4}\right)^5$$

$$=\frac{32}{x^5}-\frac{20}{x^2}+5x-\tfrac{5}{8}x^4+\tfrac{5}{128}x^7-\tfrac{1}{1024}x^{10}$$

6. $\left(\frac{2a}{x}-\frac{x^2}{(2a)^2}\right)^5$

$$=\left(\frac{2a}{x}\right)^5-5\left(\frac{2a}{x}\right)^4\left(\frac{x^2}{(2a)^2}\right)+\frac{5\cdot4}{1\cdot2}\left(\frac{2a}{x}\right)^3\left(\frac{x^2}{(2a)^2}\right)^2$$

$$-\frac{5\cdot4\cdot3}{1\cdot2\cdot3}\left(\frac{2a}{x}\right)^2\left(\frac{x^2}{(2a)^2}\right)^3+\frac{5\cdot4\cdot3\cdot2}{1\cdot2\cdot3\cdot4}\left(\frac{2a}{x}\right)\left(\frac{x^2}{(2a)^2}\right)^4$$

$$-\frac{5\cdot4\cdot3\cdot2\cdot1}{1\cdot2\cdot3\cdot4\cdot5}\left(\frac{x^2}{(2a)^2}\right)^5$$

$$=\frac{32a^5}{x^5}-\frac{20a^2}{x^2}+\frac{5x}{2a}-\frac{5x^4}{8a^4}+\frac{5x^7}{128a^7}-\frac{x^{10}}{1024a^{10}}$$

7. $(3x-2y)^6$

$$= (3x)^6 - 6(3x)^5(2y) + \frac{6\cdot5}{1\cdot2}(3x)^4(2y)^2 - \frac{6\cdot5\cdot4}{1\cdot2\cdot3}(3x)^3(2y)^3$$
$$+ \frac{6\cdot5\cdot4\cdot3}{1\cdot2\cdot3\cdot4}(3x)^2(2y)^4 - \frac{6\cdot5\cdot4\cdot3\cdot2}{1\cdot2\cdot3\cdot4\cdot5}(3x)(2y)^5$$
$$+ \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{1\cdot2\cdot3\cdot4\cdot5\cdot6}(2y)^6$$
$$= 729x^6 - 2916x^5y + 4860x^4y^2 - 4320x^3y^3 + 2160x^2y^4$$
$$- 576xy^5 + 64y^6$$

8. $\left(\frac{2x^2}{y} - \frac{\sqrt[3]{y^2}}{4}\right)^5$

$$= \left(\frac{2x^2}{y}\right)^5 - 5\left(\frac{2x^2}{y}\right)^4\left(\frac{\sqrt[3]{y^2}}{4}\right) + \frac{5\cdot4}{1\cdot2}\left(\frac{2x^2}{y}\right)^3\left(\frac{\sqrt[3]{y^2}}{4}\right)^2$$
$$- \frac{5\cdot4\cdot3}{1\cdot2\cdot3}\left(\frac{2x^2}{y}\right)^2\left(\frac{\sqrt[3]{y^2}}{4}\right)^3 + \frac{5\cdot4\cdot3\cdot2}{1\cdot2\cdot3\cdot4}\left(\frac{2x^2}{y}\right)\left(\frac{\sqrt[3]{y^2}}{4}\right)^4$$
$$- \left(\frac{\sqrt[3]{y^2}}{4}\right)^5$$
$$= \frac{32x^{10}}{y^5} - \frac{20x^8}{y^{\frac{10}{3}}} + \frac{5x^6}{y^{\frac{5}{3}}} - \frac{5}{8}x^4 + \frac{5}{128}x^2y^{\frac{5}{3}} - \frac{1}{1024}y^{\frac{10}{3}}$$

9. $\left(\sqrt{\frac{a^3}{b^2}} - \frac{\sqrt[4]{b^3}}{4a}\right)^4$

$$= \left(\frac{a^{\frac{3}{2}}}{b} - \frac{b^{\frac{3}{4}}}{4a}\right)^4$$
$$= \left(\frac{a^{\frac{3}{2}}}{b}\right)^4 - 4\left(\frac{a^{\frac{3}{2}}}{b}\right)^3\left(\frac{b^{\frac{3}{4}}}{4a}\right) + 6\left(\frac{a^{\frac{3}{2}}}{b}\right)^2\left(\frac{b^{\frac{3}{4}}}{4a}\right)^2 - 4\left(\frac{a^{\frac{3}{2}}}{b}\right)\left(\frac{b^{\frac{3}{4}}}{4a}\right)^3 + \left(\frac{b^{\frac{3}{4}}}{4a}\right)^4$$
$$= \frac{a^6}{b^4} - \frac{a^{\frac{7}{2}}}{b^{\frac{9}{4}}} + \frac{3a}{8b^{\frac{1}{2}}} - \frac{b^{\frac{5}{4}}}{16a^{\frac{3}{2}}} + \frac{b^3}{256a^4}$$

10. $(1+4x+3x^2)^4$

$$= [1+(4x+3x^2)]^4$$
$$= 1 + 4(4x+3x^2) + 6(4x+3x^2)^2 + 4(4x+3x^2)^3 + (4x+3x^2)^4$$
$$= 1 + 16x + 108x^2 + 400x^3 + 886x^4 + 1200x^5 + 972x^6 + 432x^7 + 81x^8$$

11. $(a^2-ax-2x^2)^3$

$$= [a^2-(ax+2x^2)]^3$$
$$= a^6 - 3a^4(ax+2x^2) + 3a^2(ax+2x^2)^2 - (ax+2x^2)^3$$
$$= a^6 - 3a^5x - 3a^4x^2 + 11a^3x^3 + 6a^2x^4 - 12ax^5 - 8x^6$$

12. The fourth term of

$$\left(x+\frac{1}{2x}\right)^8=\frac{8\cdot 7\cdot 6}{1\cdot 2\cdot 3}x^5\left(\frac{1}{2x}\right)^8$$
$$=7x^2$$

13. The eighth term of $\left(2-\dfrac{1}{4x^2}\right)^{10}$.

Coefficient of 8th term is same as coefficient of 4th term.

$$\therefore\ \text{8th term}=-\frac{10\cdot 9\cdot 8}{1\cdot 2\cdot 3}(2)^3\left(\frac{1}{4x^2}\right)^7$$
$$=-\frac{15}{256x^{14}}$$

14. The twelfth term of

$$\left(\frac{1}{x}-\frac{\sqrt{x}}{4}\right)^{14}=-\frac{14\cdot 13\cdot 12}{1\cdot 2\cdot 3}\left(\frac{1}{x}\right)^3\left(\frac{\sqrt{x}}{4}\right)^{11}$$
$$=-\frac{91}{4^{10}}x^{\frac{5}{2}}$$

15. The twentieth term of

$$\left(x-\frac{2}{3\sqrt[4]{x}}\right)^{23}=-\frac{23\cdot 22\cdot 21\cdot 20}{1\cdot 2\cdot 3\cdot 4}x^4\left(\frac{2}{3\sqrt[4]{x}}\right)^{19}$$
$$=-\frac{23\cdot 11\cdot 7\cdot 5\cdot 2^{19}}{3^{19}x^{\frac{3}{4}}}$$
$$=-\frac{8855\times 2^{19}}{3^{19}x^{\frac{3}{4}}}$$

16. The fourteenth term of

$$\left(\sqrt[3]{x^2}-\frac{1}{2\sqrt{x}}\right)^{17}=-\frac{17\cdot 16\cdot 15\cdot 14}{1\cdot 2\cdot 3\cdot 4}(\sqrt[3]{x^2})^4\left(\frac{1}{2\sqrt{x}}\right)^{13}$$
$$=-\frac{595}{2048x^{\frac{23}{6}}}$$

17. The $(r+1)$*th* term of

$$\left(\sqrt{x}+\sqrt[8]{\frac{3}{2x}}\right)^8=\frac{n(n-1)\cdots\cdots(n-r+1)}{1\cdot 2\cdots\cdots r}(\sqrt{x})^{n-r}\left(\sqrt[8]{\frac{3}{2x}}\right)^r$$

Put $n=8$. We have

$$\frac{8\cdot 7\cdots\cdots(9-r)}{1\cdot 2\cdots\cdots r}x^{\frac{8-r}{2}}\left(\frac{3}{2}\right)^{\frac{r}{8}}\left(\frac{1}{x}\right)^{\frac{r}{8}}=\frac{8\cdot 7\cdots\cdots(9-r)}{1\cdot 2\cdots\cdots r}\left(\frac{3}{2}\right)^{\frac{r}{8}}x^{4-\frac{5r}{6}}$$

18. The $(r+1)th$ term of

$$\left(\sqrt{\frac{1}{3x}}-\frac{\sqrt{x}}{2}\right)^{10}=\frac{10\cdot 9\cdots\cdots(11-r)}{1\cdot 2\cdots\cdots r}\left(\sqrt{\frac{1}{3x}}\right)^{10-r}\left(\frac{-\sqrt{x}}{2}\right)^{r}$$

$$=\frac{10\cdot 9\cdots\cdots(11-r)}{1\cdot 2\cdots\cdots r}(-1)^{r}\left(\frac{1}{3}\right)^{\frac{10-r}{2}}\left(\frac{1}{2}\right)^{r}x^{r-5}$$

$$=(-1)^{r}\frac{10\cdot 9\cdots\cdots(11-r)}{1\cdot 2\cdots\cdots r}\,\frac{1}{2^{r}\cdot 3^{5-\frac{r}{2}}}x^{r-5}$$

19. The $(r+3)th$ term of $\left(\frac{x}{2y}-\frac{y}{\sqrt{3x}}\right)^{12}$

Put $r+2$ instead of r in the formula. We have

$$\frac{n(n-1)\cdots\cdots[n-(r+2)+1]}{1\cdot 2\cdots\cdots(r+2)}\left(\frac{x}{2y}\right)^{n-(r+2)}\left(-\frac{y}{\sqrt{3x}}\right)^{r+2}$$

$$=\frac{n(n-1)\cdots\cdots(n-r-1)}{1\cdot 2\cdots\cdots(r+2)}\left(\frac{x}{2y}\right)^{n-r-2}\left(-\frac{y}{\sqrt{3x}}\right)^{n+2}$$

Put $n=12$.

$$\frac{12\cdot 11\cdots\cdots(11-r)}{1\cdot 2\cdots\cdots(r+2)}\left(\frac{x}{2y}\right)^{10-r}\left(-\frac{y}{\sqrt{3x}}\right)^{r+2}$$

$$=\frac{12\cdot 11\cdots\cdots(11-r)}{1\cdot 2\cdots\cdots(r+2)}(-1)^{r+2}\left(\frac{1}{2}\right)^{10-r}\left(\frac{1}{3}\right)^{\frac{r+2}{2}}x^{9-\frac{3r}{2}}y^{2r-8}$$

$$=(-1)^{r}\frac{12\cdot 11\cdots\cdots(11-r)}{1\cdot 2\cdots\cdots(r+2)}\,\frac{x^{9-\frac{3r}{2}}y^{2r-8}}{2^{10-r}3^{\frac{r+2}{2}}}$$

20. Find the middle term of $\left(\frac{3}{4x}-\sqrt{\frac{x^{8}}{2}}\right)^{12}$

The middle term is the 7th.

$$\frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\left(\frac{3}{4x}\right)^{6}\left(\sqrt{\frac{x^{8}}{2}}\right)^{6}=11\cdot 2\cdot 3\cdot 2\cdot 7\times\frac{3^{6}}{4^{6}\cdot 2^{3}}x^{8}$$

$$=\frac{11\cdot 7\cdot 3^{7}}{2^{13}}x^{8}$$

$$=\tfrac{168399}{8192}x^{8}$$

21. Find the two middle terms of $\left(\frac{a}{\sqrt{2x}}+\sqrt{\frac{3x}{4a}}\right)^{15}$

The two middle terms are the 8th and 9th.

The 8th term is

$$\frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}\left(\frac{a}{\sqrt{2x}}\right)^8\left(\sqrt{\frac{3x}{4a}}\right)^7$$

$$= 13 \cdot 11 \cdot 5 \cdot 9\,(\tfrac{1}{2})^4(\tfrac{3}{4})^{\frac{7}{2}}a^{\frac{9}{2}}x^{-\frac{1}{2}}$$

$$= \frac{13 \cdot 11 \cdot 5 \cdot 3^{\frac{11}{2}}}{2^{11}}\,\frac{a^{\frac{9}{2}}}{x^{\frac{1}{2}}} = \frac{173745\sqrt{3}}{2048}\,\frac{a^{\frac{9}{2}}}{x^{\frac{1}{2}}}$$

The 9th term is

$$13 \cdot 11 \cdot 5 \cdot 9\left(\frac{a}{\sqrt{2x}}\right)^7\left(\sqrt{\frac{3x}{4a}}\right)^8 = 13 \cdot 11 \cdot 5 \cdot 9\,(\tfrac{1}{2})^{\frac{7}{2}}(\tfrac{3}{4})^4a^8x^{\frac{1}{2}}$$

$$= \frac{13 \cdot 11 \cdot 5 \cdot 3^6}{2^{\frac{23}{2}}}a^8x^{\frac{1}{2}} = \frac{521235}{2048\sqrt{2}}a^8x^{\frac{1}{2}}$$

22. Find the rth term from the end of $\left(\frac{\sqrt[3]{x^2}}{4} - \sqrt{\frac{x^3}{2}}\right)^{11}$

The rth term from the end is the $(13 - r)$th term from the beginning. The coefficient of the rth term from the end is the same as the coefficient of the rth term from the beginning; and the former term is

$$\frac{n(n-1)\cdots\cdots(n-r+2)}{1 \cdot 2\cdots\cdots(r-1)}a^{r-1}b^{n-r+1}$$

Put $n = 11$.

$$\frac{11 \cdot 10\cdots\cdots(13-r)}{1 \cdot 2\cdots\cdots(r-1)}\left(\frac{\sqrt[3]{x^2}}{4}\right)^{r-1}\left(-\sqrt{\frac{x^3}{2}}\right)^{12-r}$$

$$= \frac{11 \cdot 10\cdots\cdots(13-r)}{1 \cdot 2\cdots\cdots(r-1)}(-1)^{12-r}\left(\frac{1}{4}\right)^{r-1}\left(\frac{1}{2}\right)^{\frac{12-r}{2}}x^{\frac{2(r-1)}{3}}x^{\frac{3(12-r)}{2}}$$

$$= (-1)^{12-r}\frac{11 \cdot 10\cdots\cdots(13-r)}{1 \cdot 2\cdots\cdots(r-1)}\left(\frac{1}{2}\right)^{\frac{3r+8}{2}}x^{\frac{104-5r}{6}}$$

$$= (-1)^r\frac{11 \cdot 10\cdots\cdots(13-r)}{1 \cdot 2\cdots\cdots(r-1)}\,\frac{x^{\frac{104-5r}{6}}}{2^{\frac{3r+8}{2}}}$$

23. In the expansion of $(a + b)^n$ show that the sum of the coefficients is 2^n.

The expansion is

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \cdots\cdots$$

If we put $a = 1$, $b = 1$, we have

$$(1+1)^n = 1 + n + \frac{n(n-1)}{2} + \cdots\cdots$$

expansion

But the right side is the sum of the coefficients in the expansion of $(a+b)^n$.

$\therefore$ $2^n =$ sum of the coefficients.

24. In the expansion of $(a+b)^n$ show that the sum of the even coefficients is equal to the sum of the odd coefficients.

The ~~expression~~ is

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \dots\dots$$

If we put $a = 1$, $b = -1$, we have,

$$(1-1)^n = 1 - n + \frac{n(n-1)}{2} - \dots\dots$$

But the right side is:

Sum of even coefficients − sum of odd coefficients in the expansion of $(a+b)^n$.

$\therefore$ $(1-1)^n =$ sum of even coefficients − sum of odd coefficients.

But $\quad (1-1)^n = 0^n = 0$.

$\therefore$ sum of even coefficients = sum of odd coefficients.

25. Expand

$$\left(x + \frac{\sqrt{-1}}{2x}\right)^4; \quad \left(\sqrt{-1} + \frac{\sqrt[3]{x}}{4\sqrt{-1}}\right)^5; \quad \left(\frac{\sqrt{-a}}{2} + \frac{1}{a\sqrt{-1}}\right)^5.$$

(1) $\left(x + \dfrac{\sqrt{-1}}{2x}\right)^4$

$$= x^4 + 4x^3\left(\frac{\sqrt{-1}}{2x}\right) + 6x^2\left(\frac{\sqrt{-1}}{2x}\right)^2 + 4x\left(\frac{\sqrt{-1}}{2x}\right)^3 + \left(\frac{\sqrt{-1}}{2x}\right)^4$$

$$= x^4 + 2\sqrt{-1}\,x^2 - \frac{3}{2} - \frac{\sqrt{-1}}{2x^2} + \frac{1}{16x^4}$$

$$= \left(x^4 - \frac{3}{2} + \frac{1}{16x^4}\right) + \left(2x^2 - \frac{1}{2x^2}\right)\sqrt{-1}$$

(2) $\sqrt{-1} + \dfrac{\sqrt[3]{x}}{4\sqrt{-1}} = \sqrt{-1} - \dfrac{\sqrt{-1}\ \sqrt[3]{x}}{4} = \sqrt{-1}\left(1 - \dfrac{\sqrt[3]{x}}{4}\right)$

$$\therefore \left(\sqrt{-1} + \frac{\sqrt[3]{x}}{4\sqrt{-1}}\right)^5 = (\sqrt{-1})^5\left(1 - \frac{\sqrt[3]{x}}{4}\right)^5 = \sqrt{-1}\left(1 - \frac{\sqrt[3]{x}}{4}\right)^5$$

$$\left(1-\frac{\sqrt[3]{x}}{4}\right)^5=1-\frac{5\sqrt[3]{x}}{4}+10\left(\frac{\sqrt[3]{x}}{4}\right)^2-10\left(\frac{\sqrt[3]{x}}{4}\right)^3+5\left(\frac{\sqrt[3]{x}}{4}\right)^4-\left(\frac{\sqrt[3]{x}}{4}\right)^5$$

$$=1-\frac{5\sqrt[3]{x}}{4}+\frac{5\sqrt[3]{x^2}}{8}-\frac{5x}{32}+\frac{5\sqrt[3]{x^4}}{256}-\frac{\sqrt[3]{x^5}}{1024}$$

$$\therefore\left(\sqrt{-1}+\frac{\sqrt[3]{x}}{4\sqrt{-1}}\right)^5$$

$$=(1-\tfrac{5}{4}x^{\frac{1}{3}}+\tfrac{5}{8}x^{\frac{2}{3}}-\tfrac{5}{32}x+\tfrac{5}{256}x^{\frac{4}{3}}-\tfrac{1}{1024}x^{\frac{5}{3}})\sqrt{-1}$$

(3) $\dfrac{\sqrt{-a}}{2}+\dfrac{1}{a\sqrt{-1}}=\dfrac{\sqrt{-a}}{2}-\dfrac{\sqrt{-1}}{a}=\sqrt{-1}\left(\dfrac{\sqrt{a}}{2}-\dfrac{1}{a}\right)$

$$\therefore\left(\frac{\sqrt{-a}}{2}+\frac{1}{a\sqrt{-1}}\right)^5=(\sqrt{-1})^5\left(\frac{\sqrt{a}}{2}-\frac{1}{a}\right)^5=\sqrt{-1}\left(\frac{\sqrt{a}}{2}-\frac{1}{a}\right)^5$$

$$\left(\frac{\sqrt{a}}{2}-\frac{1}{a}\right)^5=\left(\frac{\sqrt{a}}{2}\right)^5-5\left(\frac{\sqrt{a}}{2}\right)^4\left(\frac{1}{a}\right)+10\left(\frac{\sqrt{a}}{2}\right)^3\left(\frac{1}{a}\right)^2$$

$$-10\left(\frac{\sqrt{a}}{2}\right)^2\left(\frac{1}{a}\right)^3+5\left(\frac{\sqrt{a}}{2}\right)\left(\frac{1}{a}\right)^4-\left(\frac{1}{a}\right)^5$$

$$=\frac{a^2\sqrt{a}}{32}-\frac{5a}{16}+\frac{5}{4\sqrt{a}}-\frac{5}{2a^2}+\frac{5}{2a^3\sqrt{a}}-\frac{1}{a^5}$$

$$\therefore\left(\frac{\sqrt{-a}}{2}+\frac{1}{a\sqrt{-1}}\right)=\left(-\frac{a^{\frac{5}{2}}}{32}-\frac{5a}{16}+\frac{5}{4a^{\frac{1}{2}}}-\frac{5}{2a^2}+\frac{5}{2a^{\frac{7}{2}}}-\frac{1}{a^5}\right)\sqrt{-1}$$

26. If A is the sum of the odd terms, and B the sum of the even terms, in the expansion of $(a+b)^n$, show that

$$A^2-B^2=(a^2-b^2)^n.$$

$$(a+b)^n=a^n+na^{n-1}b+\frac{n(n-1)}{2}a^{n-2}b^2+\cdots\cdots$$

$$(a-b)^n=a^n-na^{n-1}b+\frac{n(n-1)}{2}a^{n-2}b^2-\cdots\cdots$$

Sum of odd terms in expansion of $(a+b)^n$ is

$$a^n+\frac{n(n-1)}{2}a^{n-2}b^2+\cdots\cdots=A$$

Sum of even terms in expansion of $(a+b)^n$ is

$$na^{n-1}+\frac{n(n-1)(n-2)}{2\cdot 3}a^{n-3}b^3+\cdots\cdots=B$$

$$\therefore\ (a+b)^n=A+B$$

$$(a-b)^n=A-B$$

$$\therefore\ (a+b)^n(a-b)^n=(A+B)(A-B)$$

$$(a^2-b^2)^n=A^2-B^2.$$

Exercise 41.

Expand to four terms:

1. $$(1+x)^{\frac{1}{2}}=1+\tfrac{1}{2}x+\frac{\frac{1}{2}(\frac{1}{2}-1)}{1\cdot 2}x^2+\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{1\cdot 2\cdot 3}x^3+\cdots\cdots$$
$$=1+\tfrac{1}{2}x-\tfrac{1}{8}x^2+\tfrac{1}{16}x^3-\cdots\cdots$$

2. $$(1+x)^{\frac{1}{4}}=1+\tfrac{1}{4}x+\frac{\frac{1}{4}(\frac{1}{4}-1)}{1\cdot 2}x^2+\frac{\frac{1}{4}(\frac{1}{4}-1)(\frac{1}{4}-2)}{1\cdot 2\cdot 3}x^3+\cdots\cdots$$
$$=1+\tfrac{1}{4}x-\tfrac{3}{32}x^2+\tfrac{7}{128}x^3-\cdots\cdots$$

3. $$\frac{1}{\sqrt[3]{1-x}}=(1-x)^{-\frac{1}{3}}$$
$$=1-(-\tfrac{1}{3})x+\frac{-\frac{1}{3}\cdot-\frac{4}{3}}{1\cdot 2}x^2-\frac{-\frac{1}{3}\cdot-\frac{4}{3}\cdot-\frac{7}{3}}{1\cdot 2\cdot 3}x^3+\cdots\cdots$$
$$=1+\tfrac{1}{3}x+\tfrac{2}{9}x^2+\tfrac{14}{81}x^3+\cdots\cdots$$

4. $$(1-x)^{-4}=1-(-4)x+\frac{-4\cdot-5}{1\cdot 2}x^2-\frac{-4\cdot-5\cdot-6}{1\cdot 2\cdot 3}x^3+\cdots\cdots$$
$$=1+4x+10x^2+20x^3+\cdots\cdots$$

5. $$(1+x)^{\frac{5}{2}}=1+\tfrac{5}{2}x+\frac{\frac{5}{2}\cdot\frac{3}{2}}{1\cdot 2}x^2+\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}}{1\cdot 2\cdot 3}x^2+\cdots\cdots$$
$$=1+\tfrac{5}{2}x+\tfrac{15}{8}x^2+\tfrac{5}{16}x^3+\cdots\cdots$$

6. $$\frac{1}{\sqrt[4]{a^2-x^2}}=(a^2-x^2)^{-\frac{1}{4}}$$
$$=(a^2)^{-\frac{1}{4}}-(-\tfrac{1}{4})(a^2)^{-\frac{5}{4}}(x^2)+\frac{-\frac{1}{4}\cdot-\frac{5}{4}}{1\cdot 2}(a^2)^{-\frac{9}{4}}(x^2)^2$$
$$-\frac{-\frac{1}{4}\cdot-\frac{5}{4}\cdot-\frac{9}{4}}{1\cdot 2\cdot 3}(a^2)^{-\frac{13}{4}}(x^2)^3+\cdots\cdots$$
$$=a^{-\frac{1}{2}}+\tfrac{1}{4}a^{-\frac{5}{2}}x^2+\tfrac{5}{32}a^{-\frac{9}{2}}x^4+\tfrac{15}{128}a^{-\frac{13}{2}}x^6+\cdots\cdots$$

7. $\sqrt[5]{2-3x}=(2-3x)^{\frac{1}{5}}$

$$=2^{\frac{1}{5}}-\tfrac{1}{5}2^{-\frac{4}{5}}(3x)+\frac{\frac{1}{5}\cdot-\frac{4}{5}}{1\cdot 2}2^{-\frac{9}{5}}(3x)^2-\frac{\frac{1}{5}\cdot-\frac{4}{5}\cdot-\frac{9}{5}}{1\cdot 2\cdot 3}2^{-\frac{14}{5}}(3x)^3+\cdots\cdots$$

$$=2^{\frac{1}{5}}-\frac{3}{5\cdot 2^{\frac{4}{5}}}x-\frac{9}{25\cdot 2^{\frac{4}{5}}}x^2-\frac{81}{125\cdot 2^{\frac{9}{5}}}x^3-\cdots\cdots$$

8. $\sqrt[3]{(2-x^2)^2}=(2-x^2)^{\frac{2}{3}}$

$$=2^{\frac{2}{3}}-\tfrac{2}{3}2^{-\frac{1}{3}}x^2+\frac{\frac{2}{3}\cdot-\frac{1}{3}}{1\cdot 2}2^{-\frac{4}{3}}(x^2)^2-\frac{\frac{2}{3}\cdot-\frac{1}{3}\cdot-\frac{4}{3}}{1\cdot 2\cdot 3}2^{-\frac{7}{3}}(x^2)^3+\cdots\cdots$$

$$=2^{\frac{2}{3}}-\frac{2^{\frac{2}{3}}}{3}x^2-\frac{1}{18\cdot 2^{\frac{1}{3}}}x^4-\frac{1}{81\cdot 2^{\frac{1}{3}}}x^6-\cdots\cdots$$

9. $\dfrac{1}{\sqrt[4]{(1+2x^2)^3}}=(1+2x^2)^{-\frac{3}{4}}$

$$=1-\tfrac{3}{4}(2x^2)+\frac{-\frac{3}{4}\cdot-\frac{7}{4}}{1\cdot 2}(2x^2)^2+\frac{-\frac{3}{4}\cdot-\frac{7}{4}\cdot-\frac{11}{4}}{1\cdot 2\cdot 3}(2x^2)^3$$

$$=1-\tfrac{3}{2}x^2+\tfrac{21}{8}x^4-\tfrac{77}{16}x^6+\cdots\cdots$$

10. The eighth term of

$$(1-2x)^{\frac{1}{2}}=-\frac{\frac{1}{2}\cdot-\frac{1}{2}\cdot-\frac{3}{2}\cdot-\frac{5}{2}\cdot-\frac{7}{2}\cdot-\frac{9}{2}\cdot-\frac{11}{2}}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}(2x)^7$$

$$=-\frac{3\cdot 5\cdot 7\cdot 9\cdot 11}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}x^7$$

$$=-\tfrac{33}{16}x^7$$

11. The tenth term of

$$(a-3x)^{-\frac{2}{3}}$$

$$=-\frac{-\frac{2}{3}\cdot-\frac{5}{3}\cdot-\frac{8}{3}\cdot-\frac{11}{3}\cdot-\frac{14}{3}\cdot-\frac{17}{3}\cdot-\frac{20}{3}\cdot-\frac{23}{3}\cdot-\frac{26}{3}}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9}a^{-\frac{29}{3}}(3x)^9$$

$$=\frac{2\cdot 5\cdot 8\cdot 11\cdot 14\cdot 17\cdot 20\cdot 23\cdot 26}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9}a^{-\frac{29}{3}}x^9$$

$$=\frac{11\cdot 17\cdot 10\cdot 23\cdot 13}{81}a^{-\frac{29}{3}}x^9$$

$$=\tfrac{559130}{81}a^{-\frac{29}{3}}x^9.$$

12. The $(r+1)$th term of

$$(a+x)^{\frac{1}{3}}=\frac{\frac{1}{3}\cdot-\frac{2}{3}\cdot-\frac{5}{3}\cdots\cdots\left(\frac{4}{3}-r\right)}{1\cdot 2\cdot 3\cdots\cdots r}a^{\frac{1}{3}-r}x^r$$

$$=(-1)^{r-1}\frac{2\cdot 5\cdot 8\cdots\cdots(3r-4)}{1\cdot 2\cdot 3\cdots\cdots r}\,\frac{1}{3^r}\,\frac{x^r}{a^{\frac{r-1}{3}}}$$

13. The $(r+1)$th term of

$$(a^2-4x^2)^{-\frac{5}{2}} = (-1)^r\frac{-\frac{5}{2}\cdot-\frac{7}{2}\cdot-\frac{9}{2}\cdots\cdots(-\frac{3}{2}-r)}{1\cdot2\cdot3\cdots\cdots r}(a^2)^{-\frac{5}{2}-r}(4x^2)^r$$
$$=(-1)^r(-1)^r\frac{5\cdot7\cdot9\cdots\cdots(3+2r)}{1\cdot2\cdot3\cdots\cdots r}2^r\frac{x^{2r}}{a^{2r+5}}$$
$$=\frac{5\cdot7\cdot9\cdots\cdots(3+2r)}{1\cdot2\cdot3\cdots\cdots r}2^r\frac{x^{2r}}{a^{2r+5}}$$

14. Find $\sqrt{65}$ to five decimal places

$$\sqrt{65}=\sqrt{64+1}=\sqrt{64(1+\tfrac{1}{64})}=8\sqrt{1+\tfrac{1}{64}}$$
$$\sqrt{1+\tfrac{1}{64}}=(1+\tfrac{1}{64})^{\frac{1}{2}}$$
$$=1+\tfrac{1}{2}(\tfrac{1}{64})+\frac{\frac{1}{2}\cdot-\frac{1}{2}}{1\cdot2}(\tfrac{1}{64})^2+\frac{\frac{1}{2}\cdot-\frac{1}{2}\cdot-\frac{3}{2}}{1\cdot2\cdot3}(\tfrac{1}{64})^3+\cdots\cdots$$
$$=1+\frac{1}{128}-\frac{1}{8.64^2}+\frac{1}{16.64^3}-\cdots\cdots$$
$$=1+.0078125-.0000305+.0000002$$
$$=1.0077822$$
$$\therefore\ \sqrt{65}=8(1.0077822)$$
$$=8.06225+$$

15. Find $\sqrt[7]{129}$ to six decimal places.

$$\sqrt[7]{129}=\sqrt[7]{128(1+\tfrac{1}{128})}=2\sqrt[7]{1+\tfrac{1}{128}}$$
$$\sqrt[7]{1+\tfrac{1}{128}}=(1+\tfrac{1}{128})^{\frac{1}{7}}$$
$$=1+\frac{1}{7}\ \frac{1}{128}+\frac{\frac{1}{7}\cdot-\frac{6}{7}}{1\cdot2}\left(\frac{1}{128}\right)^2+\cdots\cdots$$
$$=1+\frac{1}{7\cdot128}-\frac{3}{49\cdot128^2}+\cdots\cdots$$
$$=1+.00111607-.00000372$$
$$=1.00111235$$
$$\therefore\ \sqrt[7]{129}=2.002224+$$

16. Expand $(1-2x+3x^2)^{-\frac{2}{5}}$ to four terms.

$$(1-2x+3x^2)^{-\frac{2}{5}}=[1-(2x-3x^2)]^{-\frac{2}{5}}$$
$$=1-(-\tfrac{2}{5})(2x-3x^2)+\frac{-\frac{2}{5}\cdot-\frac{7}{5}}{1\cdot2}(2x-3x^2)^2$$
$$-\frac{-\frac{2}{5}\cdot-\frac{7}{5}\cdot-\frac{12}{5}}{1\cdot2\cdot3}(2x-3x^2)^3+\cdots\cdots$$
$$=1+\tfrac{4}{5}x-\tfrac{6}{5}x^2+\tfrac{28}{25}x^2-\tfrac{84}{25}x^3+\cdots\cdots+\tfrac{224}{125}x^3+\cdots\cdots$$
$$=1+\tfrac{4}{5}x-\tfrac{2}{25}x^2-\tfrac{196}{125}x^3-\cdots\cdots$$

17. Find the coefficient of x^4 in the expansion of

$$\frac{(1+2x)^2}{(1+3x)^3} = (1+2x)^2(1+3x)^{-3}$$

$$= (1+2x)^2\left[1-3(3x)+\frac{-3\cdot-4}{1\cdot2}(3x)^2+\frac{-3\cdot-4\cdot-5}{1\cdot2\cdot3}(3x)^3 + \frac{-3\cdot-4\cdot-5\cdot-6}{1\cdot2\cdot3\cdot4}(3x)^4+\cdots\cdots\right]$$

$$= (1+4x+4x^2)(1-9x+54x^2-270x^3+1215x^4\cdots\cdots)$$

By multiplication, coefficient of x^4 is 351.

18. By means of the expansion of $(1+x)^{\frac{1}{2}}$ show that the limit of the series

$$1+\frac{1}{2}-\frac{1}{2\cdot2^2}+\frac{1\cdot3}{2\cdot3\cdot2^3}-\frac{1\cdot3\cdot5}{2\cdot3\cdot4\cdot2^4}+\cdots\cdots$$

is $\sqrt{2}$.

$$(1+x)^{\frac{1}{2}}=1+\tfrac{1}{2}x+\frac{\frac{1}{2}\cdot-\frac{1}{2}}{1\cdot2}x^2+\frac{\frac{1}{2}\cdot-\frac{1}{2}\cdot-\frac{3}{2}}{1\cdot2\cdot3}x^3+\frac{\frac{1}{2}\cdot-\frac{1}{2}\cdot-\frac{3}{2}\cdot-\frac{5}{2}}{1\cdot2\cdot3\cdot4}x^4+\cdots\cdots$$

$$=1+\tfrac{1}{2}x-\frac{1}{2\cdot2^2}x^2+\frac{1\cdot3}{2\cdot3\cdot2^3}x^3-\frac{1\cdot3\cdot5}{2\cdot3\cdot4\cdot2^4}x^4+\cdots\cdots$$

Put $x=1$

$$(1+1)^{\frac{1}{2}}=1+\frac{1}{2}-\frac{1}{2\cdot2^2}+\frac{1\cdot3}{2\cdot3\cdot2^3}-\frac{1\cdot3\cdot5}{2\cdot3\cdot4\cdot2^4}+\cdots\cdots$$

$$\sqrt{2}=1+\frac{1}{2}-\frac{1}{2\cdot2^2}+\frac{1\cdot3}{2\cdot3\cdot2^3}-\frac{1\cdot3\cdot5}{2\cdot3\cdot4\cdot2^4}+\cdots\cdots$$

19. Find the first negative term in the expansion of $(1+x)^{\frac{11}{3}}$.

The first negative term is the sixth from the beginning, and is

$$\frac{\frac{11}{3}\cdot\frac{8}{3}\cdot\frac{5}{3}\cdot\frac{2}{3}-\frac{1}{3}}{1\cdot2\cdot3\cdot4\cdot5}x^5=-\frac{11\cdot8\cdot5\cdot2}{2\cdot3\cdot4\cdot5\cdot3^5}x^5$$

$$=-\frac{22}{3^6}x^5$$

20. Expand $\sqrt{\frac{1+x}{1-x}}$ in ascending powers of x to six terms.

$$\sqrt{\frac{1+x}{1-x}}=\frac{1+x}{\sqrt{1-x^2}}=(1+x)(1-x^2)^{-\frac{1}{2}}$$

$$(1-x^2)^{-\frac{1}{2}}=1-(-\tfrac{1}{2})x^2+\frac{-\frac{1}{2}\cdot-\frac{3}{2}}{1\cdot2}(x^2)^2-\cdots\cdots$$

$$=1+\tfrac{1}{2}x^2+\tfrac{3}{8}x^4+\cdots\cdots$$

$$\therefore\ (1+x)(1-x^2)^{-\frac{1}{2}}=1+x+\tfrac{1}{2}x^2+\tfrac{1}{2}x^3+\tfrac{3}{8}x^4+\tfrac{3}{8}x^5+\cdots\cdots$$

21. If n is a positive integer, show that the coefficient of x^{n-1}, in the expansion of $(1-x)^{-n}$, is always twice the coefficient of x^{n-2}.

$$(1-x)^{-n} = 1 + nx + \cdots\cdots + (-1)^{n-2}\frac{(-n)(-n-1)(-n-2)\cdots\cdots[-n-(n-3)]}{1\cdot 2\cdot 3\cdots\cdots(n-2)}x^{n-2}$$

$$+ (-1)^{n-1}\frac{(-n)(-n-1)\cdots\cdots[-n-(n-2)]}{1\cdot 2\cdots\cdots(n-1)}x^{n-1} + \cdots\cdots$$

$$= 1 + nx + \cdots\cdots + (-1)^{n-2}(-1)^{n-2}\frac{n(n+1)(n+2)\cdots\cdots(2n-3)}{1\cdot 2\cdot 3\cdots\cdots(n-2)}x^{n-2}$$

$$+ (-1)^{n-1}(-1)^{n-1}\frac{n(n+1)\cdots\cdots(2n-2)}{1\cdot 2\cdots\cdots(n-1)}x^{n-1} + \cdots\cdots$$

$$= 1 + nx + \cdots\cdots + \frac{n(n+1)(n+2)\cdots\cdots(2n-3)}{1\cdot 2\cdot 3\cdots\cdots n-2}x^{n-2} + \frac{n(n+1)\cdots\cdots(2n-2)}{1\cdot 2\cdots\cdots(n-1)}x^{n-1}$$

The coefficient of x^{n-1} is $\dfrac{n(n+1)\cdots\cdots(2n-3)(2n-2)}{1\cdot 2\cdots\cdots(n-2)(n-1)} = \dfrac{n(n+1)\cdots\cdots(2n-3)}{1\cdot 2\cdots\cdots(n-2)}x^2$

= twice the coefficient of x^{n-2}

22. If m and n are positive integers, show that the coefficient of x^m in $(1-x)^{-n-1}$ is the same as the coefficient of x^n in $(1-x)^{-m-1}$.

$$(1-x)^{-n-1}=1+(n+1)x+\cdots\cdots+(-1)^m\frac{(-n-1)(-n-2)\cdots\cdots[-n-1-(m-1)]}{1\cdot2\cdots\cdots m}x^m+\cdots\cdots$$

$$=1+(n+1)x+\cdots\cdots-(-1)^m(-1)^m\frac{(n+1)(n+2)\cdots\cdots(n+m)}{1\cdot2\cdots\cdots m}x^m+\cdots\cdots$$

$$=1+(n+1)x+\cdots\cdots+\frac{(n+1)(n+2)\cdots\cdots(n+m)}{1\cdot2\cdots\cdots m}x^m+\cdots\cdots$$

$$(1-x)^{-m-1}=1+(m+1)x+\cdots\cdots+(-1)^n\frac{(-m-1)(-m-2)\cdots\cdots[-m-1-(n-1)]}{1\cdot2\cdots\cdots n}x^n$$

$$=1+(m+1)x+\cdots\cdots+(-1)^n(-1)^n\frac{(m+1)(m+2)\cdots\cdots(m+n)}{1\cdot2\cdots\cdots n}x^n+\cdots\cdots$$

$$=1+(m+1)x+\cdots\cdots+\frac{(m+1)(m+2)\cdots\cdots(m+n)}{1\cdot2\cdots\cdots n}x^n+\cdots\cdots$$

The coefficients of x^m in the expansion of $(1-x)^{-n-1}$ is

$$\frac{(n+m)(n+m-1)\cdots\cdots(n+1)}{1\cdot2\cdots\cdots m}=\frac{(n+m)(n+m-1)\cdots\cdots(n+1)n(n-1)\cdots\cdots1}{1\cdot2\cdots\cdots m\cdot1\cdot2\cdots\cdots n}$$

The coefficients of x^n in the expansion of $(1-x)^{-m-1}$ is

$$\frac{(m+n)(m+n-1)\cdots\cdots(m+1)}{1\cdot2\cdots\cdots n}=\frac{(m+n)(m+n-1)\cdots\cdots(m+1)m(m-1)\cdots\cdots1}{1\cdot2\cdots\cdots n\cdot1\cdot2\cdots\cdots m}$$

∴ The two coefficients are equal.

23. Find the coefficient of x^{3r} in the expansion of $\sqrt{\frac{1-x}{1+x}}$ in ascending powers of x.

$$\sqrt{\frac{1-x}{1+x}}=\frac{1-x}{\sqrt{1-x^2}}=(1-x)(1-x^2)^{-\frac{1}{2}}$$

$$(1-x^2)^{-\frac{1}{2}}=1+\tfrac{1}{2}x^2+\frac{1\cdot3}{2\cdot4}x^4+\cdots\cdots\frac{1\cdot3\cdot5\cdots\cdots2k-1}{2\cdot4\cdot6\cdots\cdots2k}x^{2k}+\cdots\cdots$$

$$\sqrt{\frac{1-x}{1+x}}=1-x+\tfrac{1}{2}(x^2-x^3)+\frac{1\cdot3}{2\cdot4}(x^4-x^5)+\cdots\cdots$$

$$\cdots\cdots\frac{1\cdot3\cdot5\cdots\cdots2k-1}{2\cdot4\cdot6\cdots\cdots2k}(x^{2k}-x^{2k+1})+\cdots\cdots$$

If r is even, $3r$ is even, and the required coefficient is

$$\frac{1\cdot3\cdot5\cdots\cdots3r-1}{2\cdot4\cdot6\cdots\cdots3r}$$

If r is odd, $3r$ is odd, and the required coefficient is

$$-\frac{1\cdot3\cdot5\cdots\cdots3r-2}{2\cdot4\cdot6\cdots\cdots3r-1}$$

24. Prove that the coefficient of x^r in the expansion of $(1-4x)^{-\frac{1}{2}}$ is $\frac{1\cdot2\cdot3\cdots\cdots2r}{(1\cdot2\cdot3\cdots\cdots r)^2}$.

$$(1-4x)^{-\frac{1}{2}}$$

$$=1+2x+\cdots\cdots+(-1)^r\frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\cdots\cdots(\frac{1}{2}-r)}{1\cdot2\cdot3\cdots\cdots r}(4x)^r+\cdots\cdots$$

$$=1+2x+\cdots\cdots+(-1)^r(-1)^r\frac{1\cdot3\cdot5\cdots\cdots(2r-1)}{1\cdot2\cdot3\cdots\cdots r}2^rx^r+\cdots\cdots$$

∴ The coefficient of

$$x^r=\frac{1\cdot3\cdot5\cdots\cdots(2r-1)}{1\cdot2\cdot3\cdots\cdots r}2^r$$

$$=\frac{1\cdot3\cdot5\cdots\cdots(2r-1)}{1\cdot2\cdots\cdots r}\cdot\frac{(2\cdot1)\cdot(2\cdot2)\cdots\cdots(2\cdot r)}{1\cdot2\cdots\cdots r}$$

$$=\frac{1\cdot2\cdot3\cdot4\cdot5\cdots\cdots2r}{(1\cdot2\cdot3\cdot4\cdot5\cdots\cdots r)^2}$$

Exercise 44.

Find by logarithms:

25. $$\sqrt[5]{\frac{0.0075433^2 \times 78.343 \times 8172.4^{\frac{1}{3}} \times 0.00052}{64285^{\frac{1}{3}} \times 154.27^4 \times 0.001 \times 586.79^{\frac{1}{2}}}}.$$

$$\begin{aligned}
\log 0.0075433^2 &= \bar{5}.7552\\
\log 78.343 &= 1.8940\\
\log 8172.4^{\frac{1}{3}} &= 1.3044\\
\log 0.00052 &= \bar{4}.7160\\
\operatorname{colog} 64285^{\frac{1}{3}} &= \bar{2}.3973\\
\operatorname{colog} 154.27^4 &= \bar{9}.2468\\
\operatorname{colog} 0.001 &= 3.0000\\
\operatorname{colog} 586.79^{\frac{1}{2}} &= \underline{\bar{2}.6158}\\
&\overline{14}.9392 = 36.9392 - 50
\end{aligned}$$

Antilog $\bar{3}.3858 = 0.002431$

$$5\overline{)36.9292 - 50}$$
$$7.3858 - 10 = \bar{3}.3858$$

26. $$\sqrt[7]{\frac{0.03271^2 \times 53.429 \times 0.77542^3}{32.769 \times 0.000371^4}}.$$

$$\begin{aligned}
\log 0.03271^2 &= \bar{3}.0292\\
\log 53.429 &= 1.7278\\
\log 0.77542^3 &= \bar{1}.6688\\
\operatorname{colog} 32.769 &= \bar{2}.4845\\
\operatorname{colog} 0.000371^4 &= \underline{13.7224}
\end{aligned}$$

$$7\overline{)10.6327}$$
$$1.5189$$

Antilog $1.5189 = 33.03$

27. $$\sqrt[3]{\frac{7.1206 \times \sqrt{0.13274} \times 0.057389}{\sqrt{0.43468} \times 17.385 \times \sqrt{0.0096372}}}.$$

$$\begin{aligned}
\log 7.1206 &= 0.8525\\
\log \sqrt{0.13274} &= \bar{1}.5615\\
\log 0.057389 &= \bar{2}.7588\\
\operatorname{colog} \sqrt{0.43468} &= 0.1809\\
\operatorname{colog} 17.385 &= \bar{2}.7590\\
\operatorname{colog} \sqrt{0.0096372} &= \underline{1.0080}\\
&\bar{1}.1216 = 29.1216 - 30
\end{aligned}$$

Antilog $\bar{1}.7072 = 0.5095$

$$3\overline{)29.1216 - 30}$$
$$9.7072 - 10 = \bar{1}.7072$$

Find x from the equations:

28.
$$\begin{aligned} 5^x &= 12 \\ x\log 5 &= \log 12 \\ x &= \frac{\log 12}{\log 5} \\ &= \frac{1.0792}{0.6990} \\ \log 1.0792 &= 0.0331 \\ \operatorname{colog} 0.6990 &= \underline{0.1555} \\ \therefore\ \log x &= 0.1886 \\ x &= 1.544 \end{aligned}$$

29.
$$\begin{aligned} 4^x &= 40 \\ x\log 4 &= \log 40 \\ x &= \frac{\log 40}{\log 4} \\ &= \frac{1.6021}{0.6021} \\ \log 1.6021 &= 0.2047 \\ \operatorname{colog} 0.6021 &= \underline{0.2208} \\ \therefore\ \log x &= 0.4250 \\ x &= 2.66 \end{aligned}$$

30.
$$\begin{aligned} 7^x &= 25 \\ x\log 7 &= \log 25 \\ x &= \frac{\log 25}{\log 7} \\ &= \frac{1.3979}{0.8451} \\ \log 1.3979 &= 0.1454 \\ \operatorname{colog} 0.8451 &= \underline{0.0730} \\ \therefore\ \log x &= 0.2184 \\ x &= 1.654 \end{aligned}$$

31.
$$\begin{aligned} (1.3)^x &= 7.2 \\ x\log 1.3 &= \log 7.2 \\ x &= \frac{\log 7.2}{\log 1.3} \\ &= \frac{0.8573}{0.1139} \\ \log 0.8573 &= \bar{1}.9332 \\ \log 0.1139 &= \underline{\bar{1}.0569} \\ \therefore\ \log x &= 0.8773 \\ x &= 7.538 \end{aligned}$$

32.
$$\begin{aligned} (0.4)^{-x} &= 7 \\ -x\log 0.4 &= \log 7 \\ x &= \frac{\log 7}{\log 0.4} \\ &= \frac{0.8451}{9.6020-10} \\ &= \frac{0.8451}{0.3980} \\ \log 0.8451 &= \bar{1}.9270 \\ \operatorname{colog} 0.3980 &= \underline{0.4001} \\ \log x &= 0.3271 \\ \therefore\ x &= 2.124 \end{aligned}$$

33.
$$\begin{aligned} (0.9)^{\frac{1}{x^2}} &= (4.7)^{-\frac{1}{3}}. \\ \frac{1}{x^2}\log 0.9 &= -\tfrac{1}{3}\log 4.7 \\ x^2 &= -\frac{3\log 0.9}{\log 4.7} \\ &= -\frac{3(9.9542-10)}{0.6721} \\ &= \frac{0.1374}{0.6721} \\ \log 0.1374 &= \bar{1}.1380 \\ \operatorname{colog} 0.6721 &= \underline{0.1725} \\ \log(x^2) &= \bar{1}.3105 \\ \log x &= \bar{1}.6552 \\ \therefore\ x &= 0.4521 \end{aligned}$$

Exercise 45.

Find to four places the natural logarithms of:

1. 2.

$$\begin{aligned}\log_e 2 &= 2.302 \times \log_{10} 2\\ &= 2.302 \times 0.3010\end{aligned}$$

$$\begin{aligned}\log 2.302 &= 0.3621\\ \log 0.3010 &= \bar{1}.4786\\ &\overline{1.8407}\end{aligned}$$

$\therefore\ \log_e 2 = 0.6930$

2. 3.

$$\begin{aligned}\log_e 3 &= 2.302 \times \log_{10} 3\\ &= 2.302 \times 0.4771\end{aligned}$$

$$\begin{aligned}\log 2.302 &= 0.3621\\ \log 0.4771 &= \bar{1}.6786\\ &\overline{0.0407}\end{aligned}$$

$\therefore\ \log_e 3 = 1.099$

3. 100.

$$\begin{aligned}\log_e 100 &= 2.302 \times \log_{10} 100\\ &= 2.302 \times 2\\ &= 4.604\end{aligned}$$

4. 32.5.

$$\begin{aligned}\log_e 32.5 &= 2.302 \times \log_{10} 32.5\\ &= 2.302 \times 1.5119\end{aligned}$$

$$\begin{aligned}\log 2.302 &= 0.3622\\ \log 1.5119 &= 0.1798\\ &\overline{0.5420}\end{aligned}$$

$\therefore\ \log_e 32.5 = 3.483$

5. 7.89.

$$\begin{aligned}\log_e 7.89 &= 2.302 \times \log_{10} 7.89\\ &= 2.302 \times 0.8971\end{aligned}$$

$$\begin{aligned}\log 2.302 &= 0.3621\\ \log 0.8971 &= \bar{1}.9529\\ &\overline{0.3150}\end{aligned}$$

$\therefore\ \log_e 7.89 = 2.065$

6. 1.23.

$$\begin{aligned}\log_e 1.23 &= 2.302 \times \log_{10} 1.23\\ &= 2.302 \times 0.0899\end{aligned}$$

$$\begin{aligned}\log 2.302 &= 0.3621\\ \log 0.0899 &= \bar{2}.9538\\ &\overline{\bar{1}.3159}\end{aligned}$$

$\therefore\ \log_e 1.23 = 0.207$

7. 2.001.

$$\begin{aligned}\log_e 2.001 &= 2.302 \times \log_{10} 2.001\\ &= 2.302 \times 0.3012\end{aligned}$$

$$\begin{aligned}\log 2.302 &= 0.3621\\ \log 0.3012 &= \bar{1}.4789\\ &\overline{\bar{1}.8410}\end{aligned}$$

$\therefore\ \log_e 2.001 = 0.6935$

8. 0.0931.

$$\begin{aligned}\log_e 0.0931 &= 2.302 \times \log_{10} 0.0931\\ &= 2.302 \times (8.9689 - 10)\\ &= 2.302 \times (-1.0311)\end{aligned}$$

$$\begin{aligned}\log 2.302 &= 0.3621\\ \log 1.0311 &= 0.0133 -\\ &\overline{0.3754 -}\end{aligned}$$

$$\begin{aligned}\therefore\ \log_e 0.0931 &= -2.374\\ &= \bar{3}.626\end{aligned}$$

Find to four places:

9.
$$\begin{aligned}\log_2 7 &= \frac{\log_{10} 7}{\log_{10} 2}\\ &= \frac{0.8451}{0.3010}\\ \log 0.8451 &= \bar{1}.9270\\ \text{colog}\, 0.3010 &= \underline{0.5214}\\ &\ 0.4484\\ \therefore\ \log_2 7 &= 2.808\end{aligned}$$

10.
$$\begin{aligned}\log_3 4 &= \frac{\log_{10} 4}{\log_{10} 3}\\ &= \frac{0.6020}{0.4771}\\ \log 0.6020 &= \bar{1}.7796\\ \text{colog}\, 0.4771 &= \underline{0.3214}\\ &\ 0.1010\\ \therefore\ \log_3 4 &= 1.262\end{aligned}$$

11.
$$\begin{aligned}\log_4 9 &= \frac{\log_{10} 9}{\log_{10} 4}\\ &= \frac{0.9542}{0.6020}\\ \log 0.9542 &= \bar{1}.9796\\ \text{colog}\, 0.6020 &= \underline{0.2204}\\ &\ 0.2000\\ \therefore\ \log_4 9 &= 1.585\end{aligned}$$

12.
$$\begin{aligned}\log_5 7 &= \frac{\log_{10} 7}{\log_{10} 5}\\ &= \frac{0.8451}{0.6990}\\ \log 0.8451 &= \bar{1}.9270\\ \text{colog}\, 0.6990 &= \underline{0.1554}\\ &\ .0824\\ \therefore\ \log_5 7 &= 1.209\end{aligned}$$

13.
$$\begin{aligned}\log_9 8 &= \frac{\log_{10} 8}{\log_{10} 9}\\ &= \frac{0.9030}{0.9542}\\ \log 0.9030 &= \bar{1}.9557\\ \text{colog}\, 0.9542 &= \underline{0.0204}\\ &\ \bar{1}.9761\\ \therefore\ \log_9 8 &= 0.9465\end{aligned}$$

14.
$$\begin{aligned}\log_8 5 &= \frac{\log_{10} 5}{\log_{10} 8}\\ &= \frac{0.6990}{0.9031}\\ \log 0.6990 &= 1.8445\\ \text{colog}\, 0.9030 &= \underline{0.0443}\\ &\ \bar{1}.8889\\ \therefore\ \log_8 5 &= 0.7743\end{aligned}$$

15.
$$\begin{aligned}\log_7 14 &= \frac{\log_{10} 14}{\log_{10} 7}\\ &= \frac{1.1461}{0.8451}\\ \log 1.1461 &= 0.0598\\ \text{colog}\, 0.8451 &= \underline{0.0730}\\ &\ 0.1328\\ \therefore\ \log_7 14 &= 1.356\end{aligned}$$

16.
$$\begin{aligned}\log_5 102 &= \frac{\log_{10} 102}{\log_{10} 5}\\ &= \frac{2.0086}{0.6990}\\ \log 2.0086 &= 0.3029\\ \text{colog}\, 0.6990 &= \underline{0.1554}\\ &\ 0.4583\\ \therefore\ \log_5 102 &= 2.873\end{aligned}$$

17. Find the logarithm of 4 in the system of which $\frac{1}{3}$ is the base.

$$\log_{\frac{1}{3}} 4 = \frac{\log_{10} 4}{\log_{10} \frac{1}{3}} = \frac{\log_{10} 4}{-\log_{10} 3} = -\frac{0.6020}{0.4771}$$

$$\begin{aligned} \log 0.6020 &= \bar{1}.7796 \\ \text{colog}\, 0.4771 &= \underline{0.3214} \\ & \ 0.1010 \end{aligned}$$

$$\log_{\frac{1}{3}} 4 = -1.262$$

18. Find the logarithm of $\frac{7}{11}$ in the system of which 0.5 is the base.

$$\log_{0.5} \tfrac{7}{11} = \frac{\log_{10} \frac{7}{11}}{\log_{10} 0.5} = \frac{\log_{10} 7 - \log_{10} 11}{\log_{10} 0.5}$$

$$= \frac{-0.1963}{9.6990 - 1} = \frac{0.1963}{0.3010}$$

$$\begin{aligned} \log 0.1963 &= \bar{1}.2930 \\ \text{colog}\, 0.3010 &= \underline{0.5214} \\ & \ \bar{1}.8144 \end{aligned}$$

$$\therefore \log_{0.5} \tfrac{7}{11} = 0.6523$$

19. Find the base of the system in which the logarithm of 8 is $\frac{2}{3}$.

Let a be the required base.

Then $$\log_a 8 = \tfrac{2}{3}$$

$$\therefore 8 = a^{\frac{2}{3}}$$

$$a^2 = 512 = 2^9$$

$$a = 2^4\sqrt{2} = 16\sqrt{2}$$

20. Find the base of the system in which the logarithm of $\frac{2}{3}$ is $-\frac{3}{5}$.

Let a be the required base.

Then $$\log_a \tfrac{2}{3} = -\tfrac{3}{5}$$

$$a^{-\frac{3}{5}} = \tfrac{2}{3} \qquad a^{\frac{3}{5}} = \tfrac{3}{2}$$

$$a = (\tfrac{3}{2})^{\frac{5}{3}} = \frac{3}{2} \times \frac{3^{\frac{2}{3}}}{2^{\frac{2}{3}}} = \tfrac{3}{4}\sqrt[3]{18}$$

$$a = 1.968$$

Exercise 46.

1. In how many years will $100 amount to $1050, at 5 per cent compound interest?

$$A = PR^n$$

Put $A = 1050$

$P = 100$

$R = 1.05$

Then $1050 = 100 \times 1.05^n$

$$1.05^n = \tfrac{1050}{100}$$

Take the logarithms of both sides of the equation,

$$n \log 1.05 = \log 1050 - \log 100$$

$$n = \frac{\log 1050 - \log 100}{\log 1.05}$$

$$= \frac{3.0212 - 2}{0.0212} = \frac{1.0212}{0.0212}$$

$$\begin{aligned} \log 1.0212 &= 0.0091 \\ \operatorname{colog} 0.0212 &= \underline{1.6737} \\ \therefore\ \log n &= 1.6828 \\ n &= 48.18 \end{aligned}$$

$\therefore$ It will take 48.18 years.

2. In how many years will $$A$ amount to $$B$ (1) at simple interest, (2) at compound interest, r and R being used in their usual sense?

(1)

$$B = A(1 + nr)$$

$$\therefore\ n = \frac{B - A}{r}$$

$$\frac{B - A}{r} \text{ years.}$$

(2)

$$B = AR^n$$

$$R^n = \frac{B}{A}$$

$$n \log R = \log B - \log A$$

$$n = \frac{\log B - \log A}{\log R}$$

$$\frac{\log B - \log A}{\log R} \text{ years.}$$

3. Find the difference (to five places of decimals) between the amount of $1 in 2 years, at 6 per cent compound interest, according as the interest is due yearly or monthly.

(1) Let the interest be due yearly.

$$A = PR^n$$
$$P = 1$$
$$R = 1.06$$
$$n = 2$$
$$\therefore A = 1 \times (1.06)^2$$
$$= 1.1236$$

(2) Let the interest be due monthly.

$$A = P\left(1 + \frac{r}{12}\right)^{12n}$$
$$P = 1$$
$$r = 0.06$$
$$1 + \frac{r}{12} = 1.005$$
$$12n = 24$$
$$\therefore A = 1 \times 1.005^{24}$$
$$\log A = 24 \log 1.005$$
$$= 0.0528$$
$$A = 1.12926$$

$$\$1.12926 - \$1.1236 = \$0.00366.$$

The difference is \$0.00366.

4. At 5 per cent, find the amount of an annuity of $\$A$ which has been left unpaid for 4 years.

$$A' = \frac{s(R^n - 1)}{r}$$
$$s = A$$
$$R = 1.05$$
$$r = 0.05$$
$$n = 4$$

$$A' = \frac{A(1.05^4 - 1)}{0.05}$$
$$= A\frac{0.2155}{0.05}$$
$$= A \times 4.31$$

The amount is $4.31 \times A$ dollars.

5. Find the present value of an annuity of \$100 for 5 years, reckoning interest at 4 per cent.

$$P = \frac{S}{R^n} \times \frac{R^n - 1}{R - 1}$$
$$S = 100$$
$$R = 1.04$$
$$n = 5$$
$$\therefore P = \frac{100}{1.04^5} \times \frac{1.04^5 - 1}{1.04 - 1}$$

By logarithms,
$$P = \frac{100}{1.216} \times \frac{0.216}{0.04}$$
$$P = 444$$

The present value is \$444.

6. A perpetual annuity of $1000 is to be purchased, to begin at the end of 10 years. If interest is reckoned at $3\frac{1}{2}$ per cent, what should be paid for it?

Let $P =$ present value.

Then $PR^{10} =$ value at end of 10 years.

But $\frac{S}{r} =$ value of perpetual annuity.

$$\therefore PR^{10} = \frac{S}{r}$$

$$P = \frac{S}{R^{10}r}$$

$$S = 1000$$

$$R = 1.035$$

$$r = 0.035$$

$$\therefore P = \frac{1000}{1.035^{10} \times 0.035}$$

By logarithms, $p = 20270$

$20,270 should be paid.

7. A debt of $1850 is discharged by two payments of $1000 each, at the end of one and two years. Find the rate of interest paid.

Let $r =$ rate of interest.

Then $1850(1 + r) =$ amount of debt at end of first year.

$1850(1 + r) - 1000 =$ amount of debt after first payment.

$[1850(1 + r) - 1000](1 + r) =$ amount of debt at end of second year.

$$\therefore [1850(1 + r) - 1000](1 + r) = 1000$$

$$850 + 2700r + 1850r^2 = 1000$$

$$1850r^2 + 2700r = 150$$

$$37r^2 + 54r = 3$$

$$\therefore r = \frac{-27 \pm 2\sqrt{210}}{37}$$

By logarithms, $r = 0.05351$

The rate of interest is 0.05351.

8. Reckoning interest at 4 per cent, what annual premium should be paid for 30 years, in order to secure $2000 to be paid at the end of that time, the premium being due at the beginning of each year?

$$P = \frac{Ar}{R(R^n - 1)}$$

$$A = 2000$$

$$r = 0.04$$
$$R = 1.04$$
$$n = 30$$
$$\therefore\ P = \frac{2000 \times 0.04}{1.04\,(1.04^{30} - 1)}$$

By logarithms, $P = 34.40$

$\therefore$ The annual premium is \$34.40.

9. An annual premium of \$150 is paid to a life-insurance company for insuring \$5000. If money is worth 4 per cent, for how many years must the premium be paid in order that the company may sustain no loss?

$$P = \frac{Ar}{R^n - 1}$$
$$P\,(R^n - 1) = Ar$$
$$R^n = \frac{Ar + P}{P}$$
$$n \log R = \log (Ar + P) - \log P$$
$$\therefore\ n = \frac{\log (Ar + P) - \log P}{\log R}$$
$$P = 150$$
$$A = 5000$$
$$R = 1.04$$
$$r = 0.04$$
$$n = \frac{\log 350 - \log 150}{\log 1.04}$$
$$= \frac{0.3680}{0.0170} = 21.65$$

$\therefore$ The premium must be paid 21.65 years.

10. What may be paid for bonds due in 10 years, and bearing semi-annual coupons of 4 per cent each, in order to realize 3 per cent semi-annually, if money is worth 3 per cent semi-annually.

$$P\,(1 + x)^n = \frac{Sq + Sr\,(1 + q)^n - Sr}{q}$$
$$P = \frac{Sq + Sr(1 + q)^n - Sr}{q\,(1 + x)^n}$$
$$S = 100$$
$$q = 0.03$$
$$r = 0.04$$

$$x = 0.03$$
$$n = 20$$
$$\therefore\ P = \frac{3 + 4\,(1.03^{20} - 1)}{0.03\,(1.03)^{20}}$$

By logarithms, $$P = \frac{6.212}{0.05409} = 114.8$$

The bonds must be bought at 114.80.

11. When money is worth 2 per cent semi-annually, if bonds having 12 years to run, and bearing semi-annual coupons of $3\frac{1}{2}$ per cent each, are bought at $114\frac{1}{8}$, what per cent is realized on the investment?

Let $x =$ rate of interest received on the investment.

Then $$1 + x = \left(\frac{Sq + Sr(1+q)^n - Sr}{Pq}\right)^{\frac{1}{n}}$$
$$S = 100$$
$$q = 0.02$$
$$r = 0.035$$
$$n = 24$$
$$P = 114.125$$
$$\therefore\ 1 + x = \left(\frac{2 + 3.5 \times 1.02^{24} - 3.5}{114.125 \times 0.02}\right)^{\frac{1}{24}}$$
$$= \left(\frac{4.13}{2.2825}\right)^{\frac{1}{24}} = 1.025.$$
$$x = 0.025$$

$\therefore$ $2\frac{1}{2}$ per cent semi-annually is realized on the investment; that is 5 per cent per annum.

12. If \$126 is paid for bonds due in 12 years, and yielding $3\frac{1}{2}$ per cent semi-annually, what per cent is realized on the investment, provided money is worth 2 per cent semi-annually?

Let $x =$ rate of interest received on the investment.

Then $$1 + x = \left(\frac{Sq + Sr(1+q)^n - Sr}{Pq}\right)^{\frac{1}{n}}$$
$$S = 100$$
$$q = 0.02$$
$$r = 0.035$$
$$n = 24$$
$$P = 126$$

$$\therefore\ 1+x=\left(\frac{2+3.5\times 1.02^{24}-3.5}{126\times 0.02}\right)^{\frac{1}{24}}$$

$$=\left(\frac{4.13}{2.52}\right)^{\frac{1}{24}}=1.021$$

$$x=0.021$$

∴ 4.2 per cent per annum is made on the investment.

13. A person borrows \$600.25. How much must he pay annually that the whole debt may be discharged in 35 years, interest being reckoned at 4 per cent?

Let $p=$ amount of debt.

$s=$ annual payment.

Then
$$PR^n=\frac{S(R^n-1)}{r}$$

$$\therefore\ S=\frac{PrR^n}{R^n-1}$$

$$P=600.25$$
$$r=0.04$$
$$R=1.04$$
$$n=35$$

$$S=\frac{600.25\times 0.04\times 1.04^{35}}{1.04^{35}-1}$$

$$=\frac{94.5}{2.935}=32.3$$

∴ He must pay \$32.30 per year.

14. A perpetual annuity of \$100 a year is sold for \$2500. At what rate is the interest reckoned?

$$P=\frac{S}{r}$$

$$\therefore\ r=\frac{S}{P}$$

$$P=2500$$
$$S=100$$

$$\therefore\ r=\tfrac{100}{2500}$$
$$r=0.04$$

∴ The interest is reckoned at 4 per cent.

15. A perpetual annuity of $320, to begin 10 years hence, is to be purchased. If interest is reckoned at 3⅕ per cent, what should be paid for it?

Let $P =$ present value of the annuity.

Then $P(1.032)^{10} =$ its value 10 years hence.

But $\dfrac{320}{0.032} =$ its value 10 years hence.

$$\therefore P(1.032)^{10} = \frac{320}{0.032} = 10000$$

$$P = \frac{10000}{(1.032)^{10}} = 7377$$

∴ $7377 should be paid for the annuity.

16. A sum of $10,000 is loaned at 4 per cent. At the end of the first year a payment of $400 is made; and at the end of each following year a payment is made greater by 30 per cent than the preceding payment. Find in how many years the debt will be paid.

Let $n =$ number of years in which the debt will be paid.

Then

$10000(1.04)^n =$ sum to which 10,000 dollars will amount at the end of that time.

$400(1.04)^{n-1} + 400 \times 1.30(1.04)^{n-2} + 400(1.30)^2(1.04)^{n-3} + \text{etc.}$
$=$ sum to which the payments would amount at the end of that time.

$$\therefore 10000(1.04)^n = 400(1.04)^{n-1} + 400 \times 1.30(1.04)^{n-2} + 400(1.30)^2(1.04)^{n-3} + \text{etc.}$$
$$= 400[1.04^{n-1} + 1.30(1.04)^{n-2} + (1.30)^2(1.04)^{n-3} + \text{etc.}]$$

But $1.04^{n-1} + 1.30(1.04)^{n-2} + (1.30)^2(1.04)^{n-3} + \text{etc.}$ is a geometrical progression of which the ratio is $\dfrac{1.30}{1.04}$, and the sum

$$= \frac{1.04^{n-1}\left[\left(\frac{1.30}{1.04}\right)^n - 1\right]}{\frac{1.30}{1.04} - 1} = \frac{1.30^n - 1.04^n}{1.30 - 1.04}$$

$$\therefore 10000(1.04)^n = 400\,\frac{1.30^n - 1.04^n}{1.30 - 1.04}$$

$$25(1.30 - 1.04) = \frac{1.30^n - 1.04^n}{1.04^n}$$

$$25 \times 0.26 = \left(\frac{1.30}{1.04}\right)^n - 1$$

$$\left(\frac{1.30}{1.04}\right)^n = 7.5$$

$$n \log \frac{1.30}{1.04} = \log 7.5$$

$$n = \frac{\log 7.5}{\log 1.30 - \log 1.04}$$

$$= \frac{0.8751}{0.0969} = 9.032$$

∴ **The debt will be paid in 9.032 years.**

17. A man with a capital of $100,000 spends every year $9000. If the current rate of interest is 5 per cent, in how many years will he be ruined?

Let n **= number of years in which he will be ruined.**

Then

100000(1.05)^n = sum to which his capital will amount in n years.

9000(1.05)$^{n-1}$ + 9000(1.05)$^{n-2}$ + etc.

= sum to which his expenditure will amount in n years.

$$\therefore\ 9000(1.05)^{n-1} + 9000(1.05)^{n-2} + \text{etc.} = 100000(1.05)^n$$

$$9000\,\frac{1.05^{n-1}\left(\frac{1}{1.05^n} - 1\right)}{\frac{1}{1.05} - 1} = 100000(1.05)^n$$

$$9000\,\frac{\left(\frac{1}{1.05^n} - 1\right)}{\left(\frac{1}{1.05} - 1\right)1.05} = 100000$$

$$\frac{\frac{1}{1.05^n} - 1}{1 - 1.05} = \frac{100}{9}$$

$$\frac{1}{1.05^n} = 1 - \frac{5}{9} = \frac{4}{9}$$

$$1.05^n = \tfrac{9}{4}$$

$$n \log 1.05 = \log 9 - \log 4$$

$$\therefore\ n = \frac{\log 9 - \log 4}{\log 1.05}$$

$$= \frac{0.3521}{0.0212} = 16.23$$

∴ **He will be ruined in 16.23 years.**

18. Find the amount of $365 at compound interest for 20 years, at 5 per cent.

$$A = 365(1.05)^{20} = 969$$

$969 is the amount.

Exercise 47.

1. How many numbers of five figures each can be formed with the digits 1, 2, 3, 4, 5, no digit being repeated?

The 5 digits can be arranged in $\underline{|5}$ ways, and each arrangement gives a number. Hence, $\underline{|5}$, or 120, numbers can be formed.

2. How many *even* numbers of four figures each can be formed with the digits 1, 2, 3, 4, 5, 6, no digit being repeated?

The last place can be filled in 3 ways, by 2, 4, or 6. The first place in 5 ways; and so on.

Hence there are $5 \times 4 \times 3 \times 3 = 180$ numbers.

3. How many *odd* numbers between 1000 and 5000 can be formed with the figures 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, no figure being repeated? How many of these numbers will be divisible by 5?

The last place can be filled only by 1, 3, 5, 7, or 9.

The first place can be filled only by 1, 2, 3, or 4.

If 5, 7, or 9 occupies the last place, the first figure may be 1, 2, 3, or 4.

But, if 1 or 3 occupy the last place, only 2, 3, and 4, or 1, 2, and 4, can occupy the first place.

The first and last places can therefore be filled in $4+4+4+3+3$, or 18 ways.

These 2 places having been filled, the remaining 2 places can be filled from the remaining 8 digits in 56 ways.

Hence there are in all $18 \times 56 = 1008$ numbers.

Of these, those that are divisible by 5 must have 5 for the last figure.

The first figure may be 1, 2, 3, or 4.

These two places having been filled, the remaining figure can be selected in $\frac{\underline{|8}}{\underline{|6}} = 56$ ways.

Hence $4 \times 56 = 224$ of the numbers are divisible by 5.

4. How many three-lettered words can be made from the alphabet, no letter being repeated in the same word?

$$26 \cdot 25 \cdot 24 = 15{,}600$$

5. In how many ways can 4 persons, A, B, C, D, sit at a round table?

A can take any one of the four seats: 4 possibilities.

A being seated, B can take any one of the three remaining seats: 3 possibilities.

A and B being seated, C can take either of the two remaining seats: 2 possibilities.

D must take the remaining seat: 1 possibility.

Hence there are in all $4 \times 3 \times 2 \times 1 = 24$ ways.

If however only the *order* in which A, B, C, and D sit, not the particular seats they occupy, be regarded, then A may take any seat, and the others can then arrange themselves in $3 \times 2 \times 1 = 6$ ways.

6. In how many ways can 6 persons form a ring?

Any one of the 6 being selected as a starting point, his neighbor on either side, say on the right, can be selected in 5 ways, the second man's right-hand neighbor in 4 ways, and so on.

Hence there are in all $5 \times 4 \times 3 \times 2 \times 1 = 120$ different ways.

The 120 arrangements can be divided into pairs, such that in each pair the order of succession from right to left in the one arrangement is the same in the order of succession from left to right in the other. If such a pair be regarded as a single arrangement, there will be only 60 ways.

7. How many words can be made with 9 letters, 3 letters remaining inseparable and keeping the same order?

The three inseparable letters may be regarded as one letter.

There are therefore practically 7 letters.

Hence there are $\underline{|7} = 5040$ words.

8. What will be the answer to the preceding problem if the 3 inseparable letters can be arranged in any order?

In each of the $\underline{|7}$ words of the last problem, the 3 inseparable letters may be arranged among themselves in 6 ways.

Hence there are $6 \times \underline{|7} = 30{,}240$ words in this case.

9. A captain, having under his command 60 men, wishes to form a guard of 8 men. In how many different ways can the guard be formed?

$$\frac{\lfloor\underline{60}}{\lfloor\underline{52}\,\lfloor\underline{8}}$$

10. A detachment of 30 men must furnish each night a guard of 4 men. For how many nights can a different guard be formed, and how many times will each soldier serve?

$\frac{\lfloor\underline{30}}{\lfloor\underline{26}\,\lfloor\underline{4}} = 27{,}405$ nights.

Any soldier being selected, the remaining 3 can be chosen from the remaining 29 in $\frac{\lfloor\underline{29}}{\lfloor\underline{26}\,\lfloor\underline{3}}$ different ways.

Each soldier therefore serves $\frac{\lfloor\underline{29}}{\lfloor\underline{26}\,\lfloor\underline{3}} = 3654$ times.

11. Out of 12 Democrats and 16 Republicans, how many different committees can be formed, each committee consisting of 3 Democrats and 4 Republicans?

The 3 Democrats can be selected in $\frac{\lfloor\underline{12}}{\lfloor\underline{9}\,\lfloor\underline{3}}$, or 220 ways.

The 4 Republicans can be selected in $\frac{\lfloor\underline{16}}{\lfloor\underline{12}\,\lfloor\underline{4}}$, or 1820 ways.

Any 3 Democrats can serve with any 4 Republicans.
Hence $220 \times 1820 = 400{,}400$ different committees can be formed.

12. Out of 26 Republicans and 14 Democrats, how many different committees can be formed, each committee consisting of 10 Republicans and 8 Democrats?

The 10 Republicans can be selected in $\frac{\lfloor\underline{26}}{\lfloor\underline{16}\,\lfloor\underline{10}}$ ways.

The 8 Democrats can be selected in $\frac{\lfloor\underline{14}}{\lfloor\underline{8}\,\lfloor\underline{6}}$ ways.

Any 10 Republicans can serve with any 8 Democrats.

Hence $\frac{\lfloor\underline{26}}{\lfloor\underline{16}\,\lfloor\underline{10}} \times \frac{\lfloor\underline{14}}{\lfloor\underline{8}\,\lfloor\underline{6}}$ different committees can be formed.

13. There are m different things of one kind and n different things of another kind; how many different sets can be made, each set containing r things of the first kind and s of the second?

The r things of the first kind can be selected in $\dfrac{\underline{|m}}{\underline{|m-r}\,\underline{|r}}$ ways.

The s things of the second kind can be selected in $\dfrac{\underline{|n}}{\underline{|n-s}\,\underline{|s}}$ ways.

Any r things of the first kind can go with any s things of the second kind.

Hence there are $\dfrac{\underline{|m}}{\underline{|m-r}\,\underline{|r}} \times \dfrac{\underline{|n}}{\underline{|n-s}\,\underline{|s}}$ different sets.

14. With 12 consonants and 6 vowels, how many different words can be formed consisting of 3 different consonants and 2 different vowels, any arrangement of letters being considered a word?

The 3 consonants can be selected in $\dfrac{\underline{|12}}{\underline{|3}\,\underline{|9}}$, or 220 ways.

The 2 vowels can be selected in $\dfrac{\underline{|6}}{\underline{|4}\,\underline{|2}}$, or 15 ways.

Hence 3 consonants and 2 vowels can be selected in 15×220, or 3300 ways.

When any 3 consonants and 2 vowels have been selected, they may be arranged in $\underline{|5}$, or 120 ways.

Hence in all $120 \times 3300 = 396{,}000$ words can be formed.

15. With 10 consonants and 6 vowels, how many words can be formed, each word containing 5 consonants and 4 vowels?

The 5 consonants can be selected in $\dfrac{\underline{|10}}{\underline{|5}\,\underline{|5}}$, or 252 ways.

The 4 vowels can be selected in $\dfrac{\underline{|6}}{\underline{|4}\,\underline{|2}}$, or 15 ways.

Hence 5 consonants and 4 vowels can be selected in 15×252, or 3780 ways.

When any 5 consonants and 4 vowels have been selected, they may be arranged in $\underline{|9}$ different ways.

Hence in all, $3780 \times \underline{|9}$ words can be formed.

16. How many words can be formed with 20 consonants and 6 vowels, each word containing 3 consonants and 2 vowels, the vowels occupying the second and fourth places?

The 3 consonants can be selected in $\frac{\underline{|20}}{\underline{|17}\,\underline{|3}}$, or 1140 ways.

The 2 vowels can be selected in $\frac{\underline{|6}}{\underline{|4}\,\underline{|2}}$, or 15 ways.

Any 3 selected consonants can occupy the first, third, and fifth places in $3 \times 2 \times 1$, or 6 ways.

Any 2 selected vowels can occupy the second and fourth places in 2 ways.

Hence in all there are $2 \times 6 \times 15 \times 1140$ words.

17. An assembly of stockholders, composed of 40 merchants, 20 lawyers, and 10 physicians, wishes to elect a commission of 4 merchants, 1 physician, and 2 lawyers. In how many ways can the commission be formed?

The 4 merchants can be selected in $\frac{\underline{|40}}{\underline{|36}\,\underline{|4}}$ ways.

The 2 lawyers can be selected in $\frac{\underline{|20}}{\underline{|18}\,\underline{|2}}$, or 190 ways.

The 1 physician can be selected in 10 ways.

Any 4 merchants can serve with any 2 lawyers and with any 1 physician.

Hence the commission can be formed in $190 \times \frac{\underline{|40}}{\underline{|36}\,\underline{|4}} \times 10 = 173{,}641{,}000$.

18. Of 8 men forming a boat's crew, one is selected as stroke. How many arrangements of the rest are possible? When the 4 men who row on each side are decided on, how many arrangements are still possible?

(1) There remain 7 men and 7 places. The 7 men can fill the 7 places in $\underline{|7} = 5040$ different ways.

(2) The 3 men who row with the stroke can be arranged in $\underline{|3}$, or 6 ways. The 4 men on the opposite side can be arranged in $\underline{|4}$ or 24 different ways.

Hence 6×24, or 144 different arrangements are possible.

19. A boat's crew consists of 8 men. Either A or B must row stroke. Either B or C must row bow. D can pull only on the starboard side. In how many ways can the crew be seated?

If A rows stroke, either B or C must row bow: 2 possibilities.

If B rows stroke, C must row bow: 1 possibility.

Hence stroke and bow can be selected in 3 ways.

There remain 6 men, and 3 places on each side to be filled.

Since D can only row on the starboard side, there are only 5 men who can row on the port side.

From these 5, 3 can be selected to row on the port side in $\frac{\underline{|5}}{\underline{|3}\,\underline{|2}} = 10$ ways.

The remaining 2 row with D on the starboard side.

The 3 men on the port side can be seated in $3 \times 2 \times 1 = 6$ ways.

The 3 men on the starboard side can be seated in $3 \times 2 \times 1 = 6$ ways.

Hence the crew can be seated in $3 \times 10 \times 6 \times 6 = 1080$ ways.

20. A boat's crew consists of 8 men. Of these, 3 can row only on the port side, and 2 only on the starboard side. In how many ways can the crew be seated?

3 must row on the port side and 2 must row on the starboard side.

There remain 3 who can row on either side.

From these 3, the fourth man on the port side can be chosen in 3 ways.

The remaining 2 row on the starboard side.

The 4 men on the port side can be seated in $\underline{|4} = 24$ ways.

The 4 men on the starboard side can be seated in $|4 = 24$ ways.

Hence there are $3 \times 24 \times 24 = 1728$ ways of seating the crew.

21. Of a base ball nine, either A or B must pitch; either B or C must catch; D, E, and F play in the field. In how many ways can the nine be arranged?

A pitcher, B catcher, D, E, F can be arranged in $\underline{|3}$ ways, the other 4 men in $\underline{|4}$ ways; total, $\underline{|3} \times \underline{|4} = 144$ ways.

A pitcher, C catcher,	144 ways.
B pitcher, C catcher,	144 ways.
Total,	432 ways.

22. How many signals may be made with 8 flags of different colors, which can be hoisted either singly, or any number at a time one above another?

With 8 flags	$8\times 7\times 6\times 5\times 4\times 3\times 2\times 1$ signals.
With 7 flags	$8\times 7\times 6\times 5\times 4\times 3\times 2$
With 6 flags	$8\times 7\times 6\times 5\times 4\times 3$
With 5 flags	$8\times 7\times 6\times 5\times 4$
With 4 flags	$8\times 7\times 6\times 5$
With 3 flags	$8\times 7\times 6$
With 2 flags	8×7
With 1 flag	8
In all	109,600 signals.

23. Of 30 things, how many must be taken together, in order that having that number for selection, there may be the greatest possible variety of choice?

15 must be taken.

24. The number of combinations of $n+2$ objects, taken 4 at a time, is to the number of combinations of n objects, taken 2 at a time, as 11 is to 1. Find n.

$n+2$ objects taken 4 at a time give $\frac{\underline{|n+2}}{\underline{|n-2}\,\underline{|4}}$ combinations.

n objects taken 2 at a time give $\frac{\underline{|n}}{\underline{|n-2}\,\underline{|2}}$ combinations.

$$\therefore \frac{\underline{|n+2}}{\underline{|n-2}\,\underline{|4}} : \frac{\underline{|n}}{\underline{|n-2}\,\underline{|2}} = 11 : 1$$

$$\frac{\underline{|n+2}}{\underline{|n-2}\,\underline{|4}} = \frac{11\,\underline{|n}}{\underline{|n-2}\,\underline{|2}}$$

$$\frac{(n+2)(n+1)n(n-1)}{4\times 3\times 2\times 1} = \frac{11\,n(n-1)}{2\times 1}$$

$$(n+2)(n+1) = 132$$

$$n^2 + 3n - 130 = 0$$

$$(n-10)(n+13) = 0$$

$$\therefore n = 10$$

25. The number of combinations of n things, taken r together, is 3 times the number taken $r-1$ together, and half the number taken $r+1$ together. Find n and r.

The number of combinations of n things r at a time is $\frac{\underline{|n}}{\underline{|n-r}\,\underline{|r}}$.

The number of combinations of n things $r-1$ at a time is

$$\frac{\lfloor\underline{n}}{\lfloor\underline{n-r+1}\,\lfloor\underline{r-1}}$$

The number of combinations of n things $r+1$ at a time is

$$\frac{\lfloor\underline{n}}{\lfloor\underline{n-r-1}\,\lfloor\underline{r+1}}$$

$$\therefore\ \frac{\lfloor\underline{n}}{\lfloor\underline{n-r}\,\lfloor\underline{r}} = 3\,\frac{\lfloor\underline{n}}{\lfloor\underline{n-r+1}\,\lfloor\underline{r-1}} \qquad (1)$$

$$\frac{\lfloor\underline{n}}{\lfloor\underline{n-r}\,\lfloor\underline{r}} = \tfrac{1}{2}\,\frac{\lfloor\underline{n}}{\lfloor\underline{n-r-1}\,\lfloor\underline{r+1}} \qquad (2)$$

From (1),

$$\frac{\lfloor\underline{n-r+1}}{\lfloor\underline{n-r}} = 3\,\frac{\lfloor\underline{r}}{\lfloor\underline{r-1}}$$

$$n-r+1=3r$$

$$n-4r+1=0 \qquad (3)$$

From (2),

$$\frac{\lfloor\underline{r+1}}{\lfloor\underline{r}} = \tfrac{1}{2}\,\frac{\lfloor\underline{n-r}}{\lfloor\underline{n-r-1}}$$

$$r+1=\tfrac{1}{2}(n-r)$$

$$2r+2=n-r$$

$$n-3r-2=0 \qquad (4)$$

From (3) and (4),

$$n=11$$

$$r=3$$

26. At a game of cards, 3 being dealt to each person, any one can have 425 times as many hands as there are cards in the pack. How many cards are there in the pack?

Let n = number of cards in the pack.

Then

$$\frac{\lfloor\underline{n}}{\lfloor\underline{n-3}\,\lfloor\underline{3}} = 425n$$

$$\frac{n(n-1)(n-2)}{3\times2\times1} = 425n$$

$$n^3-3n^2+2n=2550n$$

$$n^3-3n^2-2548n=0$$

$$n(n^2-3n-2548)=0$$

But n cannot be 0.

$$\therefore\ n^2-3n-2548=0$$

$$(n-52)(n+49)=0$$

$$\therefore\ n=52$$

27. It is proposed to divide 15 objects into lots, each lot containing 3 objects. In how many ways can the lots be made?

$$\frac{|\underline{15}}{|\underline{12}\,|\underline{3}} = 455 \text{ ways.}$$

Exercise 48.

1. How many three-lettered words can be made from the 6 vowels, when repetitions are allowed?

The first letter can be selected in 6 ways.

The second letter can be selected in 6 ways.

The third letter can be selected in 6 ways.

Hence the 3 letters can be selected in $6 \times 6 \times 6 = 216$ ways.

2. A railway signal has 3 arms, and each arm may take 4 different positions, including the position of rest. How many signals in all can be made?

Each arm can take 4 positions.

Hence $4 \times 4 \times 4 - 1 = 63$ different signals can be made.

3. In how many different orders can a row of 7 white balls, 2 red balls, and 3 black balls be arranged?

There are in all 12 balls, and these admit of $|\underline{12}$ arrangements.

But given 7 white, 2 red, and 3 black balls, arranged in any order, the 7 white balls can be interchanged among themselves in $|\underline{7}$ ways, the 2 red balls in $|\underline{2}$ ways, and the 3 black balls in $|\underline{3}$ ways, without changing the appearance of the row.

Hence every distinct order gives $|\underline{7} \times |\underline{2} \times |\underline{3}$ of the $|\underline{12}$ arrangements.

Hence there are $\dfrac{|\underline{12}}{|\underline{7} \times |\underline{2} \times |\underline{3}} = 7920$ different orders.

4. In how many ways can the letters of the word *Mathematics*, taken all together, be arranged?

The word contains 2 *a*'s, 2 *m*'s, and 2 *t*'s.

The 11 letters can be arranged in $|\underline{11}$ ways.

But given the 11 letters arranged in any order, the 2 *a*'s may be interchanged, as likewise the 2 *m*'s and the 2 *t*'s. Every distinct order of the letters gives therefore $2 \times 2 \times 2 = 8$ of the $|\underline{11}$ arrangements.

Hence there are $\dfrac{|\underline{11}}{8}$ distinct orders or words.

5. How many different signals can be made with 10 flags, of which 3 are white, 2 red, and the rest blue, always hoisted all together, and one above another?

The 10 flags can be arranged in $\underline{|10}$ ways.

But, given the flags arranged in any order, the 3 white flags can be interchanged among themselves in $\underline{|3}$ ways, the 2 red flags in $\underline{|2}$ ways, and the 5 blue flags in $\underline{|5}$ ways, without changing the signal. Every distinct signal therefore corresponds to $\underline{|3} \times \underline{|2} \times \underline{|5}$ of the $\underline{|10}$ arrangements.

Hence the number of distinct arrangements is $\dfrac{\underline{|10}}{\underline{|3}\,\underline{|2}\,\underline{|5}} = 2520.$

6. How many signals can be made with 7 flags, of which 2 are red, 1 white, 3 blue, and 1 yellow, always displayed all together and one above another?

The 7 flags can be arranged in $\underline{|7}$ ways.

But, given the flags arranged in any order, the 2 red flags can be interchanged with each other in 2 ways, and the 3 blue flags in $\underline{|3}$ ways, without changing the signal. Each distinct signal therefore corresponds to $2 \times \underline{|3} = 12$ of the $\underline{|7}$ arrangements.

Hence the number of distinct signals is $\dfrac{\underline{|7}}{12} = 420.$

7. In how many ways can 5 letters be selected from *a*, *b*, *c*, *d*, *e*, *f*, if each letter may be taken once, twice, up to five times, in making the selection?

All letters alike,	6 selections.
4 letters alike,	$6 \times 5 = 30$ selections.
3 letters alike, 2 different,	$6 \times \dfrac{5 \times 4}{2} = 60$ selections.
3 letters alike, 2 alike,	$6 \times 5 = 30$ selections.
2 letters alike, 3 different,	$6 \times \dfrac{5 \times 4 \times 3}{1 \times 2 \times 3} = 60$ selections.
2 letters alike, 2 alike,	$\dfrac{6 \times 5}{2} \times 4 = 60$ selections.
5, all different,	$\dfrac{6 \times 5 \times 4 \times 3 \times 2}{1 \times 2 \times 3 \times 4 \times 5} = 6$ selections.
Total,	252 selections.

8. In how many ways can 6 rugs be selected at a shop where 2 kinds of rugs are sold?

6 of the first kind may be selected, or 5 of the first kind and 1 of the second, or 4 of the first and 2 of the second, etc.

In all 7 selections are possible.

9. How many dominos are there in a set numbered from double blank to double ten?

As many as there are pairs of numbers from 0 to 10 inclusive.

There are 11 numbers to choose from.

The first number can be selected in 11 ways.

The second number can be selected in 11 ways.

Hence the two numbers can be selected in 121 ways.

But this mode of selection gives every domino except the doubles twice. Thus 0, 1 appears also as 1, 0.

There are 11 doubles.

Hence there are $\frac{121-11}{2}+11=66$ dominos.

10. In how many ways can 3 letters be selected from n different letters, when repetitions are allowed?

Three letters alike,	n	selections.
Two letters alike,	$n(n-1)=n^2-n$	selections.
Three letters different,	$\frac{n(n-1)(n-2)}{1\times2\times3}=\frac{n^3-3n^2+2n}{6}$	selections.
Total number,	$\frac{n^3+3n^2+2n}{6}=\frac{n(n+1)(n+2)}{6}$	selections.

11. Five flags of different colors can be hoisted either singly, or any number at a time, one above another. How many different signals can be made with them?

If all are hoisted together,	$5\times4\times3\times2\times1$ signals.
If 4 are hoisted together,	$5\times4\times3\times2$
If 3 are hoisted together,	$5\times4\times3$
If 2 are hoisted together,	5×4
If 1 is hoisted,	5
In all,	325 signals.

12. If there are m kinds of things, and 1 thing of the first kind, 2 of the second, 3 of the third, and so on, in how many ways can a selection be made?

$$(p+1)(q+1)(r+1)\cdots\cdots 1$$

Here $p=1,\ q=2,\ r=3$, etc.

Hence the number of selections is:

$$2\cdot 3\cdot 4\cdots\cdots(m+1)-1=\underline{|m+1}-1$$

13. How many selections of 6 letters each can be made from the letters in *Democracy*? How many arrangements of 6 letters each?

The word Democracy contains 9 letters, among which are 2 *c*'s.

Selections:

(*a*) A selection of 6 letters may contain both *c*'s.

Then the remaining 4 letters can be selected from the remaining 7 in $\dfrac{\underline{|7}}{\underline{|4}\,\underline{|3}}=35$ ways.

(*b*) A selection of 6 letters may contain only one *c*.

Then the remaining 5 letters can be selected from the 7 available in $\dfrac{\underline{|7}}{\underline{|5}\,\underline{|2}}=21$ ways.

(*c*) Or, a selection of 6 letters may contain no *c*'s.

Then the 6 letters can be selected from the 7 available in $\dfrac{\underline{|7}}{\underline{|6}\,\underline{|1}}=7$ ways.

Hence the number of selections is $35+21+7=63$.

Arrangements:

(*a*) Each selection can be arranged in $\dfrac{\underline{|6}}{\underline{|2}}=360$ ways.

$$35\times 360=12600 \text{ arrangements.}$$

(*b*) Each selection can be arranged in $\underline{|6}=720$ ways.

$$720\times 21=15120 \text{ arrangements.}$$

(*c*) Each selection can be arranged in $\underline{|6}=720$ ways.

$$7\times 720=5040 \text{ arrangements.}$$

Hence the number of arrangements is $12600+15120+5040=32{,}760$.

14. If of $p+q+r$ things, p are alike, and q are alike, and the rest different, show that the total number of selections is

$$(p+1)(q+1)2^r-1.$$

Of the p things none may be taken, or 1, 2, 3, ….. p may be taken.

$(p+1)$ possibilities.

Of the q things none may be taken, or 1, 2, ….. q may be taken.

$(q+1)$ possibilities.

Of the r things any one may be either taken or not taken.

2^r possibilities.

∴ There are in all $(p+1)(q+1)2^r$ possibilities. But the case when no things are taken must be excluded.

Hence then $(p+1)(q+1)2^r-1$ selections possible.

15. Show that the total number of arrangements of $2n$ letters, of which some are a's and the rest b's, is greatest when the number of a's is equal to the number of b's.

Let $x =$ number of a's.
Then $2n-x =$ number of b's.
The number of arrangements is $\dfrac{\underline{|2n}}{\underline{|2n-x}\,\underline{|x}}.$

But (§ 206) this number is greatest when $2n-x=x$, or $x=n$, $2n-x=n$.

16. If in a given number the prime factor a occurs m times, the prime factor b, n times, the prime factor c, p times, find the number of different divisors of the given number.

There are as many divisors as there are selections of $m+n+p$ things, of which m are equal to a, n to b, and p to c.
Hence there are $(m+1)(n+1)(p+1)-2$ divisors.
This does not include the number itself or 1.

Exercise 49.

1. The chance of an event happening is $\frac{4}{7}$. What are the odds in favor of the event?

The chance of its not happening is $1-\frac{4}{7}=\frac{3}{7}$.
The odds in its favor are therefore 4 to 3.

2. If the odds are 10 to 1 against an event, what is the chance of its happening?

There is 1 chance out of 11 that the event will happen. Hence the chance of its happening is $\frac{1}{11}$.

3. The odds against an event are 3 to 1. What is the chance of the event happening?

There is 1 chance in 4 that the event will happen. Hence the chance of its happening is $\frac{1}{4}$.

4. The chance of an event happening is $\frac{2}{9}$. Find the odds against the event.

There are 2 chances in 9 that the event will happen, and 7 in 9 that it will not happen.
Hence the odds against the event are 7 to 2.

5. In one throw with a pair of dice what number is most likely to be thrown? Find the odds against throwing that number.

12 can be thrown in 1 way,	6 and 6.
11 can be thrown in 2 ways,	6 and 5.
10 can be thrown in 3 ways,	6 and 4, or 5 and 5.
9 can be thrown in 4 ways,	6 and 3, or 5 and 4.
8 can be thrown in 5 ways,	6 and 2, 5 and 3, or 4 and 4.
7 can be thrown in 6 ways,	6 and 1, 5 and 2, or 4 and 3.
6 can be thrown in 5 ways,	5 and 1, 4 and 2, or 3 and 3.
5 can be thrown in 4 ways,	4 and 1, or 3 and 2.
4 can be thrown in 3 ways,	3 and 1, or 2 and 2.
3 can be thrown in 2 ways,	2 and 1.
2 can be thrown in 1 way,	1 and 1.
In all, 36 ways.	

7 can be thrown in 6 ways, and is therefore most likely to be thrown. There are 30 chances that it will not be thrown. Hence the odds against it are 30 to 6.

6. Find the chance of throwing doublets in one throw with a pair of dice.

There are 36 throws, 6 of which are doublets.

Hence the chance of throwing doublets is $\frac{6}{36} = \frac{1}{6}$.

7. If 4 cards are drawn from a pack of 52 cards, what is the chance that there will be one of each suit?

The chance that the first card is a heart is $\frac{13}{52} = \frac{1}{4}$.

The chance that the second card is a diamond is $\frac{13}{51}$.

The chance that the third card is a spade is $\frac{13}{50}$.

The chance that the fourth card is a club is $\frac{13}{49}$.

Hence the chance that 4 cards of different suits are drawn in this order is $\frac{13^4}{52 \cdot 51 \cdot 50 \cdot 49}$.

But there are 24 orders in which 4 cards of different suits may be drawn.

Hence the chance that there will be one card of each suit is

$$\frac{24 \cdot 13^4}{52 \cdot 51 \cdot 50 \cdot 49} = \frac{13^3}{17 \cdot 25 \cdot 49} = \frac{1}{10} \text{ nearly.}$$

8. If 4 cards are drawn from a pack of 52 cards, what is the chance that they will all be hearts?

The chance that the first will be a heart is $\frac{13}{52}$.

The chance that the second will be a heart is $\frac{12}{51}$.

The chance that the third will be a heart is $\frac{11}{50}$.

The chance that the fourth will be a heart is $\frac{10}{49}$.

∴ The chance that all the four cards will be hearts is

$$\frac{13 \cdot 12 \cdot 11 \cdot 10}{52 \cdot 51 \cdot 50 \cdot 49} = \frac{11}{5 \cdot 17 \cdot 49}.$$

9. If 10 persons stand in a line, what is the chance that 2 assigned persons will stand together?

If we call the two persons A and B, the chances that A will stand at the beginning of the line is $\frac{1}{10}$, and, if A stands at the beginning of the line, the chance that B will have the second place is $\frac{1}{9}$.

Hence the chance that A and B occupy these two positions together is $\frac{1}{10} \times \frac{1}{9} = \frac{1}{90}$.

Also the chance that A and B will occupy the tenth and ninth places is $\frac{1}{90}$.

If A occupies any of the 8 remaining positions, B may stand on either side of him.

The chance that A occupies one of these 8 positions is $\frac{8}{10}$.

The chance in such case that B occupies one of the 2 places beside A is $\frac{2}{9}$.

Hence the chance that both events occur together is $\frac{8}{10} \times \frac{2}{9} = \frac{16}{90}$.

Hence the chance that A and B stand together is $\frac{2}{90} + \frac{16}{90} = \frac{18}{90} = \frac{1}{5}$.

10. If 10 persons form a ring, what is the chance that 2 assigned persons will stand together?

Call the 2 assigned persons A and B.

Then if A occupies any position, B may stand on either side of him.

The chance that B does stand in one of these two positions is $\frac{2}{9}$.

Hence the chance that A and B stand together is $\frac{2}{9}$.

11. Three balls are to be drawn from an urn containing 5 black, 3 red, and 2 white balls. What is the chance of drawing 1 red and 2 black balls?

There are 10 balls in the urn.

3 balls can be selected from these in $\dfrac{\underline{|10}}{\underline{|7}\,\underline{|3}} = 120$ ways.

1 red ball can be selected in 3 ways.

2 black balls can be selected from 5 in $\dfrac{\underline{|5}}{\underline{|3}\,\underline{|2}} = 10$ ways.

Hence the required chance is $\frac{30}{120} = \frac{1}{4}$.

12. In a bag are 5 white and 4 black balls. If 4 balls are drawn out, what is the chance that they will be all of the same color?

There are 9 balls in the bag.

4 balls can be selected from these in $\dfrac{\underline{|9}}{\underline{|5}\,\underline{|4}} = 126$ ways.

4 white balls can be selected from 5 in 5 ways.

4 black balls can be selected in 1 way.

Hence 4 balls all of the same color can be selected in 6 ways.

Hence the required chance is $\frac{6}{126} = \frac{1}{21}$.

13. If 2 tickets are drawn from a package of 20 tickets marked 1, 2, 3, ·····, what is the chance that both will be marked with *odd* numbers?

There are 10 odd and 10 even numbers.

The chance that the first ticket is marked with an odd number is $\frac{1}{2}$.

If the first ticket be odd, there remain 10 even and 9 odd numbers in the package.

Hence the chance that the second ticket is odd is $\frac{9}{19}$.

And the chance that both are odd is $\frac{1}{2} \times \frac{9}{19} = \frac{9}{38}$.

14. A bag contains 3 white, 4 black, and 5 red balls; 3 balls are drawn. Find the odds against the 3 being of three different colors.

The bag contains 12 balls.

3 balls can be selected from them in $\frac{\lfloor 12}{\lfloor 9 \, \lfloor 3} = 220$ ways.

1 white ball can be selected in 3 ways.

1 black ball can be selected in 4 ways.

1 red ball can be selected in 5 ways.

Hence 3 balls of 3 different colors can be selected in $3 \times 4 \times 5 = 60$ ways.

And the odds against the 3 balls being of 3 different colors is 160 to 60, or 8 to 3.

15. Show that the odds are 35 to 1 against throwing 16 in a single throw with 3 dice.

The total number of throws with one dice is 6, with 3 dice 6^3 or 216.

Of these the following give 16.

6, 6, and 4, which may happen in 3 ways.

6, 5, and 5, which may happen in 3 ways.

In all 6 ways.

Hence the odds are 210 to 6, or 35 to 1 that 16 will not be thrown.

16. There are 10 tickets numbered 1, 2, ····· 9, 0. Three tickets are drawn at random. Find the chance of drawing a total of 22.

Three tickets can be selected from the 10 in $\frac{\lfloor 10}{\lfloor 7 \, \lfloor 3} = 120$ ways.

Of these sets of 3, only 9, 8, 5, and 9, 7, 6 will give 22.

Hence the chance of drawing a total of 22 is $\frac{2}{120} = \frac{1}{60}$.

17. Find the probability of throwing 15 in one throw with 3 dice.

The total number of throws with 3 dice is 216.

Of these the following give 15.

6, 6, and 3, which may happen in 3 ways.
6, 5, and 4, which may happen in 6 ways.
5, 5, and 5, which may happen in 1 way.

In all, 10 ways.

Hence the chance of throwing 15 is $\frac{10}{216} = \frac{5}{108}$.

18. With 3 dice, what are the relative chances of throwing a doublet and a triplet?

There are 6 triplets.

There are 6 doublets. But since the doublet includes only 2 of the 3 dice, it may be thrown in 3 ways, and moreover the third die may be thrown in any one of the remaining 5 ways.

Hence a doublet can be thrown in $6 \times 3 \times 5 = 90$ ways.

Hence there are 15 times as many chances of throwing a doublet as of throwing a triplet.

19. If 3 cards are drawn from a pack of 52 cards, what is the chance that they will be king, queen, and knave?

3 cards can be selected from 52 in $\frac{|\underline{52}}{|\underline{49}\,|\underline{3}} = 22100$ ways.

A king can be selected in 4 ways, a queen in 4 ways, and a knave in 4 ways.

Hence there are $4 \times 4 \times 4$, or 64, combinations of king, queen, and knave.

Therefore the chance that the 3 cards will be king, queen, and knave is $\frac{64}{22100}$, or $\frac{16}{5525}$.

Exercise 50.

1. One of two events must happen. If the chance of one is $\frac{2}{3}$ that of the other, find the odds on the first.

Since the sum of the 2 chances is 1, and one is $\frac{2}{3}$ of the other, it follows that the one chance is $\frac{3}{5}$, and the other $\frac{2}{5}$. The odds are 3 to 2 in favor of the first event,

2. There are three events, A, B, C, one of which must happen. The odds are 3 to 8 on A, and 2 to 5 on B. Find the odds on C.

The chances are $\frac{3}{11}$ or $\frac{21}{77}$ that A will happen.
The chances are $\frac{2}{7}$ or $\frac{22}{77}$ that B will happen.
Hence the chances that C will happen is $1 - \frac{21}{77} - \frac{22}{77} = \frac{34}{77}$.
Hence the odds are 43 to 34 against C.

3. In one bag are 9 balls and in another 6; and in each bag the balls are marked 1, 2, 3, etc. What is the chance that on drawing one ball from each bag the two balls will have the same number?

The chance that 1 will be drawn from the first bag is $\frac{1}{9}$.
The chance that 1 will be drawn from the second bag is $\frac{1}{6}$.
Therefore the chance that 2 1's will be drawn is $\frac{1}{9} \times \frac{1}{6}$, or $\frac{1}{54}$.
Similarly, the chance that 2 2's will be drawn is $\frac{1}{54}$, etc.
Hence the chance that two equal numbers will be drawn is $6 \times \frac{1}{54}$, or $\frac{1}{9}$.

4. What is the chance of throwing at least one ace in 2 throws with one die?

The chance that the first throw will not give an ace is $\frac{5}{6}$.
The chance that the second throw will not give an ace is $\frac{5}{6}$.
Therefore the chance that neither throw will give an ace is $\frac{5}{6} \times \frac{5}{6}$, or $\frac{25}{36}$.
Hence the chance that at least one ace will be thrown is $1 - \frac{25}{36}$, or $\frac{11}{36}$.

5. Find the probability of throwing a number greater than 9 in a single throw with a pair of dice.

The total number of throws with 2 dice is 36.
Of these the following give a number greater than 9:

6 and 6,	1 way.
6 and 5,	2 ways.
6 and 4,	2 ways.
5 and 5,	1 way.
In all,	6 ways.

Hence the chance of throwing a number greater than 9 is $\frac{6}{36}$, or $\frac{1}{6}$.

6. The chance that A can solve a certain problem is $\frac{1}{4}$, and the chance that B can solve it is $\frac{2}{3}$. What is the chance that the problem will be solved if both try?

The chance that A cannot solve it is $\frac{3}{4}$.
The chance that B cannot solve it is $\frac{1}{3}$.

Therefore the chance that neither of them solve it is $\frac{3}{4} \times \frac{1}{3}$, or $\frac{1}{4}$.

Hence the chance that at least one of them succeeds in solving it is $1 - \frac{1}{4}$, or $\frac{3}{4}$.

7. A, B, C have equal claims for a prize. A says to B, "You and C draw lots, and the winner shall draw lots with me for the prize." Is this fair?

No. Since A draws lots with the winner, he has 1 chance in 2 of winning the prize, whereas he is only entitled to 1 chance in 3.

8. A bag contains 5 tickets numbered 1, 2, 3, 4, 5. Three tickets are drawn at random, the tickets not being replaced after drawing. Find the chance of drawing a total of 10.

3 tickets can be selected from 5 in $\dfrac{|\underline{5}}{|\underline{3}\,|\underline{2}} = 10$ ways.

Of these 10 sets only 5, 4, and 1 and 5, 3, and 2 will give a total of 10.

Hence the chance of drawing a total of 10 is $\frac{2}{10}$, or $\frac{1}{5}$.

9. A bag contains 10 tickets, 5 marked 1, 2, 3, 4, 5, and 5 blank. Three tickets are drawn at random, each being replaced before the next is drawn. Find the probability of drawing a total of 10.

The only combination which gives 10 are 5, 5, and 0; 5, 4, and 1; 5, 3, and 2; 4, 4, and 2; and 4, 3, and 3.

5, 5, and 0 may be drawn in 3 ways; that is, the 0 may be drawn either the first, the second, or the third time.

The chance that 5 will be drawn the first time is $\frac{1}{10}$.

The chance that 5 will be drawn the second time is $\frac{1}{10}$.

The chance that 0 will be drawn the third time is $\frac{1}{2}$.

Hence the chance that 5, 5, and 0 will be drawn in this order is $\frac{1}{10} \times \frac{1}{10} \times \frac{1}{2} = \frac{1}{200}$.

And the chance that this combination will be drawn in some other order is also $\frac{1}{200}$.

Hence the chance that 5, 5, and 0 will be drawn is $\frac{3}{200}$.

Similarly, the chance that 5, 4, and 1 will be drawn is

$$6 \times \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} = \frac{6}{1000}.$$

The chance that 5, 3, and 2 will be drawn is $\frac{6}{1000}$.

The chance that 4, 4, and 2 will be drawn is $\frac{3}{1000}$.

The chance that 4, 3, and 3 will be drawn is $\frac{3}{1000}$.

Hence the chance of drawing a total of 10 is

$$\frac{3}{200} + \frac{6}{1000} + \frac{6}{1000} + \frac{3}{1000} + \frac{3}{1000} = \frac{33}{1000}.$$

10. Find the probability of drawing in the previous example a total of 10 when the tickets are not replaced.

If the tickets are not replaced, the only combinations which give 10 are 5, 4, and 1, and 5, 3, and 2.

5, 4, and 1 can be drawn in 6 ways.

The chance that 5 will be drawn the first time is $\frac{1}{10}$.

5 not being replaced, the chance of drawing 4 the second time is $\frac{1}{9}$.

And the chance of drawing 1 the third time is $\frac{1}{8}$.

Hence the chance of drawing 5, 4, and 1 in this order is

$$\frac{1}{10} \times \frac{1}{9} \times \frac{1}{8} = \frac{1}{720}.$$

Therefore the chance of drawing 5, 4, and 1 in any order is

$$6 \times \frac{1}{720} = \frac{1}{120}.$$

And similarly, the chance of drawing 5, 3, and 2 is $\frac{1}{120}$.

Hence the chance of drawing a total of 10 is $\frac{1}{60}$.

11. A bag contains four $10 gold pieces, and six silver dollars. A person is entitled to draw 2 coins at random. Find the value of his expectation.

The chance that he will draw 2 $10 gold pieces is $\frac{4}{10} \times \frac{3}{9} = \frac{2}{15}$.

His expectation from this event is therefore $\frac{2}{15}$ of $20, or $$2\frac{2}{3}$.

The chance that he will draw 1 $10 gold piece and 1 silver dollar is $\frac{4}{10} \times \frac{6}{9} + \frac{6}{10} \times \frac{4}{9} = \frac{8}{15}$.

His expectation from this event is therefore $\frac{8}{15}$ of $11, or $$5\frac{13}{15}$.

The chance that he will draw 2 silver dollars is $\frac{6}{10} \times \frac{5}{9} = \frac{1}{3}$.

His expectation from this event is therefore $\frac{1}{3}$ of $2, or $$\frac{2}{3}$.

Hence his total expectation is $$2\frac{2}{3}$ + $$5\frac{13}{15}$ + $$\frac{2}{3}$, or $9.20.

12. Six $5 pieces, four $3 pieces, and five coins which are either all gold dollars or all silver dimes are thrown together into a bag. Assuming that the unknown coins are equally likely to be dimes or dollars, what is a fair price to pay for the privilege of drawing at random a single coin?

The chance of drawing a $5 piece is $\frac{6}{15}$, or $\frac{2}{5}$.

The expectation from this event is therefore $\frac{2}{5}$ of $5, or $2.

The chance of drawing a $3 piece is $\frac{4}{15}$.

The expectation from this event is therefore $\frac{4}{15}$ of $3, or $$\frac{4}{5}$.

The chance of drawing one of the unknown coins is $\frac{5}{15}$, or $\frac{1}{3}$.

The chance that the coin is a dollar is $\frac{1}{2}$, and the expectation in this event is $\frac{1}{2} \times \frac{1}{3}$, or $\frac{1}{6}$ of 1 dollar, or $$\frac{1}{6}$.

The chance that the coin is a dime is $\frac{1}{2}$, and the expectation in this event is $\frac{1}{2} \times \frac{1}{3}$, or $\frac{1}{6}$ of 1 dime, or $\$\frac{1}{60}$.

Hence the total expectation is $\$2 + \$\frac{4}{5} + \$\frac{1}{6} + \$\frac{1}{60} = \$2\frac{59}{60} = \$2.98\frac{1}{3}$, and this is a fair price to pay.

13. A bag contains six \$5 pieces, and four other coins which have all the same value. The expectation of drawing at random 2 coins is worth \$8.40. Find the value of each of the unknown coins.

Let $\$x$ be the value of each of the unknown coins.

The chance of drawing 2 \$5 pieces is $\frac{6}{10} \times \frac{5}{9}$, or $\frac{1}{3}$.

The expectation from this event is therefore $\frac{1}{3}$ of \$10, or $\$3\frac{1}{3}$.

The chance of drawing 1 \$5 piece and 1 of the unknown coins is $\frac{6}{10} \times \frac{4}{9} + \frac{4}{10} \times \frac{6}{9} = \frac{8}{15}$.

The expectation from this event is therefore $\frac{8}{15}$ of $5 + x$ dollars, or $2\frac{2}{3} + \frac{8x}{15}$ dollars.

The chance of drawing 2 of the unknown coins is $\frac{4}{10} \times \frac{3}{9} = \frac{2}{15}$.

The expectation from this event is therefore $\frac{2}{15}$ of $2x$ dollars, or $\frac{4x}{15}$ dollars.

Hence the total expectation is $3\frac{1}{3} + 2\frac{2}{3} + \frac{8x}{15} + \frac{4x}{15}$, or $6 + \frac{4x}{5}$ dollars.

$$\therefore\ 6 + \frac{4x}{5} = 8\frac{2}{5}$$

$$x = 3$$

That is, the unknown coins are worth \$3 each.

14. Find the probability of throwing at least one ace in 4 throws with a single die.

The probability that no ace will be thrown in the 4 trials is $(\frac{5}{6})^8$, or $\frac{625}{1296}$.

Hence the probability that at least one ace will be thrown is $1 - \frac{625}{1296}$, or $\frac{671}{1296}$.

15. A copper is tossed 3 times. Find the chance that it will fall heads once and tails twice.

The chance that it will fall heads the first time and tails the second and third times is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$, or $\frac{1}{8}$.

Similarly, the chance that it falls heads the second time and tails the first and third, or that it falls heads the third time and tails the first and second is in each case $\frac{1}{8}$.

Hence the chance that it falls heads once and tails twice is $\frac{3}{8}$.

16. What is the chance of throwing double sixes at least once in 3 throws with a pair of dice?

The chance that double sixes will be thrown the first time is $\frac{1}{36}$.

The chance that they will not be thrown the first time is $\frac{35}{36}$.

The chance that they will not be thrown the second time is $\frac{35}{36}$.

The chance that they will not be thrown the third time is $\frac{35}{36}$.

Hence the chance that will not be thrown at all is $(\frac{35}{36})^3$.

And the chance that they will be thrown at least once is $1-(\frac{35}{36})^3$, or $\frac{3781}{46656}$.

17. Two bags contain each 4 black and 3 white balls. A ball is drawn at random from the first bag, and if it be white, it is put into the second bag, and a ball drawn at random from that bag. Find the odds against drawing two white balls.

The chance that the ball first drawn is white is $\frac{3}{7}$.

If a white ball be drawn and placed in the second bag, the chance of drawing a white ball from this bag will be $\frac{4}{8}$, or $\frac{1}{2}$.

Hence the chance that 2 white balls will be drawn is $\frac{1}{2}\times\frac{3}{7}$, or $\frac{3}{14}$.

And the chance that 2 white balls will not be drawn is $\frac{11}{14}$.

The odds are therefore 11 to 3 against drawing 2 white balls.

18. A and B play at chess, and A wins on an average 2 games out of 3. Find the chance of A's winning exactly 4 games out of the first 6, drawn games being disregarded.

A is to lose 2 games. The 2 games can be selected from the 6 in $\frac{\underline{|6}}{\underline{|4}\,\underline{|2}}$, or 15 ways.

The chance that A will lose any given game is $\frac{1}{3}$.

The chance that A will lose any other given game is $\frac{1}{3}$.

Hence the chance that A will lose 2 given games is $\frac{1}{9}$.

The chance that A will win a given game is $\frac{2}{3}$.

Hence the chance that A will win the 4 remaining games is $(\frac{2}{3})^4=\frac{16}{81}$.

Hence the chance that A will lose 2 given games and win the 4 others is $\frac{1}{9}\times\frac{16}{81}$, or $\frac{16}{729}$.

But the 2 games can be selected in 15 ways.

Hence the chance that A will lose some 2 games and win the other 4 is $15\times\frac{16}{729}$, or $\frac{80}{243}$.

19. At tennis A on an average beats B 2 games out of 3. If they play one set, find the chance that A will win by the score of 6 to 2.

The chance that A will win any particular game is $\frac{2}{3}$.

The chance that B will win any particular game is $\frac{1}{3}$.

A must win 5 games out of the first 7, and in addition the last game. The required chance is

$$\frac{7\times 6}{1\times 2}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)=\frac{21\times 2^5}{3^7}=\frac{7\times 2^5}{3^6}=\cancel{\frac{226}{729}}. \qquad \frac{448}{2187}$$

20. A and B, two players of equal skill, are playing tennis. A wants 2 games to complete the set, and B wants 3 games. Find the chance that A will win the set.

A may win the set as follows:

(1) He may win the next 2 games. The chance of this is $\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$.

(2) He may win the next game, lose the second, and win the third. The chance of this is $\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$.

(3) He may win the next game, lose the second and third, and win the fourth. The chance of this is $\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{16}$.

(4) He may lose the next game, and win the second and third. The chance of this is $\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$.

(5) He may lose the next game, win the second, lose the third, and win the fourth. The chance of this is $\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{16}$.

(6) He may lose the next two games and win the third and fourth. The chance of this is $\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{16}$.

Hence A's total chance of winning is $\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{8}+\frac{1}{16}+\frac{1}{16}=\frac{11}{16}$.

21. If n coins are tossed up, what is the chance that one, and only one, will turn up head?

The chance that a given one will turn up head is $\frac{1}{2}$.

The chance that all the others will turn up tail is $(\frac{1}{2})^{n-1}$.

Hence the chance that both events will occur together is $\frac{1}{2}\times(\frac{1}{2})^{n-1}$, or $(\frac{1}{2})^n$.

But a coin can be selected in n ways.

Hence the chance that some coin will turn up head while all the others turn up tail is $n(\frac{1}{2})^n$, or $\frac{n}{2^n}$.

22. A bag contains n balls. A person takes out one ball, and then replaces it. He does this n times. What is the chance that he has had in his hand every ball in the bag?

He must draw a different ball each time.

After drawing the first ball, he can only draw the second time one of the $n-1$ remaining balls.

The chance of drawing one of these is $\frac{n-1}{n}$.

After drawing the first and second balls, he can draw the third time only one of the $n-2$ remaining balls.

The chance of drawing one of these is $\frac{n-2}{n}$, etc.

Hence the chance that he will have drawn every ball in the n drawings is $\frac{(n-1)(n-2)\cdots\cdots 1}{n^{n-1}}$.

23. If on an average 9 ships out of 10 return safe to port, what is the chance that out of 5 ships expected at least 3 will safely return?

At least 3 will return if 3 return and 2 are lost, or if 4 return and 1 is lost, or if all 5 return.

The chance that a given 3 will return and other 2 be lost is

$$\left(\frac{9}{10}\right)^3 \times \left(\frac{1}{10}\right)^2, \text{ or } \frac{9^3}{10^5}.$$

But 3 can be selected from 5 in $\frac{\lfloor 5}{\lfloor 3 \lfloor 2} = 10$ ways.

Hence the chance that some 3 will return and the other 2 be lost is

$$10 \times \frac{9^3}{10^5}, \text{ or } \frac{9^3}{10^4}.$$

The chance that 4 given ships will return and the other 1 will be lost is

$$\left(\frac{9}{10}\right)^4 \times \frac{1}{10}, \text{ or } \frac{9^4}{10^5}.$$

But 4 can be selected from 5 in 5 ways.

Hence the chance that some 4 will return and the other be lost is

$$5 \times \frac{9^4}{10^5}.$$

The chance that all will return is $\left(\frac{9}{10}\right)^5$.

Hence the chance that at least 3 will return is

$$\frac{9^3}{10^4} + \frac{5 \times 9^4}{10^5} + \frac{9^5}{10^5}, \text{ or } \frac{12393}{12500}.$$

24. At tennis A beats B on an average 2 games out of 3; if the score is 4 games to 3 in B's favor, find the chance of A's winning 6 games before B does.

A may win 6 games before B in the following ways:

(1) He may win the next 3 games. The chance of this is $(\frac{2}{3})^3$, or $\frac{8}{27}$.

(2) He may win the next 2 games, lose the third, and win the fourth. The chance of this is $(\frac{2}{3})^2 \times \frac{1}{3} \times \frac{2}{3}$, or $\frac{8}{81}$.

(3) He may win the next game, lose the second, and win the third and fourth. The chance of this is $\frac{2}{3} \times \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3}$, or $\frac{8}{81}$.

(4) He may lose the next game and win the second, third, and fourth. The chance of this is $\frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3}$, or $\frac{8}{81}$.

Hence his total chance of winning 6 games before B does is

$$\tfrac{8}{27} + \tfrac{8}{81} + \tfrac{8}{81} + \tfrac{8}{81}, \text{ or } \tfrac{16}{27}.$$

25. A bets B \$10 to \$1 that he will throw heads at least once in 3 trials. What is B's expectation? What would have been a fair bet?

The chance of throwing tails in the first trial is $\frac{1}{2}$.

The chance of throwing tails every time in 3 trials is $(\frac{1}{2})^3$, or $\frac{1}{8}$.

Hence the chance that B will win is $\frac{1}{8}$, and B's expectation from this event is $\frac{1}{8}$ of \$10, or \$1.25.

But the chance of B's losing is $\frac{7}{8}$, and his negative expectation in this case is $\frac{7}{8}$ of \$1, or \$0.87½.

Hence B's expectation is \$1.25 – \$0.87½, or \$0.37½.

Since A's chance is to B's as 7 to 1, a fair bet would have been \$7 to \$1 against B. In this case B's expectation would have been $\frac{1}{8}$ of \$7 – $\frac{7}{8}$ of \$1, or 0. That is, he would have paid a fair price for his chance, and could not expect either to gain or to lose.

26. A draws 5 times (replacing) from a bag containing 3 white and 7 black balls, drawing each time one ball; every time he draws a white ball he is to receive \$1, and every time he draws a black ball he is to pay 50 cents. What is his expectation?

The chance that he will draw 5 white balls is $(\frac{3}{10})^5$; and his expectation from this event is $(\frac{3}{10})^5 \times 5$ dollars, or $\dfrac{243}{2 \times 10^4}$ dollars.

He can draw 4 white balls and 1 black one in 5 ways.

The chance that any given one of these ways will occur is $(\frac{3}{10})^4 \times \frac{7}{10}$, or $\dfrac{567}{10^5}$.

Hence the chance that some one of these ways will occur is $5 \times \frac{567}{10^5}$. If one of these ways occurs, he receives $3.50. Hence his expectation from this event is $5 \times \frac{567}{10^5} \times 3\frac{1}{2}$, or $\frac{3969}{4 \times 10^4}$ dollars.

He can draw 3 white balls and 2 black ones in 10 ways.

The chance that any given one of these ways occurs is $(\frac{3}{10})^3 \times (\frac{7}{10})^2$, or $\frac{1323}{10^5}$.

Hence the chance that some one of these ways will occur is $10 \times \frac{1323}{10^5}$, or $\frac{1323}{10^4}$, and his expectation from this event is $\frac{1323}{10^4} \times 2$ dollars, or $\frac{2646}{10^4}$ dollars.

He can draw 2 white balls and 3 black balls in 10 ways.

The chance that some one of these ways will happen is

$$10 \times \left(\frac{3}{10}\right)^2 \times \left(\frac{7}{10}\right)^3, \text{ or } \frac{3087}{10^4},$$

and the expectation from this event is

$$\frac{3087}{10^4} \times \frac{1}{2} \text{ dollars, or } \frac{3087}{2 \times 10^4} \text{ dollars.}$$

He can draw 1 white ball and 4 black balls in 5 ways.

The chance that some one of these ways will happen is

$$5 \times \frac{3}{10} \times \left(\frac{7}{10}\right)^4, \text{ or } \frac{7203}{2 \times 10^4},$$

and the expectation from this event is

$$\frac{7203}{2 \times 10^4} \times (-1) \text{ dollars, or } -\frac{7203}{2 \times 10^4}.$$

The chance of drawing 5 black balls is $\left(\frac{7}{10}\right)^5$, or $\frac{16807}{10^5}$, and the expectation from this event is $\frac{16807}{10^5} \times (-2\frac{1}{2})$ dollars, or $-\frac{16807}{4 \times 10^4}$ dollars.

Hence his total expectation is

$$\frac{243}{2 \times 10^4} + \frac{3969}{4 \times 10^4} + \frac{2646}{10^4} + \frac{3087}{2 \times 10^4} - \frac{7203}{2 \times 10^4} - \frac{16807}{4 \times 10^4},$$

or $-\frac{1}{4}$ dollar, or -25 cents. That is, the expectation is that he will lose 25 cents.

Same.

The probability that he will draw a white ball the first time is $\frac{3}{10}$, and his expectation from this event is $\frac{3}{10}$ of $1, or 30 cents.

The probability that he will draw a black ball the first time is $\frac{7}{10}$, and his expectation from this event is $\frac{7}{10} \times (-50)$ cents, or -35 cents.

Hence his total expectation for the first trial is $30 - 35$, or -5 cents. That is, he must expect to lose 5 cents.

Hence in 5 trials the expectation is that he will lose 5×5, or 25 cents.

27. From a bag containing 2 eagles, 3 dollars, and 3 quarter-dollars, A is to draw 1 coin and then B 3 coins; and A, B, and C are to divide equally the value of the remainder. What are their expectations?

In the final division A, B, and C receive equal shares.

Consider their expectations from this division.

There are $43.75 in the bag.

A and B are to draw out 4 coins at first.

4 coins can be selected from 8 in $\frac{|\underline{8}}{|\underline{4}\,|\underline{4}} = 70$ ways.

A and B may draw:	Probability.	Amount remaining.	Expectation for the division.
(1) 2 eagles, 2 dollars	$\frac{3}{70}$	$1.75	$\frac{3}{70} \times \$1.75 = \$\frac{21}{280}$.
(2) 2 eagles, 1 dollar, 1 quarter	$\frac{9}{70}$	$2.50	$\frac{9}{70} \times \$2.50 = \$\frac{9}{280}$.
(3) 2 eagles, 2 quarters	$\frac{3}{70}$	$3.25	$\frac{3}{70} \times \$3.25 = \$\frac{39}{280}$.
(4) 1 eagle, 3 dollars	$\frac{2}{70}$	$10.75	$\frac{2}{70} \times \$10.75 = \$\frac{43}{140}$.
(5) 1 eagle, 2 dollars, 1 quarter	$\frac{18}{70}$	$11.50	$\frac{18}{70} \times \$11.50 = \$\frac{207}{70}$.
(6) 1 eagle, 1 dollar, 2 quarters	$\frac{18}{70}$	$12.25	$\frac{18}{70} \times \$12.25 = \$\frac{63}{20}$.
(7) 1 eagle, 3 quarters	$\frac{2}{70}$	$13.00	$\frac{2}{70} \times \$13.00 = \$\frac{13}{35}$.
(8) 3 dollars, 1 quarter	$\frac{3}{70}$	$20.50	$\frac{3}{70} \times \$20.50 = \$\frac{123}{140}$.
(9) 2 dollars, 2 quarters	$\frac{9}{70}$	$21.25	$\frac{9}{70} \times \$21.25 = \$\frac{133}{56}$.
(10) 1 dollar, 3 quarters	$\frac{3}{70}$	$22.00	$\frac{3}{70} \times \$22.00 = \$\frac{33}{35}$.

Total expectation for the division $\$\frac{3244}{280}$, or $\$11\frac{41}{70}$.

Hence the expectation of each from the division is $\frac{1}{3}$ of $\$11\frac{41}{70}$, or $\$3.86\frac{4}{21}$.

Hence C's entire expectation is $\$3.86\frac{4}{21}$.

Again, A may draw an eagle. The probability of this is $\frac{2}{8}$, or $\frac{1}{4}$. The corresponding expectation is $2.50.

Or, he may draw a dollar. The probability of this is $\frac{3}{8}$. The corresponding expectation is $37\frac{1}{2}$ cents.

Or, he may draw a quarter. The probability of this is $\frac{3}{8}$. The corresponding expectation is $9\frac{3}{8}$ cents.

Hence A's expectation from his draw is $\$2.96\frac{7}{8}$.

And A's total expectation is \$6.83$\frac{11}{168}$.
A's and C's expectations together are \$10.69$\frac{43}{168}$.
Hence B's expectation is \$23.75 − \$17.52$\frac{9}{28}$ = \$6.22$\frac{19}{28}$.

Exercise 51.

1. An even number greater than 6 has been thrown with 2 dice. What is the chance that doublets were thrown?

Even numbers greater than 6 may be thrown as follows:

6 and 2, 2 ways.
6 and 4, 2 ways.
6 and 6, 1 way.
5 and 3, 2 ways.
5 and 5, 1 way.
4 and 4, 1 way.
In all, 9 ways, of which 3 give doublets.

Hence the chance that doublets were thrown is $\frac{3}{9}$, or $\frac{1}{3}$.

2. A number divisible by 3 has been thrown with 2 dice. What is the chance that the number was odd?

The number must have been 3, 6, 9, or 12.
3 may be thrown as 1 and 2, 2 ways.
6 may be thrown as 1 and 5, 2 and 4, or 3 and 3, 5 ways.
9 may be thrown as 3 and 6, or 4 and 5, 4 ways.
12 may be thrown as 6 and 6, 1 way.
In all, 12 ways.
Of these, 6 give 3 or 5.
Hence the chance that the number was odd is $\frac{6}{12}$, or $\frac{1}{2}$.

3. Fourteen has been thrown with 3 dice. Find the chance that one, and only one, of the dice turned up a six.

14 can be thrown only as follows:

6, 6, and 2, 3 ways.
6, 5, and 3, 6 ways.
6, 4, and 4, 3 ways.
5, 5, and 4, 3 ways.
In all, 15 ways.

6 occurs once, and only once, in 9 ways.
Hence the required chance is $\frac{9}{15}$, or $\frac{3}{5}$.

4. An even number greater than 10 has been thrown with 3 dice. Find the chance that the number was 14.

Even numbers greater than 10 can be thrown as follows:

$$12 \left\{\begin{array}{ll} 6, 5, \text{ and } 1 & 6 \text{ ways} \\ 6, 4, \text{ and } 2 & 6 \text{ ways} \\ 6, 3, \text{ and } 3 & 3 \text{ ways} \\ 5, 5, \text{ and } 2 & 3 \text{ ways} \\ 5, 4, \text{ and } 3 & 6 \text{ ways} \\ 4, 4, \text{ and } 4 & 1 \text{ way} \end{array}\right\} 25 \text{ ways.}$$

$$14 \left\{\begin{array}{ll} 6, 6, \text{ and } 2 & 3 \text{ ways} \\ 6, 5, \text{ and } 3 & 6 \text{ ways} \\ 6, 4, \text{ and } 4 & 3 \text{ ways} \\ 5, 5, \text{ and } 4 & 3 \text{ ways} \end{array}\right\} 15 \text{ ways.}$$

$$16 \left\{\begin{array}{ll} 6, 3, \text{ and } 4 & 3 \text{ ways} \\ 6, 5, \text{ and } 5 & 3 \text{ ways} \end{array}\right\} 6 \text{ ways.}$$

$$18 \left\{\begin{array}{ll} 6, 6, \text{ and } 6 & 1 \text{ way} \end{array}\right\} 1 \text{ way.}$$

Hence even numbers above 10 can be thrown in 47 ways, of which 15 ways give 14.

Hence the chance that the number was 14 is $\frac{15}{47}$.

5. From a bag containing 6 white and 2 black balls a person draws 3 balls at random and places them in a second bag. A second person then draws from the second bag 2 balls and finds them to be both white. Find the chance that the third ball in the second bag is white.

From 8 balls 3 can be drawn in $\frac{\underline{|8}}{\underline{|3}\,\underline{|5}} = 56$ ways; 3 white balls in $\frac{\underline{|6}}{\underline{|3}\,\underline{|3}} = 20$ ways.

2 white balls and 1 black ball in 30 ways.

Chance that the balls in the second bag will be all white, and that from them 2 white balls will be drawn, is $\frac{20}{56} \times 1 = \frac{5}{14}$.

Chance that there will be 2 white balls and 1 black ball in the second bag, and that from them 2 white balls will be drawn, is $\frac{30}{56} \times \frac{1}{3} = \frac{5}{28}$.

Odds in favor of 3 white balls are as $\frac{5}{14} : \frac{5}{28}$, or as $2 : 1$.

Chance of 3 white balls is $\frac{2}{3}$.

6. A bag contains 4 balls, each of which is equally likely to be white or black. A person is to receive $12 if all four are white. Find the value of his expectation.

Suppose he draws 2 balls and finds them to be both white. What is now the value of his expectation?

(1) $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times 12 = \frac{3}{4}$ dollar $= 75$ cents.

(2) There are left 2 balls in doubt. Chance that both are white is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$. Answer is $\frac{1}{4} \times 12 = \$3$.

7. A and B obtain the same answer to a certain problem. It is found that A obtains a correct answer 11 times out of 12, and B 9 times out of 10. If it is 100 to 1 against their making the same mistake, find the chance that the answer they both obtain is correct.

There are but two hypotheses; the answer is right or it is wrong. In the latter case A and B must have made the same mistake.

First hypothesis. Chance that A and B will get the same answer is $\frac{11}{12} \times \frac{9}{10}$, or $\frac{33}{40}$.

Second hypothesis. Chance that A and B will get the same answer is $\frac{1}{12} \times \frac{1}{10} \times \frac{1}{101}$, or $\frac{1}{12120}$.

The odds that the answer is correct is as $\frac{33}{40}$ to $\frac{1}{12120}$, or as 9999 to 1. The chance that the answer is correct is $\frac{9999}{10000}$.

8. From a pack of 52 cards one has been lost; from the imperfect pack 2 cards are drawn and found to be both spades. Required the chance that the missing card is a spade.

If the missing card is not a spade, the chance that 2 spades will be drawn is $\frac{13}{51} \times \frac{12}{50}$.

If the missing card is a spade, the chance that 2 spades will be drawn is $\frac{12}{51} \times \frac{11}{50}$.

Hence the odds against the missing card being a spade are as $\frac{13}{51} \times \frac{12}{50}$ to $\frac{12}{51} \times \frac{11}{50}$, or as 13 to 11. The chance that the missing card is a spade is $\frac{11}{24}$.

9. A speaks truth 9 times out of ten, and B 11 times out of 12. There is a certain event which must either happen or fail, and is of itself twice as likely to happen as to fail. A says that the event happened, and B that it failed. Find the odds for the event happening.

If the event happened, A tells the truth, and B lies; if it did not happen, A lies, and B tells the truth. The chance of the first is $\frac{2}{3} \times \frac{9}{10} \times \frac{1}{12}$. The chance of the second is $\frac{1}{3} \times \frac{1}{10} \times \frac{11}{12}$. The odds in favor of happening are as $2 \times 9 \times 1$ to $1 \times 1 \times 11$, or 18 to 11. The chance that the event did happen is $\frac{18}{29}$.

Exercise 52.

1. Find the values of:

$$\frac{1}{4+}\frac{1}{3+}\frac{1}{2};\quad \frac{1}{2+}\frac{1}{3+}\frac{1}{7};\quad \frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{4+}\frac{1}{5}.$$

$$\frac{1}{4+}\frac{1}{3+}\frac{1}{2}.$$

Quotients $= 4, 3, 2.$

Convergents $= \frac{0}{1}, \frac{1}{4}, \frac{3}{13}, \frac{7}{30}.$

$$\frac{1}{2+}\frac{1}{3+}\frac{1}{7}.$$

Quotients $= 2, 3, 7.$

Convergents $= \frac{0}{1}, \frac{1}{2}, \frac{3}{7}, \frac{22}{51}.$

$$\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{4+}\frac{1}{5}.$$

Quotients $= 1, 2, 1, 4, 5.$

Convergents $= \frac{0}{1}, \frac{1}{1}, \frac{2}{3}, \frac{3}{4}, \frac{14}{19}, \frac{73}{99}.$

2. Find continued fractions for $\frac{123}{157}$; $\frac{159}{47}$; $\frac{103}{71}$; $\frac{67}{177}$; $\sqrt{5}$; $\sqrt{11}$; $4\sqrt{6}$; and find the fourth convergent to each.

$$\frac{123}{157}=\frac{1}{1+}\frac{1}{3+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{2}.$$

Quotients $= 1, 3, 1, 1, 1.$

Convergents $= \frac{0}{1}, \frac{1}{1}, \frac{3}{4}, \frac{4}{5}, \frac{7}{9}.$

$$\frac{159}{47}=3+\frac{1}{2+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{3}.$$

Quotients $= 2, 1, 1, 1.$

Convergents $= \frac{0}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{5}, \frac{3}{8}.$

$\therefore$ Fourth convergent $= 3\frac{3}{8} = \frac{27}{8}.$

$$\frac{103}{71}=1+\frac{1}{2+}\frac{1}{4+}\frac{1}{1+}\frac{1}{1+}\frac{1}{3}.$$

Quotients $= 2, 4, 1, 1.$

Convergents $= \frac{0}{1}, \frac{1}{2}, \frac{4}{9}, \frac{5}{11}, \frac{9}{20}.$

$\therefore$ Fourth convergent $= 1\frac{9}{20} = \frac{29}{20}.$

$$\frac{67}{177}=\frac{1}{2+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{3+}\frac{1}{1+}\frac{1}{4}.$$

Quotients $= 2, 1, 1, 1.$

Convergents $= \frac{0}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{5}, \frac{3}{8}.$

Let $\sqrt{5}=2+\frac{1}{x}$

$\therefore\ x=\frac{1}{\sqrt{5}-2}$

$=\sqrt{5}+2$

$=4+\frac{1}{x}$

$\therefore\ \sqrt{5}=2+\frac{\dot{1}}{4}$

Quotients $=4, 4, 4, 4.$

Convergents $=\frac{0}{1}, \frac{1}{4}, \frac{4}{17}, \frac{17}{72}, \frac{72}{305}.$

$\therefore$ Fourth convergent $=2\frac{72}{305}=\frac{682}{305}.$

Let $\sqrt{11}=3+\frac{1}{x}$

$\therefore\ x=\frac{1}{\sqrt{11}-3}$

$=\frac{\sqrt{11}+3}{2}$

Let $\frac{\sqrt{11}+3}{2}=3+\frac{1}{y}$

$\therefore\ y=\frac{2}{\sqrt{11}-3}$

$=\sqrt{11}+3$

$=6+\frac{1}{x}$

$\therefore\ \sqrt{11}=3+\frac{\dot{1}}{3+}\frac{\dot{1}}{6}$

Quotients $=3, 6, 3, 6.$

Convergents $=\frac{0}{1}, \frac{1}{3}, \frac{6}{19}, \frac{19}{60}, \frac{120}{379}.$

$\therefore$ Fourth convergent $=3\frac{120}{379}=\frac{1257}{379}.$

Let $4\sqrt{6}=9+\frac{1}{x}$

$\therefore\ x=\frac{1}{4\sqrt{6}-9}$

$=\frac{4\sqrt{6}+9}{15}$

Let $\frac{4\sqrt{6}+9}{15}=1+\frac{1}{y}$

$\therefore\ y=\frac{15}{4\sqrt{6}-6}$

$=\frac{2\sqrt{6}+3}{2}$

Let $\frac{2\sqrt{6}+3}{2}=3+\frac{1}{z}$

$\therefore\ z=\frac{2}{2\sqrt{6}-3}$

$=\frac{4\sqrt{6}+6}{15}$

Let $\frac{4\sqrt{6}+6}{15}=1+\frac{1}{u}$

$\therefore\ u=\frac{15}{4\sqrt{6}-9}$

$=4\sqrt{6}+9$

$=18+\frac{1}{x}$

$\therefore\ 4\sqrt{6}=9+\frac{\dot{1}}{1+}\frac{1}{3+}\frac{1}{1+}\frac{\dot{1}}{18}$

Quotients $=1, 3, 1, 18.$

Convergents $=\frac{0}{1}, \frac{1}{1}, \frac{3}{4}, \frac{4}{5}, \frac{75}{94}.$

$\therefore$ Fourth convergent $=9\frac{75}{94}=\frac{921}{94}.$

3. Find continued fractions for $\frac{47}{257}$; $\frac{457}{204}$; $\frac{2065}{4626}$; $\frac{2991}{568}$; and find the third convergent to each.

$\frac{47}{257}=\frac{1}{5+}\frac{1}{2+}\frac{1}{7+}\frac{1}{3}.$

Quotients $=5, 2, 7.$

Convergents $=\frac{0}{1}, \frac{1}{5}, \frac{2}{11}, \frac{15}{82}.$

$\frac{2065}{4626}=\frac{1}{2+}\frac{1}{4+}\frac{1}{6+}\frac{1}{8+}\frac{1}{10}.$

Quotients $=2, 4, 6, 8.$

Convergents $=\frac{0}{1}, \frac{1}{2}, \frac{4}{9}, \frac{25}{56}.$

$\frac{457}{204}=2+\frac{1}{4+}\frac{1}{6+}\frac{1}{8}.$

Quotients $=4, 6, 8.$

Convergents $=\frac{0}{1}, \frac{1}{4}, \frac{6}{25}, \frac{49}{204}.$

$\therefore$ Third convergent $=2\frac{49}{204}=\frac{457}{204}.$

$\frac{2991}{568}=5+\frac{1}{3+}\frac{1}{1+}\frac{1}{3+}\frac{1}{5+}\frac{1}{7}.$

Quotients $=3, 1, 3, 5.$

Convergents $=\frac{0}{1}, \frac{1}{3}, \frac{1}{4}, \frac{4}{15}.$

$\therefore$ Third convergent $=5\frac{4}{15}=\frac{79}{15}.$

4. Find continued fractions for $\sqrt{21}$; $\sqrt{22}$; $\sqrt{33}$; $\sqrt{55}$.

Let $\sqrt{21} = 4 + \frac{1}{x}$

$$\begin{aligned}\therefore x &= \frac{1}{\sqrt{21}-4}\\ &= \frac{\sqrt{21}+4}{5}\end{aligned}$$

Let $\frac{\sqrt{21}+4}{5} = 1 + \frac{1}{y}$

$$\begin{aligned}\therefore y &= \frac{5}{\sqrt{21}-1}\\ &= \frac{\sqrt{21}+1}{4}\end{aligned}$$

Let $\frac{\sqrt{21}+1}{4} = 1 + \frac{1}{z}$

$$\begin{aligned}\therefore z &= \frac{4}{\sqrt{21}-3}\\ &= \frac{\sqrt{21}+3}{3}\end{aligned}$$

Let $\frac{\sqrt{21}+3}{3} = 2 + \frac{1}{u}$

$$\begin{aligned}\therefore u &= \frac{3}{\sqrt{21}-3}\\ &= \frac{\sqrt{21}+3}{4}\end{aligned}$$

Let $\frac{\sqrt{21}+3}{4} = 1 + \frac{1}{v}$

$$\begin{aligned}\therefore v &= \frac{4}{\sqrt{21}-1}\\ &= \frac{\sqrt{21}+1}{5}\end{aligned}$$

Let $\frac{\sqrt{21}+1}{5} = 1 + \frac{1}{w}$

$$\begin{aligned}\therefore w &= \frac{5}{\sqrt{21}-4}\\ &= \sqrt{21}+4\\ &= 8 + \frac{1}{x}\end{aligned}$$

$$\therefore \sqrt{21} = 4 + \frac{\dot{1}}{1+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{\dot{1}}{8}$$

Let $\sqrt{22} = 4 + \frac{1}{x}$

$$\begin{aligned}\therefore x &= \frac{1}{\sqrt{22}-4}\\ &= \frac{\sqrt{22}+4}{6}\end{aligned}$$

Let $\frac{\sqrt{22}+4}{6} = 1 + \frac{1}{y}$

$$\begin{aligned}\therefore y &= \frac{6}{\sqrt{22}-2}\\ &= \frac{\sqrt{22}+2}{3}\end{aligned}$$

Let $\frac{\sqrt{22}+2}{3} = 2 + \frac{1}{z}$

$$\begin{aligned}\therefore z &= \frac{3}{\sqrt{22}-4}\\ &= \frac{\sqrt{22}+4}{2}\end{aligned}$$

Let $\frac{\sqrt{22}+4}{2} = 4 + \frac{1}{u}$

$$\begin{aligned}\therefore u &= \frac{2}{\sqrt{22}-4}\\ &= \frac{\sqrt{22}+4}{3}\end{aligned}$$

Let $\frac{\sqrt{22}+4}{3} = 2 + \frac{1}{v}$

$$\begin{aligned}\therefore v &= \frac{3}{\sqrt{22}-2}\\ &= \frac{\sqrt{22}+2}{6}\end{aligned}$$

Let $\frac{\sqrt{22}+2}{6} = 1 + \frac{1}{w}$

$$\begin{aligned}\therefore w &= \frac{6}{\sqrt{22}-4}\\ &= \sqrt{22}+4\\ &= 8 + \frac{1}{x}\end{aligned}$$

$$\therefore \sqrt{22} = 4 + \frac{\dot{1}}{1+}\,\frac{1}{2+}\,\frac{1}{4+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{\dot{1}}{8}$$

Let $\sqrt{33} = 5 + \frac{1}{x}$

$$\therefore x = \frac{1}{\sqrt{33} - 5} = \frac{\sqrt{33} + 5}{8}$$

Let $\frac{\sqrt{33} + 5}{8} = 1 + \frac{1}{y}$

$$\therefore y = \frac{8}{\sqrt{33} - 3} = \frac{\sqrt{33} + 3}{3}$$

Let $\frac{\sqrt{33} + 3}{3} = 2 + \frac{1}{z}$

$$\therefore z = \frac{3}{\sqrt{33} - 3} = \frac{\sqrt{33} + 3}{8}$$

Let $\frac{\sqrt{33} + 3}{8} = 1 + \frac{1}{u}$

$$\therefore u = \frac{8}{\sqrt{33} - 5} = \sqrt{33} + 5 = 10 + \frac{1}{x}$$

$$\therefore \sqrt{33} = 5 + \frac{\dot{1}}{1} \, \frac{1}{+2} \, \frac{1}{+1} \, \frac{\dot{1}}{+10}$$

Let $\sqrt{55} = 7 + \frac{1}{x}$

$$\therefore x = \frac{1}{\sqrt{55} - 7} = \frac{\sqrt{55} + 7}{6}$$

Let $\frac{\sqrt{55} + 7}{6} = 2 + \frac{1}{y}$

$$\therefore y = \frac{6}{\sqrt{55} - 5} = \frac{\sqrt{55} + 5}{5}$$

Let $\frac{\sqrt{55} + 5}{5} = 2 + \frac{1}{z}$

$$\therefore z = \frac{5}{\sqrt{55} - 5} = \frac{\sqrt{55} + 5}{6}$$

Let $\frac{\sqrt{55} + 5}{6} = 2 + \frac{1}{u}$

$$\therefore u = \frac{6}{\sqrt{55} - 7} = \sqrt{55} + 7 = 14 + \frac{1}{x}$$

$$\therefore \sqrt{55} = 7 + \frac{\dot{1}}{2} \, \frac{1}{+2} \, \frac{1}{+2} \, \frac{\dot{1}}{+14}$$

5. Obtain convergents, with only two figures in the denominator, that approach nearest to the values of: $\sqrt{7}$; $\sqrt{10}$; $\sqrt{15}$; $\sqrt{17}$; $\sqrt{18}$; $\sqrt{20}$; $3 - \sqrt{5}$; $2 + \sqrt{11}$.

$$\sqrt{7} = 2 + \frac{\dot{1}}{1} \, \frac{\dot{1}}{+1} \, \frac{\dot{1}}{+1} \, \frac{\dot{1}}{+4}.$$

Quotients $= 1, 1, 1, 4, 1, 1, 1, 4.$

Convergents $= \frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{9}{14}, \frac{11}{17}, \frac{20}{31}, \frac{31}{48}.$

$\therefore 2\frac{31}{48}$ is the required convergent.

$$\sqrt{10} = 3 + \frac{\dot{1}}{6}.$$

Quotients $= 6, 6, 6.$

Convergents $= \frac{0}{1}, \frac{1}{6}, \frac{6}{37}.$

$\therefore 3\frac{6}{37}$ is the required convergent.

$$\sqrt{15}=3+\frac{\dot{1}}{1+}\frac{\dot{1}}{6}.$$

Quotients $= 1, 6, 1, 6, 1, 6.$

Convergents $= \frac{0}{1}, \frac{1}{1}, \frac{6}{7}, \frac{7}{8}, \frac{48}{55}, \frac{55}{63}.$

$\therefore 3\frac{55}{63}$ is the required convergent.

$$\sqrt{18}=4+\frac{\dot{1}}{4+}\frac{\dot{1}}{8}.$$

Quotients $= 4, 8, 4.$

Convergents $= \frac{0}{1}, \frac{1}{4}, \frac{8}{33}.$

$\therefore 4\frac{8}{33}$ is the required convergent.

$$\sqrt{17}=4+\frac{\dot{1}}{8}.$$

Quotients $= 8, 8, 8.$

Convergents $= \frac{0}{1}, \frac{1}{8}, \frac{8}{65}.$

$\therefore 4\frac{8}{65}$ is the required convergent.

$$\sqrt{20}=4+\frac{\dot{1}}{2+}\frac{\dot{1}}{8}.$$

Quotients $= 2, 8, 2, 8.$

Convergents $= \frac{0}{1}, \frac{1}{2}, \frac{8}{17}, \frac{17}{36}.$

$\therefore 4\frac{17}{36}$ is the required convergent.

$$3-\sqrt{5}=3-\left(2+\frac{\dot{1}}{4}\right)$$
$$=1-\frac{\dot{1}}{4}.$$

Quotients $= 4, 4, 4, 4.$

Convergents $= \frac{0}{1}, \frac{1}{4}, \frac{4}{17}, \frac{17}{72}.$

$\therefore \frac{17}{72}$ is the required convergent.

$$2+\sqrt{11}=2+\left(3+\frac{\dot{1}}{3+}\frac{\dot{1}}{6}\right)$$
$$=5+\frac{\dot{1}}{3+}\frac{\dot{1}}{6}.$$

Quotients $= 3, 6, 3, 6.$

Convergents $= \frac{0}{1}, \frac{1}{3}, \frac{6}{19}, \frac{19}{60}.$

$\therefore 5\frac{19}{60}$ is the required convergent.

6. Find the proper fraction which, if converted into a continued fraction, will have quotients 1, 7, 5, 2.

Quotients $= 1, 7, 5, 2.$

Convergents $= \frac{0}{1}, \frac{1}{1}, \frac{7}{8}, \frac{36}{41}, \frac{79}{90}.$

7. Find the next convergent when the two preceding convergents are $\frac{3}{17}$ and $\frac{19}{89}$, and the next quotient is 5.

$$\frac{u_3}{v_3}=\frac{m_3u_2+u_1}{m_3v_2+v_1}$$

Put $\quad m_3=5,\ u_1=3,\ v_1=17,\ u_2=19,\ v_2=89$

Then $\quad \frac{u_3}{v_3}=\frac{5\times 19+3}{5\times 89+17}=\frac{98}{462}=\frac{7}{33}$

8. If the pound troy is the weight of 22.8157 cubic inches of water, and the pound avoirdupois of 27.7274 cubic inches of water, find a fraction with denominator less than 100 which shall differ from their ratio by less than 0.0001.

$$\frac{228157}{277274}=\frac{1}{1+}\frac{1}{4+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{4+}\frac{1}{1+}\frac{1}{1+}\frac{1}{82+}\frac{1}{1+}\frac{1}{5+}\frac{1}{3}.$$

Quotients $= 1, 4, 1, 1, 1, 4, 1, 1, 82.$
Convergents $= \frac{0}{1}, \frac{1}{1}, \frac{4}{5}, \frac{5}{6}, \frac{9}{11}, \frac{14}{17}, \frac{65}{79}, \frac{79}{96}, \frac{144}{175}.$
If we stop with $\frac{79}{96}$, the error is $< \frac{79}{96} - \frac{144}{175}$
or $< \frac{1}{16800}$
$\therefore$ $\frac{79}{96}$ is the required fraction.

9. The ratio of the diagonal to a side of a square being $\sqrt{2}$, find a fraction with denominator less than 100 which shall differ from their ratio by less than 0.0001.

$$\sqrt{2} = 1 + \frac{\dot{1}}{2}.$$

Quotients $= 2, 2, 2, 2, 2, 2, 2.$
Convergents $= \frac{0}{1}, \frac{1}{2}, \frac{2}{5}, \frac{5}{12}, \frac{12}{29}, \frac{29}{70}, \frac{70}{169}.$
If we stop with $\frac{29}{70}$, the error is $< \frac{29}{70} - \frac{70}{169}$
or $< \frac{1}{11830}.$
$\therefore$ $1\frac{29}{70}$ is the fraction required.

10. The ratio of the circumference of a circle to its diameter being approximately the ratio of 3.14159265 : 1, find the first three convergents to this ratio, and determine to how many decimal places each may be depended upon as agreeing with the true value.

$$\frac{314159265}{100000000} = 3 + \frac{1}{7+} \frac{1}{15+} \frac{1}{1+}$$

First convergent $= \frac{22}{7} = 3.142+$
Second convergent $= \frac{333}{106} = 3.14150+$
Third convergent $= \frac{355}{113} = 3.1415920+$

11. In two scales of which the zero-points coincide the distances between consecutive divisions of the one are to the corresponding distances of the other as 1 : 1.06577. Find what division points most nearly coincide.

Let n divisions of the first scale equal as nearly as possible m divisions of the other. Then $\frac{m}{n}$ is a convergent of $\frac{1}{1.06577}$.

$$\frac{100000}{106577} = \frac{1}{1+} \frac{1}{15+} \frac{1}{4+} \frac{1}{1+} \frac{1}{8+} \frac{1}{11+} \frac{1}{2+} \frac{1}{1+} \frac{1}{1+} \frac{1}{2}.$$

Quotients $= 1, 15, 4, 1, 8, 11.$
Convergents $= \frac{0}{1}, \frac{1}{1}, \frac{15}{16}, \frac{61}{65}, \frac{76}{81}, \frac{669}{713}.$
If we stop with $\frac{76}{81}$, the error is $< \frac{669}{713} - \frac{76}{81}$
or $< \frac{1}{57753}.$

That is, division 76 of the second scale nearly coincides in the division 81 of the first scale, the distance between them being less than $\frac{1}{5753}$ of one division of the first scale.

12. Find the surd values of:

$$1+\frac{\dot{1}}{4+}\frac{\dot{1}}{2};\quad 3+\frac{\dot{1}}{1+}\frac{\dot{1}}{6};\quad \frac{\dot{1}}{3+}\frac{1}{1+}\frac{\dot{1}}{6};\quad 1+\frac{\dot{1}}{2+}\frac{1}{3+}\frac{\dot{1}}{4}.$$

Let $x=\frac{\dot{1}}{4+}\frac{\dot{1}}{2}$

$$\therefore\ x=\cfrac{1}{4+\cfrac{1}{2+x}}$$

$$=\frac{2+x}{9+x}$$

$\therefore\ 4x^2+8x-2=0$

$2x^2+4x-1=0$

$$x=\frac{-4\pm\sqrt{24}}{4}$$

$$\therefore\ x=-1\pm\sqrt{\tfrac{3}{2}}$$

$\therefore$ The required value

$$=1-1+\sqrt{\tfrac{3}{2}}$$

$$=\sqrt{\tfrac{3}{2}}$$

Let $x=\frac{\dot{1}}{3+}\frac{\dot{1}}{1+}\frac{\dot{1}}{6}$

$$=\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{6+x}}}$$

$$=\frac{7+x}{27+4x}$$

$\therefore 4x^2+26x-7=0$

$$x=\frac{-26\pm2\sqrt{197}}{8}$$

$$x=\frac{-13+\sqrt{197}}{4}$$

$\therefore$ The required value

$$=\frac{-13+\sqrt{197}}{4}$$

Let $x=\frac{\dot{1}}{1+}\frac{\dot{1}}{6}$

$$\therefore\ x=\cfrac{1}{1+\cfrac{1}{6+x}}$$

$$=\frac{6+x}{7+x}$$

$\therefore\ x^2+6x-6=0$

$$x=\frac{-6\pm\sqrt{60}}{2}$$

$$\therefore\ x=-3+\sqrt{15}$$

$\therefore$ The required value

$$=3-3+\sqrt{15}$$

$$=\sqrt{15}$$

Let $x=\frac{\dot{1}}{2+}\frac{1}{3+}\frac{\dot{1}}{4}$

$$=\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+x}}}$$

$$=\frac{13+3x}{30+7x}$$

$\therefore 7x^2+27x-13=0$

$$x=\frac{\sqrt{1093}-27}{14}$$

$\therefore$ The required value is

$$\frac{\sqrt{1093}-27}{14}$$

13. Prove that

$$\left(a+\frac{\dot{1}}{b+}\,\frac{\dot{1}}{a}\right)\left(\frac{\dot{1}}{b+}\,\frac{\dot{1}}{a}\right)=\frac{a}{b}.$$

Let

$$x=\frac{\dot{1}}{b+}\,\frac{\dot{1}}{a}$$

$$\therefore\ x=\cfrac{1}{b+\cfrac{1}{a+x}}$$

$$x=\frac{a+x}{bx+ab+1}$$

$$bx^2+abx-a=0$$

$$x=\frac{-ab\pm\sqrt{a^2b^2+4ab}}{2b}$$

$$\therefore\ x=\frac{-ab+\sqrt{a^2b^2+4ab}}{2b}$$

$$\left(a+\frac{\dot{1}}{b+}\,\frac{\dot{1}}{a}\right)\left(\frac{\dot{1}}{b+}\,\frac{\dot{1}}{a}\right)=\frac{ab+\sqrt{a^2b^2+4ab}}{2b}\times\frac{-ab+\sqrt{a^2b^2+4ab}}{2b}$$

$$=\frac{4ab}{4b^2}=\frac{a}{b}$$

14. Show that the ratio of the diagonal of a cube to its edge may be nearly expressed by 97 : 56. Find the greatest possible value of the error made in taking this ratio for the true ratio.

The ratio is $\sqrt{3}:1$.

$$\sqrt{3}=1+\frac{\dot{1}}{1+}\,\frac{\dot{1}}{2}.$$

Quotients $=1, 2, 1, 2, 1, 2, 1, 2, 1.$

Convergents $=\frac{0}{1}, \frac{1}{1}, \frac{2}{3}, \frac{3}{4}, \frac{8}{11}, \frac{11}{15}, \frac{30}{41}, \frac{41}{56}, \frac{112}{153}.$

If we stop with $\frac{41}{56}$, the error is $<\frac{1}{8568}$.

Hence $\frac{97}{56}$ differs from $\sqrt{3}$ by an amount $<\frac{1}{8568}$.

15. Find a series of fractions converging to the ratio of 5 hours 48 minutes 51 seconds to 24 hours.

5 hours 48 minutes 51 seconds = 20931 seconds.

24 hours = 86400 seconds.

$$\frac{20931}{86400}=\frac{1}{4+}\,\frac{1}{7+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2}.$$

Quotients $=4, 7, 1, 4, 1, 1, 1, 1, 3, 2, 1, 2.$

Convergents $=\frac{0}{1}, \frac{1}{4}, \frac{7}{29}, \frac{8}{33}, \frac{39}{161}, \frac{47}{194}, \frac{86}{355}, \frac{133}{549}, \frac{219}{904}, \frac{790}{3261}, \frac{1799}{7426}, \frac{2589}{10687}, \frac{6977}{28800}.$

16. Find a series of fractions converging to the ratio of a cubic yard to a cubic meter, if a cubic yard is $\frac{76453}{100000}$ of a cubic meter.

$$\frac{76453}{100000}=\frac{1}{1}+\frac{1}{3}+\frac{1}{4}+\frac{1}{19}+\frac{1}{2}+\frac{1}{3}+\frac{1}{1}+\frac{1}{1}+\frac{1}{5}+\frac{1}{1}+\frac{1}{2}.$$

Quotients $= 1, 3, 4, 19, 2, 3, 1, 1, 5, 1, 2$.

Convergents $= \frac{0}{1}, \frac{1}{1}, \frac{3}{4}, \frac{13}{17}, \frac{250}{327}, \frac{513}{671}, \frac{1789}{2340}, \frac{2302}{3011}, \frac{4091}{5351}, \frac{22757}{29766}, \frac{26848}{35117}, \frac{76453}{100000}$.

Exercise 53.

1. Add together 435, 624, 737 (scale of eight).

$$\begin{array}{r} 435 \\ 624 \\ 737 \\ \hline 2220 \end{array}$$

2. From 32,413 subtract 15,542 (scale of six).

$$\begin{array}{r} 32413 \\ 15542 \\ \hline 12431 \end{array}$$

3. Multiply 6431 by 35 (scale of seven).

$$\begin{array}{r} 6431 \\ 35 \\ \hline 45115 \\ 25623 \\ \hline 334345 \end{array}$$

4. Multiply 4685 by 3843 (scale of nine).

$$\begin{array}{r} 4685 \\ 3483 \\ \hline 15276 \\ 42154 \\ 21072 \\ 15276 \\ \hline 17832126 \end{array}$$

5. Divide 102,432 by 36 (scale of seven).

$$\begin{array}{l} 36)102432(1625 \\ 36 \\ \overline{334} \\ 321 \\ \overline{133} \\ 105 \\ \overline{252} \\ 252 \\ \overline{} \end{array}$$

6. Find H.C.F. of 2541 and 3102 (scale of seven).

$$\begin{array}{l} 2541)3102(1 \\ 2541 \\ \overline{231)2541(11} \\ 231 \\ \overline{231} \\ 231 \\ \overline{} \end{array}$$

H.C.F. = 231.

7. Extract the square root of 33,224 (scale of six).

$$\begin{array}{r|r} \multicolumn{2}{r}{\dot{3}3\dot{2}2\dot{4}\ \underline{|152}} \\ \multicolumn{2}{r}{1} \\ \cline{2-2} 25 & 232 \\ & 221 \\ \cline{2-2} 542 & 1124 \\ & 1124 \\ \hline \end{array}$$

8. Extract the square root of 300,114 (scale of five).

$$\begin{array}{r|l}
 & 3\dot{0}0\dot{1}1\dot{4}\;\lfloor\underline{342} \\
 & 14 \\ \hline
114 & 1101 \\
 & 1021 \\ \hline
1232 & 3014 \\
 & 3014 \\ \hline
\end{array}$$

9. Change 624 from the scale of ten to the scale of five.

$$\begin{array}{r|l}
5 & 624 \\ \hline
5 & 124 \ldots 4 \\ \hline
5 & 24 \ldots 4 \\ \hline
 & 4 \ldots 4
\end{array}$$

4444.

10. Change 3516 from the scale of seven to the scale of ten.

$$\begin{array}{r|l}
10 & 3516 \\ \hline
10 & 242 \ldots 7 \\ \hline
10 & 15 \ldots 8 \\ \hline
 & 1 \ldots 2
\end{array}$$

1287.

11. Change 3721 from the scale of eight to the scale of six.

$$\begin{array}{r|l}
6 & 3721 \\ \hline
6 & 515 \ldots 3 \\ \hline
6 & 67 \ldots 3 \\ \hline
6 & 11 \ldots 1 \\ \hline
 & 1 \ldots 3
\end{array}$$

13133.

12. Change 4535 from the scale of seven to the scale of nine.

$$\begin{array}{r|l}
9 & 4535 \\ \hline
9 & 350 \ldots 5 \\ \hline
9 & 26 \ldots 2 \\ \hline
 & 2 \ldots 2
\end{array}$$

2225.

13. Change 32.15 from the scale of six to the scale of nine.

$$\begin{array}{l}
9\,\underline{)\,32} \\
\;2 \ldots 2 \\
\,22.
\end{array}$$

$$.15 = \tfrac{1}{6} + \tfrac{5}{36} = \tfrac{11}{36}$$

$$\begin{array}{r}
11 \\
\underline{9} \\
36)99(2 \\
\underline{72} \\
27 \\
\underline{9} \\
36)243(6 \\
\underline{216} \\
27
\end{array}$$

$\therefore\ \tfrac{11}{36} = .266 +$ in the scale of 9.

$32.15 = 22.266 +$

14. Express $\tfrac{25}{128}$ (scale of ten) by a radix-fraction in the scale of four.

$$\begin{aligned}
\tfrac{25}{128} \times 4 &= \tfrac{100}{128} = \tfrac{25}{32}. \\
\tfrac{25}{32} \times 4 &= \tfrac{100}{32} = 3 + \tfrac{1}{8} \\
\tfrac{1}{8} \times 4 &= \tfrac{1}{2} \\
\tfrac{1}{2} \times 4 &= 2
\end{aligned}$$

.0302.

15. Express $\frac{43}{108}$ (scale of ten) by a radix-fraction in the scale of six.

$$\begin{aligned}\tfrac{43}{108} &= \tfrac{36}{108} + \tfrac{6}{108} + \tfrac{2}{216}\\ &= \frac{2}{6} + \frac{2}{6^2} + \frac{2}{6^3} = .222\end{aligned}$$

16. Multiply 31.24 by 0.31 (scale of five).

$$\begin{array}{r} 31.24\\ 0.31\\ \hline 3124\\ 14432\\ \hline 20.7444\end{array}$$

17. In what scale is this true? $21 \times 36 = 746$.

Let $x =$ the radix.

Then

$$\begin{aligned}(2x+1)(3x+6) &= 7x^2 + 4x + 6\\ x^2 - 11x &= 0\\ \therefore\ x &= 11\end{aligned}$$

18. In what scale is the square of 23 expressed by 540?

Let $x =$ the radix.

Then

$$\begin{aligned}(2x+3)^2 &= 5x^2 + 4x\\ x^2 - 8x - 9 &= 0\\ (x-9)(x+1) &= 0\\ \therefore\ x &= 9\end{aligned}$$

19. In what scale are 212, 1101, 1220 in arithmetical progression?

Let $x =$ the radix.

Tnen

$$\begin{aligned}2x^2 + x + 2 + x^3 + 2x^2 + 2x &= 2(x^3 + x^2 + 1)\\ x^3 - 2x^2 - 3x &= 0\\ x^2 - 2x - 3 &= 0\\ (x-3)(x+1) &= 0\\ \therefore\ x &= 3\end{aligned}$$

20. Show that 1,234,321 is a perfect square in any scale (radix greater than four).

Let x be any radix > 4.

Then in that scale

$$\begin{aligned}1234321 &= x^6 + 2x^5 + 3x^4 + 4x^3 + 3x^2 + 2x + 1\\ &= (x^3 + x^2 + x + 1)^2 = 1111^2.\end{aligned}$$

21. Which of the weights 1, 2, 4, 8, ····· pounds must be selected to weigh 345 pounds, only one weight of each kind being used?

Reduce 345 to the scale of 2.

$$\begin{aligned}345 &= 256 + 64 + 16 + 8 + 1\\ &= 2^8 + 2^6 + 2^4 + 2^3 + 1\\ &= 101011001 \text{ in the scale of 2.}\end{aligned}$$

$\therefore$ 256 lbs., 64 lbs., 16 lbs., 8 lbs., 1 lb. must be selected.

22. If two numbers are formed by the same digits in different orders, prove that the difference of the numbers is divisible by $r-1$.

Let the digits be $a, b, c, d, \cdots\cdots$, and let r be the radix.

Then let the one number be $ar^n + br^{n-1} + cr^{n-2} + \cdots\cdots$
and let the other be $lr^n + \cdots\cdots + ar^k \cdots\cdots + br^i + \cdots\cdots$

The difference is $ar^n - ar^k + br^{n-1} - br^i + \cdots\cdots$

But $r^n - r^k$, $r^{n-1} - r^i \cdots\cdots$ are all divisible by $r-1$. Hence the difference of the numbers is also divisible by $r-1$.

Exercise 54.

Find the least number by which each of the following numbers must be multiplied in order that the product may be a square number:

1. $2625 = 125 \times 21 = 5^3 \times 7 \times 3$.
Hence the required multiplier is $5 \times 7 \times 3$, or 105.

2. $3675 = 25 \times 147 = 5^2 \times 3 \times 7^2$.
Hence the required multiplier is 3.

3. $4374 = 2 \times 9 \times 243 = 2 \times 3^7$.
Hence the required multiplier is 2×3, or 6.

4. $74088 = 8 \times 27 \times 343 = 2^3 \times 3^3 \times 7^3$.
Hence the required multiplier is $2 \times 3 \times 7$, or 42.

5. If m and n are positive integers, both odd or both even, show that $m^2 - n^2$ is divisible by 4.

$m^2 - n^2 = (m+n)(m-n)$.
If m and n are both odd or both even,
$m+n$ and $m-n$ are both even.
$\therefore$ $(m+n)(m-n)$ is divisible by 4.

6. Show that $n^2 - n$ is always even.

If n is even, n^2 and n are both even.
If n is odd, n^2 and n are both odd.
In both cases their difference is even.

7. Show that $n^3 - n$ is divisible by 6, if n is even; and by 24 if n is odd.

$$n^3 - n = n(n+1)(n-1) = (n+1)n(n-1).$$

One of the three factors must be divisible by 3.

If n is even, it is the only even factor.

But if n is odd, $n+1$ and $n-1$ are both even; and since they are successive even numbers, one of them is divisible by 4, so that their product is divisible by 8.

Hence, if n is even, 2×3, or 6, is a factor; and if n is odd, 8×3, or 24, is a factor.

8. Show that $n^5 - n$ is divisible by 240 if n is odd.

$$n^5 - n = n(n^4 - 1) = (n+1)n(n-1)(n^2+1).$$

One of the factors $n+1$, n, $n-1$ is divisible by 3.

And, since n is odd, $(n+1)(n-1)$ is divisible by 8, and n^2+1 is divisible by 2.

Also the last digit of n is either 1, 3, 5, 7, or 9.

If it is 5, n is divisible by 5.

If it is 1, $n-1$ has 0 for its last digit.

If it is 3, n^2+1 has 0 for its last digit.

If it is 7, n^2+1 has 0 for its last digit.

If it is 9, $n+1$ has 0 for its last digit.

Hence in every case one of the factors is divisible by 5.

Hence $3 \times 8 \times 2 \times 5$, or 240, is a factor.

9. Show that $n^7 - n$ is divisible by 42 if n is even; and by 168 if n is odd.

$$\begin{aligned}
n^7 - n &= n(n^6 - 1) \\
&= n(n^3+1)(n^3-1) \\
&= n(n+1)(n^2-n+1)(n-1)(n^2+n+1) \\
&= (n+1)n(n-1)(n^2-n+1)(n^2+n+1) \\
&= (n+1)n(n-1)(n^2-n-6+7)(n^2+n-6+7) \\
&= (n+1)n(n-1)[(n-3)(n+2)+7][(n+3)(n-2)+7]
\end{aligned}$$

$(n+1)n(n-1)$ is always divisible by 3.

If n is even, n is the only factor divisible by 2.

Of the 7 consecutive numbers $n-3$, $n-2$, $n-1$, n, $n+1$, $n+2$, $n+3$, one must be divisible by 7. The product is divisible by 7.

Hence $3 \times 2 \times 7$, or 42, is a factor.

If n is odd, $(n+1)(n-1)$ is divisible by 8.

In this case, $8 \times 3 \times 7$, or 168, is a factor.

10. Show that $n(n+1)(n+5)$ is divisible by 6.

If n is even, it is the only even factor.

If n is odd, $n+1$ and $n+5$ are both even.

Hence the product is always divisible by 2.

If n is not divisible by 3, it must be of the form $3m + 1$ or $3m + 2$.

Then $n + 1$ and $n + 5$ are of the forms,

$$3m + 2,\ 3m + 6,\ \text{or}\ 3m + 3,\ 3m + 7$$

In each case 3 is a factor.

Hence 2×3, or 6, is always a factor of the product.

11. Show that every cube number is of one of the forms, $9n, 9n - 1, 9n + 1$.

Every number is of one of the forms:

$$\begin{aligned}
&3m - 1,\ 3m,\ 3m + 1\\
(3m - 1)^3 &= 27m^3 - 27m^2 + 9m - 1\\
&= 9(3m^3 - 3m^2 + m) - 1\\
&= 9n - 1\\
(3m)^3 &= 27m^3\\
&= 9(3m^3)\\
&= 9n\\
(3m + 1)^3 &= 27m^3 + 27m^2 + 9m + 1\\
&= 9(3m^3 + 3m^2 + m) + 1\\
&= 9n + 1
\end{aligned}$$

12. Show that every cube number is of one of the forms, $7n, 7n - 1, 7n + 1$.

Every number is of one of the forms:

$$7m - 3,\ 7m - 2,\ 7m - 1,\ 7m,\ 7m + 1,\ 7m + 2,\ 7m + 3.$$

$$\begin{aligned}
(7m \pm 3)^3 &= 7^3m^3 \pm 3^2 \times 7^2m^2 + 3^3 \times 7m \pm 27\\
&= 7(7^2m^3 \pm 3^2 \times 7m + 3^3m \pm 4) \mp 1\\
&= 7n \mp 1\\
(7m \pm 2) &= 7^3m^3 \pm 6 \times 7^2m^2 + 12 \times 7m \pm 8\\
&= 7(7^2m^3 \pm 6 \times 7m^2 + 12m \pm 1) \pm 1\\
&= 7n \pm 1\\
(7m \pm 1)^3 &= 7^3m^3 \pm 3 \times 7^2m^2 + 3 \times 7m \pm 1\\
&= 7(7^2m^3 \pm 3 \times 7m^2 + 3m) \pm 1\\
&= 7n \pm 1\\
(7m)^3 &= 343m^3 = 7n
\end{aligned}$$

Hence every cube number is of one of the forms: $7n, 7n - 1, 7n + 1$.

13. Show that every number which is both a square and a cube is of the form $7n$ or $7n + 1$.

Every number is of one of the forms:

$$7m-3,\ 7m-2,\ 7m-1,\ 7m,\ 7m+1,\ 7m+2,\ 7m+3.$$

$$\begin{aligned}(7m \pm 3)^2 &= 49m^2 \pm 42m \pm 9\\ &= 7(7m^2 \pm 6m + 1) + 2\\ &= 7n + 2\\ (7m \pm 2)^2 &= 49m^2 \pm 28m + 4\\ &= 7(7m^2 \pm 4m) + 4\\ &= 7n + 4\\ (7m \pm 1)^2 &= 49m^2 \pm 14m + 1\\ &= 7(7m^2 \pm 2m) + 1\\ &= 7n + 1\\ (7m)^2 &= 49m^2 = 7n\end{aligned}$$

∴ Every square number is of one of the forms:

$$7n,\ 7n+1,\ 7n+2, \text{ or } 7n+4.$$

But a cube number is of one of the forms:

$$7n,\ 7n-1,\ 7n+1.$$

Hence if a number is both a square and a cube, it must be of one of the forms:

$$7n,\ 7n+1.$$

14. Show that in the scale of ten every perfect fourth power ends in one of the figures 0, 1, 5, 6.

Every square number is of one of the forms:

$$5n,\ 5n-1,\ 5n+1.$$

Hence every fourth power is of one of the forms:

$$(5n)^2,\ (5n-1)^2,\ (5n+1)^2.$$

$$\begin{aligned}(5n)^2 &= 25n(n)\\ (5n \pm 1)^2 &= 25n^2 \pm 10n + 1\\ &= 5(5n^2 \pm 2n) + 1\end{aligned}$$

Hence every fourth power is of one of the forms:

$$5n,\ 5n+1,$$

and consequently must end in 0, 1, 5, or 6.

Exercise 55.

Find the limiting values of:

1. $\dfrac{(4x^2-3)(1-2x)}{7x^3-6x+4}$ when x becomes infinitesimal.

$$\frac{(4x^2-3)(1-2x)}{7x^3-6x+4} = \frac{-8x^3+4x^2+6x-3}{7x^3-6x+4}$$

Put $x = 0$; the fraction becomes $-\frac{3}{4}$.

2. $\dfrac{(x^2-5)(x^2+7)}{x^4+35}$ when x becomes infinite.

$$\frac{(x^2-5)(x^2+7)}{x^4+35}=\frac{\left(1-\frac{5}{x^2}\right)\left(1+\frac{7}{x^2}\right)}{1+\frac{35}{x^4}}$$

Put $x=\infty$; the fraction becomes 1.

3. $\dfrac{(x+2)^3}{x^2+4}$ when x becomes infinitesimal.

Put $x=0$; the fraction becomes $\frac{8}{4}$, or 2.

4. $\dfrac{x^2-8x+15}{x^2-7x+12}$ when x approaches 3.

$$\frac{x^2-8x+15}{x^2-7x+12}=\frac{(x-3)(x-5)}{(x-3)(x-4)}=\frac{x-5}{x-4}$$

As x approaches 3, the fraction approaches $\dfrac{-2}{-1}$, or 2.

5. $\dfrac{x^2-9}{x^2+9x+18}$ when x approaches -3.

$$\frac{x^2-9}{x^2+9x+18}=\frac{(x+3)(x-3)}{(x+3)(x+6)}=\frac{x-3}{x+6}$$

As x approaches -3, the fraction approaches $\dfrac{-6}{3}$, or -2.

6. $\dfrac{x(x^2+4x+3)}{x^3+3x^2+5x+3}$ when x approaches -1.

$$\frac{x(x^2+4x+3)}{x^3+3x^2+5x+3}=\frac{x(x+1)(x+3)}{(x+1)(x^2+2x+3)}=\frac{x(x+3)}{x^2+2x+3}$$

As x approaches -1, the fraction approaches $\dfrac{-2}{2}$, or -1.

7. $\dfrac{x^3+x^2-2}{x^3+2x^2-2x-1}$ when x approaches 1.

$$\frac{x^3+x^2-2}{x^3+2x^2-2x-1}=\frac{(x-1)(x^2+2x+2)}{(x-1)(x^2+3x+1)}=\frac{x^2+2x+2}{x^2+3x+1}$$

As x approaches 1 the fraction approaches $\frac{5}{5}$, or 1.

8. $\dfrac{4x+\sqrt{x-1}}{2x-\sqrt{x+1}}$ when x approaches 1.

Put $x=1$; the fraction becomes $\dfrac{4}{2-\sqrt{2}}$, or $2(2+\sqrt{2})$.

9. $\dfrac{x-1}{\sqrt{x^2-1}+\sqrt{x-1}}$ when x approaches 1.

$$\frac{x-1}{\sqrt{x^2-1}+\sqrt{x-1}}=\frac{\sqrt{x-1}\sqrt{x-1}}{\sqrt{x-1}(\sqrt{x+1}+1)}$$
$$=\frac{\sqrt{x-1}}{\sqrt{x+1}+1}$$

Put $x=1$; the fraction becomes $\dfrac{0}{\sqrt{2}+1}$, or 0.

10. $\dfrac{x^2-4}{\sqrt{x+2}-\sqrt{3x-2}}$ when x approaches 2.

$$\frac{x^2-4}{\sqrt{x+2}-\sqrt{3x-2}}=\frac{(x^2-4)(\sqrt{x+2}+\sqrt{3x-2})}{x+2-(3x-2)}$$
$$=-\frac{(x+2)(\sqrt{x+2}+\sqrt{3x-2})}{2}$$

Put $x=2$; the fraction becomes $\dfrac{-4(2+2)}{2}$, or -8.

11. $\dfrac{\sqrt{x-a}+\sqrt{x}-\sqrt{a}}{\sqrt{x^2-a^2}}$ when x approaches a.

$$\frac{\sqrt{x-a}+\sqrt{x}-\sqrt{a}}{\sqrt{x^2-a^2}}=\frac{1}{\sqrt{x+a}}+\frac{\sqrt{x}-\sqrt{a}}{\sqrt{x^2-a^2}}$$
$$=\frac{1}{\sqrt{x+a}}+\frac{(\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})}{\sqrt{x^2-a^2}(\sqrt{x}+\sqrt{a})}$$
$$=\frac{1}{\sqrt{x+a}}+\frac{\sqrt{x-a}}{\sqrt{x+a}(\sqrt{x}+\sqrt{a})}$$

Put $x=a$; the fraction becomes $\dfrac{1}{\sqrt{2a}}$.

12. If x approaches a as a limit, and n is a positive integer, show that the limit of x^n is a^n.

Since the limit of x is a, $x = a + x'$, where x' is a variable which can be made less than any assigned number.

And $x^n = (a + x')^n = a^n + na^{n-1}x' + \frac{n(n-1)}{2}a^{n-2}x'^2 + \cdots\cdots + x'^n.$

But as x' approaches 0, each term on the right except a^n also approaches 0; hence the sum approaches 0, as nearly as we please, since the limit of the sum is the sum of the limits. Hence, as x approaches a, x^n approaches a^n as nearly as we please. Hence a^n is the limit of x^n.

13. If x approaches a as a limit, and a is not 0, show that the limit of x^n is a^n, where n is a negative integer.

Let $n = -m$, when m is a positive integer.

Since the limit of x is a, $x = a + x'$ where x' is a variable which can be made less than any assigned number.

And
$$x^{-m} = \frac{1}{x^m} = \frac{1}{(a + x')^m}$$
$$= \frac{1}{a^m + ma^{m-1}x' + \frac{m(m-1)}{2}a^{m-2}x'^2 + \cdots\cdots + x'^m}.$$

But as x' approaches 0, all the terms of the denominator except a^m also approach 0, as nearly as we please; hence their sum approaches 0, as nearly as we please. Hence, as x approaches a, x^n approaches $\frac{1}{a^m}$, or a^n.

Exercise 56.

Determine whether the following infinite series are convergent or divergent:

1. $1 + \frac{1}{2} + \frac{1}{3^2} + \frac{1}{4^3} + \cdots\cdots$

The sum of all terms after the second is

$$\frac{1}{3^2} + \frac{1}{4^3} + \frac{1}{5^4} + \cdots\cdots$$

But this is less than

$$\frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots\cdots$$

But this is a convergent series (Art. 373).

Hence the given series is convergent.

2. $1+\frac{1^2}{\underline{|2}}+\frac{2^3}{\underline{|3}}+\frac{3^4}{\underline{|4}}+\cdots\cdots$

$$n_n=\frac{(n-1)^n}{\underline{|n}}=\frac{n-1}{n}\,\frac{(n-1)^{n-1}}{\underline{|n-1}}$$

But $\qquad (n-1)^{n-1}>\underline{|n-1}$

For $\qquad (n-1)^{n-1}=(n-1)(n-1)\cdots\cdots(n-1 \text{ factors})$

and $\qquad \underline{|n-1}=(n-1)(n-2)\cdots\cdots 1$

Hence $\qquad \frac{(n-1)^{n-1}}{\underline{|n-1}}>1$

Also $\qquad \frac{n-1}{n}=1-\frac{1}{n}$

As n is indefinitely increased, $1-\frac{1}{n}$ approaches 1 as a limit.

Hence the limit of n_n, as n is indefinitely increased, is not 0, but is greater than 1. Therefore the given series is divergent.

3. $1+\frac{2^2}{\underline{|2}}+\frac{3^3}{\underline{|3}}+\frac{4^4}{\underline{|4}}+\cdots\cdots$

$$n_n=\frac{n^n}{\underline{|n}}$$

But $\qquad n^n>\underline{|n}$

$\therefore$ n_n is always greater than 1.

Hence the series is divergent.

4. $\frac{2}{1^2}+\frac{3}{2^2}+\frac{4}{3^2}+\frac{5}{4^2}+\cdots\cdots$

$$=\frac{1}{1^2}+\frac{2}{2^2}+\frac{3}{3^2}+\frac{4}{4^2}+\cdots\cdots+\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\cdots$$

But $\qquad \frac{1}{1^2}+\frac{2}{2^2}+\frac{3}{3^2}+\frac{4}{4^2}+\cdots\cdots=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\cdots$

And this is a divergent series.

Hence the given series is also divergent.

5. $1+\frac{1}{1^2}+\frac{2^2}{3^2}+\frac{3^2}{4^2}+\cdots\cdots$

The sum of n terms after the nth,

$\frac{n^2}{(n+1)^2}+\frac{(n+1)^2}{(n+2)^2}+\cdots\cdots$ is greater than $n\times\frac{n^2}{(n+1)^2}$ or $\frac{n}{\left(1+\frac{1}{n}\right)^2}$.

As n increases, this sum increases. The series is therefore divergent.

6. $\frac{1}{1^m}+\frac{1^m}{2^m}+\frac{2^m}{3^m}+\frac{3^m}{4^m}+\cdots\cdots$

$$n_n=\left(\frac{n-1}{n}\right)^m=\left(1-\frac{1}{n}\right)^m.$$

$\therefore$ The limit of n_n is 1. Hence the series is divergent.

Exercise 57.

1. Find the fiftieth term of 1, 3, 8, 20, 43, ...

The series is 1 3 8 20 43 .....
First differences, 2 5 12 23 .....
Second differences, 3 7 11 .....
Third differences, 4 4 .....
Fourth differences, 0 .....

$$\begin{aligned} a &= 1,\ b=2,\ c=3,\ d=4,\ e=0.\\ a_{50} &= a+49b+49\times 24c+49\times 47\times 8d\\ &= 1+98+49\times 72+49\times 47\times 32\\ &= 77323 \end{aligned}$$

2. Find the sum of the series 4, 12, 29, 55, ... to 20 terms.

The series is 4 12 29 55 .....
First differences, 8 17 26 .....
Second differences, 9 9 .....
Third differences, 0 .....

$$\begin{aligned} a &= 4,\ b=8,\ c=9,\ d=0.\\ s_{20} &= 20a+190b+20\times 19\times 3c\\ &= 80+1520+10260\\ &= 11860 \end{aligned}$$

3. Find the twelfth term of 4, 11, 28, 55, 92, ...

The series is 4 11 28 55 92 .....
First differences, 7 17 27 37 .....
Second differences, 10 10 10 .....
Third differences, 0 0 .....

$$\begin{aligned} a &= 4,\ b=7,\ c=10,\ d=0.\\ a_{12} &= a+11b+55c\\ &= 4+77+550\\ &= 631 \end{aligned}$$

4. Find the sum of the series 43, 27, 14, 4, -3, ... to 12 terms.

The series is 43 27 14 4 -3 .....
First differences, -16 -13 -10 -7 .....
Second differences, 3 3 3 .....
Third differences, 0 0 .....

$$a = 43,\ b = -16,\ c = 3,\ d = 0$$

$$\begin{aligned} s_{12} &= 12a + 66b + 220c \\ &= 516 - 1056 + 660 \\ &= 120 \end{aligned}$$

5. Find the seventh term of 1, 1.235, 1.471, 1.708, ...

The series is 1 1.235 1.471 1.708 .....
First differences, .235 .236 .237 .....
Second differences, .001 .001 .....
Third differences, 0 .....

$$a = 1,\ b = .235,\ c = .001,\ d = 0.$$

$$\begin{aligned} a_7 &= a + 6b + 15c \\ &= 1 + 1.41 + .015 \\ &= 2.425 \end{aligned}$$

6. Find the sum of the series 70, 66, 62.3, 58.9, ... to 15 terms.

The series is 70 66 62.3 58.9 .....
First differences, -4 -3.7 -3.4 .....
Second differences, .3 .3 .....
Third differences, 0 .....

$$a = 70,\ b = -4,\ c = .3,\ d = 0.$$

$$\begin{aligned} s_{15} &= 15a + 105b + 455c \\ &= 1050 - 420 + 136.5 \\ &= 766.5 \end{aligned}$$

7. Find the eleventh term of 343, 337, 326, 310, ...

The series is 343 337 326 310 .....
First differences, -6 -11 -16 .....
Second differences, -5 -5 .....
Third differences, 0 .....

$$a = 343,\ b = -6,\ c = -5,\ d = 0.$$

$$\begin{aligned} a_{11} &= a + 10b + 45c \\ &= 343 - 60 - 225 \\ &= 58 \end{aligned}$$

8. Find the sum of the series $7 \times 13, 6 \times 11, 5 \times 9, \cdots\cdots$ to 9 terms.

The series is	91		66		45		28		.....
First differences,		−25		−21		−17		.....	
Second differences,			4		4		.....		
Third differences,				0		.....			

$$a = 91, \; b = -25, \; c = 4, \; d = 0.$$

$$\begin{aligned} s_9 &= 9a + 36b + 84c \\ &= 819 - 900 + 336 \\ &= 255. \end{aligned}$$

9. Find the sum of n terms of the series

$$3 \times 8, \; 6 \times 11, \; 9 \times 14, \; 12 \times 17, \; \cdots\cdots$$

The series is	24		66		126		204	.....
First differences,		42		60		78	.....	
Second differences,			18		18	.....		
Third differences,				0	.....			

$$a = 24, \; b = 42, \; c = 18, \; d = 0.$$

$$\begin{aligned} s_n &= na + \frac{n(n-1)}{2}b + \frac{n(n-1)(n-2)}{6}c \\ &= 24n + 21n(n-1) + 3n(n-1)(n-2) \\ &= 3n^3 + 12n^2 + 9n \\ &= 3n(n+1)(n+3). \end{aligned}$$

10. Find the sum of n terms of the series 1, 6, 15, 28, 45, ...

The series is	1		6		15		28		45	.....
First differences,		5		9		13		17	.....	
Second differences,			4		4		4	.....		
Third differences,				0		0				

$$a = 1, \; b = 5, \; c = 4, \; d = 0.$$

$$\begin{aligned} s_n &= na + \frac{n(n-1)}{2}b + \frac{n(n-1)(n-2)}{6}c \\ &= n + \frac{5n(n-1)}{2} + \frac{2n(n-1)(n-2)}{3} \\ &= \frac{4n^3 + 3n^2 - n}{6} \\ &= \frac{n(n+1)(4n-1)}{6}. \end{aligned}$$

11. Show that the sum of the cubes of the first n natural numbers is the square of the sum of the numbers.

The series of cubes is	1		8		27		64		125
First differences,		7		19		37		61	
Second differences,			12		18		24		
Third differences,				6		6			
Fourth difference,					0				

$a = 1,\ b = 7,\ c = 12,\ d = 6,\ e = 0.$

$$Sn = na + \frac{n(n-1)}{2}b + \frac{n(n-1)(n-2)}{6}c + \frac{n(n-1)(n-2)(n-3)}{24}d$$

$$= n + \frac{7n(n-1)}{2} + 2n(n-1)(n-2) + \frac{n(n-1)(n-2)(n-3)}{4}$$

$$= \frac{n^4 + 2n^3 + n^2}{4}$$

$$= \left\{\frac{n(n+1)}{2}\right\}^2$$

12. Determine the number of shot in the side of the base of a triangular pile which contains 286 shot.

$$\frac{n(n+1)(n+2)}{6} = 286$$

$$n(n+1)(n+2) = 1716$$

$$= 11 \times 12 \times 13$$

$\therefore n = 11$ is one solution.

The other solutions are imaginary.

Hence there are 11 shot in the side of the base.

13. The number of shot in the upper course of a square pile is 169, and in the lower course 1089. How many shot are there in the pile?

The lower course has $\sqrt{1089}$, or 33 shot on a side.

If the pile were full it would contain

$$\frac{n(n+1)(2n+1)}{6} = \frac{33 \times 34 \times 67}{6} = 12529$$

The upper course has $\sqrt{169}$ or 13 shot on a side.

The next course above would have 12 shot on a side.

The number of shot lacking from the whole pile is

$$\frac{n(n+1)(2n+1)}{6} = \frac{12 \times 13 \times 25}{6} = 650 \text{ shot.}$$

Hence there are 12529 — 650, or 11879 shot in the pile.

14. Find the number of shot in a rectangular pile having 17 shot in one side of the base and 42 in the other.

$$s = \frac{n}{6}(n+1)(3n' - n + 1)$$

$$n = 17,\ n' = 42$$

$$\therefore\ s = \tfrac{17}{6} \times 18 \times 110 = 5610$$

15. Find the number of shot in the five lower courses of a triangular pile which has 15 in one side of the base.

If the pile were complete the number of shot would be:

$$\frac{n(n+1)(n+2)}{6} = \frac{15 \times 16 \times 17}{6} = 680$$

The number of shot in one side of the sixth course from the bottom would be 10. Hence the number in the ten upper courses is

$$\frac{10 \times 11 \times 12}{6} = 220$$

Hence there are 680 — 220, or 460 shot in the five lower courses.

16. The number of shot in a triangular pile is to the number in a square pile, of the same number of courses, as 22 : 41. Find the number of shot in each pile.

$$\frac{n(n+1)(n+2)}{6} : \frac{n(n+1)(2n+1)}{6} = 22 : 41$$

$$\therefore\ n + 2 : 2n + 1 = 22 : 41$$

$$\therefore\ n = 20$$

$$\frac{n(n+1)(n+2)}{6} = 1540$$

$$\frac{n(n+1)(2n+1)}{6} = 2870$$

There are 1540 shot in the triangular pile, and 2870 in the square pile.

17. Find the number of shot required to complete a rectangular pile having 15 and 6 shot, respectively, in the sides of its upper course.

The next course above would have 14 and 5 shot in its sides.

Hence the number of shot required to complete the pile is:

$$s = \frac{n}{6}(n+1)(3n' - n + 1)$$

$$= \tfrac{5}{6} \times 6 \times 38 = 190$$

18. How many shot must there be in the lowest course of a triangular pile that 10 courses of the pile, beginning at the base, may contain 37,020 shot?

Let $n =$ the number of shot in one side of the lowest course.
Then $n - 10 =$ the number of shot in one side of the eleventh course.
The number of shot in the complete pile is:

$$\frac{n(n+1)(n+2)}{6}$$

The number in the courses above the tenth is:

$$\frac{(n-10)(n-9)(n-8)}{6}$$

$$\therefore \frac{n(n+1)(n+2)}{6} - \frac{(n-10)(n-9)(n-8)}{6} = 37020$$

$$\frac{30n^2 - 240n + 720}{6} = 37020$$

$$n^2 - 8n + 24 = 7404$$

$$n^2 - 8n = 7380$$

$$\therefore n = 90$$

Hence there are 90 shot on a side in the lowest course, and $\frac{90}{2}(90+1)$, or 4095, shot in the lowest course.

19. Find the number of shot in a complete rectangular pile of 15 courses which has 20 shot in the longest side of its base.

$$s = \frac{n}{6}(n+1)(3n' - n + 1)$$

$$n = 15 \qquad n' = 20$$

$$\therefore s = \frac{15}{6} \times 16 \times 46 = 1840$$

There are 1840 shot in the pile.

20. Find the number of shot in the bottom of a square pile which contains 2600 more shot than a triangular pile of the same number of courses.

Let $n =$ number of courses in each pile.

Then $\frac{n(n+1)(n+2)}{6} =$ number of shot in triangular pile.

$\frac{n(n+1)(2n+1)}{6} =$ number of shot in square pile.

$$\therefore \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)(n+2)}{6} = 2600$$

$$\therefore\ (n-1)n(n+1)=15600$$
$$(n-1)n(n+1)=24\times 25\times 26$$

$\therefore\ n=25$ is one solution.

The other two solutions are imaginary.

$\therefore$ The number of shot in the bottom row of the square pile is 25.

21. Find the number of shot in a complete square pile in which the number of shot in the base and the number in the fifth course above differ by 225.

Let $n =$ number of shot in the side of the bottom course.

Then $n-5 =$ number of shot in the side of the fifth course above.

$n^2 =$ number of shot in bottom course.

$(n-5)^2 =$ number of shot in fifth course above.

$$\therefore\ n^2-(n-5)^2=225$$
$$10n-25=225$$
$$n=25$$

Also

$$s=\frac{n(n+1)(2n+1)}{6}$$
$$=\frac{25\times 26\times 51}{6}$$
$$=5525$$

There are 5525 shot in the pile.

22. Find the number of shot in a rectangular pile which has 600 in the lowest course and 11 in the top row.

$$nn'=600$$
$$n'-n+1=p=11$$
$$\therefore\ n=20$$
$$n'=30$$

Also

$$s=\frac{n}{6}(n+1)(3n'-n+1)$$
$$=\frac{20}{6}\times 21\times 71=4970$$

There are 4970 shot in the pile.

Exercise 58.

Given the declination of the Moon at the following times. Find the declination at each hour in the afternoon of Dec. 1.

			°	′	″
1890.	Dec. 1.	Noon,	N. 22	46	53
		Midnight,	21	28	49
	Dec. 2.	Noon,	19	57	25
		Midnight,	18	13	57
	Dec. 3.	Noon,	16	19	44
		Midnight,	14	15	59

$22^\circ\,46'\,53'' = 82013''$ $\qquad 18^\circ\,13'\,57'' = 65637''$

$21^\circ\,28'\,49'' = 77329''$ $\qquad 16^\circ\,19'\,44'' = 58784''$

$19^\circ\,57'\,25'' = 71845''$ $\qquad 14^\circ\,15'\,59'' = 51359''$

The series is	82013	77329	71845	65637	58784	51359
First differences,	−4684	−5484	−6208	−6853	−7425	
Second differences,	−800	−724	−645	−572		
Third differences,	76	79	73			
Fourth differences,	3	−6				
Fifth difference,	−9					

$$a = 82013,\ b = -4684,\ c = -800,\ d = 76,\ e = 3,\ f = -9$$

Then $s_1 = a + \frac{1}{12}b - 0.0382c + 0.0244d$. All terms after the fourth may be neglected.

$$\begin{aligned}
&= 82013 - 390.3 + 30.56 + 1.85\\
&= 81655 = 22^\circ\,40'\,55''\\
s_2 &= a + \tfrac{1}{6}b - 0.0694c + 0.0424d\\
&= 82013 - 780.6 + 55.52 + 3.22\\
&= 81291 = 22^\circ\,34'\,51''\\
s_3 &= a + \tfrac{1}{4}b - 0.0937c + 0.0547d\\
&= 82013 - 1171 + 74.96 + 4.16\\
&= 80921 = 22^\circ\,28'\,41''\\
s_4 &= a + \tfrac{1}{3}b - 0.1111c + 0.0617d\\
&= 82013 - 1561.1 + 88.88 + 4.69\\
&= 80546 = 22^\circ\,22'\,26''\\
s_5 &= a + \tfrac{5}{12}b - 0.1215c + 0.0641d\\
&= 82013 - 1951.6 + 97.2 + 4.87\\
&= 80163 = 22^\circ\,16'\,3''\\
s_6 &= a + \tfrac{1}{2}b - 0.1250c + 0.0625d\\
&= 82013 - 2342 + 100 + 4.75\\
&= 79776 = 22^\circ\,9'\,36''\\
s_7 &= a + \tfrac{7}{12}b - 0.1215c + 0.0574d\\
&= 82013 - 2732.3 - 97.2 + 4.36\\
&= 79383 = 22^\circ\,3'\,3''
\end{aligned}$$

$$\begin{aligned}
s_8 &= a + \tfrac{2}{3}b - 0.1111\,c + 0.0494\,d \\
&= 82013 - 3122.6 + 88.88 + 3.75 \\
&= 78982 = 21^\circ\,56'\,22'' \\
s_9 &= a + \tfrac{3}{4}b - 0.0937\,c + 0.0391\,d \\
&= 82013 - 3513 + 74.96 + 2.97 \\
&= 78578 = 21^\circ\,49'\,38'' \\
s_{10} &= a + \tfrac{5}{6}b - 0.0694\,c + 0.0270\,d \\
&= 82013 - 3903.3 + 55.52 + 2.05 \\
&= 78168 = 21^\circ\,42'\,48'' \\
s_{11} &= a + \tfrac{11}{12}b - 0.0382\,c + 0.0138\,d \\
&= 82013 - 4293.7 + 30.56 + 1.05 \\
&= 77750 = 21^\circ\,35'\,50'' \\
s_{12} &= a + b \\
&= 82013 - 4684 \\
&= 77329
\end{aligned}$$

Exercise 59.

Write down the general term, and sum to n terms, and to an infinite number of terms, the following series:

1. $\frac{1}{1 \times 4} + \frac{1}{2 \times 5} + \frac{1}{3 \times 6} + \cdots\cdots$

The nth term is $\frac{1}{n(n+3)}$.

The series equals:

$$\begin{aligned}
&\tfrac{1}{3}\left(1 - \tfrac{1}{4}\right) + \tfrac{1}{3}\left(\tfrac{1}{2} - \tfrac{1}{5}\right) + \tfrac{1}{3}\left(\tfrac{1}{3} - \tfrac{1}{6}\right) + \cdots\cdots \tfrac{1}{3}\left(\tfrac{1}{n} - \tfrac{1}{n+3}\right) + \cdots\cdots \\
&\quad = \tfrac{1}{3}\Big(1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots\cdots + \tfrac{1}{n} - \tfrac{1}{4} - \tfrac{1}{5} - \tfrac{1}{6} - \cdots\cdots \\
&\qquad\qquad - \tfrac{1}{n} - \tfrac{1}{n+1} - \tfrac{1}{n+2} - \tfrac{1}{n+3} + \cdots\cdots\Big) \\
&\quad = \tfrac{1}{3}\left(1 + \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{n+1} - \tfrac{1}{n+2} - \tfrac{1}{n+3} + \cdots\cdots\right) \\
&\quad = \tfrac{11}{18}
\end{aligned}$$

2. $\frac{1}{1 \times 3} + \frac{1}{2 \times 4} + \frac{1}{3 \times 5} + \cdots\cdots$

The nth term is $\frac{1}{n(n+2)}$.

The series equals:

$$\frac{1}{2}\left(1-\frac{1}{3}\right)+\frac{1}{2}\left(\frac{1}{2}-\frac{1}{4}\right)+\frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots\cdots+\frac{1}{2}\left(\frac{1}{n}-\frac{1}{n+2}\right)+\cdots\cdots$$

$$=\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots\cdots+\frac{1}{n}-\frac{1}{3}-\frac{1}{4}-\frac{1}{5}-\cdots\cdots\right.$$

$$\left.-\frac{1}{n}-\frac{1}{n+1}-\frac{1}{n+2}+\cdots\cdots\right)$$

$$=\frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}+\cdots\cdots\right)$$

$$=\tfrac{3}{4}$$

3. $\dfrac{1}{1\times 5}+\dfrac{1}{5\times 9}+\dfrac{1}{9\times 13}+\cdots\cdots$

The nth term is $\dfrac{1}{(4n-3)(4n+1)}$.

The series equals:

$$\frac{1}{4}\left(1-\frac{1}{5}\right)+\frac{1}{4}\left(\frac{1}{5}-\frac{1}{9}\right)+\frac{1}{4}\left(\frac{1}{9}-\frac{1}{13}\right)+\cdots\cdots+\frac{1}{4}\left(\frac{1}{4n-3}-\frac{1}{4n+1}\right)+\cdots\cdots$$

$$=\frac{1}{4}\left(1+\frac{1}{5}+\frac{1}{9}+\cdots\cdots+\frac{1}{4n-3}-\frac{1}{5}-\frac{1}{9}-\cdots\cdots\right.$$

$$\left.-\frac{1}{4n-3}-\frac{1}{4n+1}+\cdots\cdots\right)$$

$$=\frac{1}{4}\left(1-\frac{1}{4n+1}+\cdots\cdots\right)$$

$$=\tfrac{1}{4}$$

4. $\dfrac{6}{2\times 7}+\dfrac{6}{7\times 12}+\dfrac{6}{12\times 17}+\cdots\cdots$

The nth term is $\dfrac{6}{(5n-3)(5n+2)}$.

The series equals

$$\frac{6}{5}\left(\frac{1}{2}-\frac{1}{7}\right)+\frac{6}{5}\left(\frac{1}{7}-\frac{1}{12}\right)+\cdots\cdots+\frac{6}{5}\left(\frac{1}{5n-3}-\frac{1}{5n+2}\right)+\cdots\cdots$$

$$=\frac{6}{5}\left(\frac{1}{2}+\frac{1}{7}+\cdots\cdots+\frac{1}{5n-3}-\frac{1}{7}-\frac{1}{12}-\cdots\cdots\right.$$

$$\left.-\frac{1}{5n-3}-\frac{1}{5n+2}+\cdots\cdots\right)$$

$$=\frac{6}{5}\left(\frac{1}{2}-\frac{1}{5n+2}+\cdots\cdots\right)$$

$$=\tfrac{6}{10}=\tfrac{3}{5}$$

5. $\frac{1}{5\times 11}+\frac{1}{8\times 14}+\frac{1}{11\times 17}+\cdots\cdots$

The nth term is $\frac{1}{(3n+2)(3n+8)}$.

The series equals:

$$\frac{1}{6}\left(\frac{1}{5}-\frac{1}{11}\right)+\frac{1}{6}\left(\frac{1}{8}-\frac{1}{14}\right)+\cdots\cdots+\frac{1}{6}\left(\frac{1}{3n+2}-\frac{1}{3n+8}\right)+\cdots\cdots$$

$$=\frac{1}{6}\left(\frac{1}{5}+\frac{1}{8}+\frac{1}{11}+\cdots\cdots+\frac{1}{3n+2}-\frac{1}{11}-\frac{1}{14}-\cdots\cdots\right.$$

$$\left.-\frac{1}{3n+2}-\frac{1}{3n+5}-\frac{1}{3n+8}\cdots\cdots\right)$$

$$=\frac{1}{6}\left(\frac{1}{5}+\frac{1}{8}-\frac{1}{3n+5}-\frac{1}{3n+8}+\cdots\cdots\right)=\frac{13}{240}$$

6. $\frac{1}{3\times 8}+\frac{1}{6\times 12}+\frac{1}{9\times 16}+\cdots\cdots$

The nth term is $\frac{1}{3n(4n+4)}=\frac{1}{12n(n+1)}$.

The series equals:

$$\frac{1}{12}\left(\frac{1}{2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots\cdots\right)$$

$$=\tfrac{1}{12}[(1-\tfrac{1}{2})+(\tfrac{1}{2}-\tfrac{1}{3})+(\tfrac{1}{3}-\tfrac{1}{4})+\cdots\cdots]$$

$$=\tfrac{1}{12}\times 1=\tfrac{1}{12}$$

NOTE. For 7 and 8 see 1 and 6 respectively.

Exercise 60.

Resolve into partial fractions:

1. $\frac{7x+1}{(x+4)(x-5)}$.

Assume $$\frac{7x+1}{(x+4)(x-5)}\equiv\frac{A}{x+4}+\frac{B}{x-5}$$

then $$7x+1\equiv(A+B)x-5A+4B$$

$$\therefore\ A+B=7$$

$$4B-5A=1$$

$$A=3$$

$$B=4$$

$$\therefore\ \frac{7x+1}{(x+4)(x-5)}\equiv\frac{3}{x+4}+\frac{4}{x-5}$$

2. $\dfrac{6}{(x+3)(x+4)}$.

Assume $$\frac{6}{(x+3)(x+4)} \equiv \frac{A}{x+3} + \frac{B}{x+4}$$

then $$6 \equiv (A+B)x + 4A + 3B$$

$$\therefore\ A + B = 0$$
$$4A + 3B = 6$$
$$\therefore\ A = 6$$
$$B = -6$$
$$\therefore\ \frac{6}{(x+3)(x+4)} \equiv \frac{6}{x+3} - \frac{6}{x+4}$$

3. $\dfrac{5x-1}{(2x-1)(x-5)}$.

Assume $$\frac{5x-1}{(2x-1)(x-5)} \equiv \frac{A}{2x-1} + \frac{B}{x-5}$$

then $$5x - 1 \equiv (A+2B)x - 5A - B$$

$$\therefore\ A + 2B = 5$$
$$5A + B = 1$$
$$\therefore\ A = -\tfrac{1}{3},\ B = \tfrac{8}{3}$$
$$\therefore\ \frac{5x-1}{(2x-1)(x-5)} \equiv \frac{-\frac{1}{3}}{2x-1} + \frac{\frac{8}{3}}{x-5}$$
$$\equiv \frac{8}{3(x-5)} - \frac{1}{3(2x-1)}$$

4. $\dfrac{x-2}{x^2-3x-10} \equiv \dfrac{x-2}{(x-5)(x+2)}$.

Assume $$\frac{x-2}{(x-5)(x+2)} \equiv \frac{A}{x-5} + \frac{B}{x+2}$$

then $$x - 2 \equiv (A+B)x + 2A - 5B$$

$$\therefore\ A + B = 1$$
$$2A - 5B = -2$$
$$\therefore\ A = \tfrac{3}{7},\ B = \tfrac{4}{7}$$
$$\therefore\ \frac{x-2}{x^2-3x-10} \equiv \frac{3}{7(x-5)} + \frac{4}{7(x+2)}$$

5. $\dfrac{3}{x^3-1} \equiv \dfrac{3}{(x-1)(x^2+x+1)}$.

Assume $$\frac{3}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

then $$3 \equiv (A+B)x^2 + (A-B+C)x + A - C$$

$$\therefore A+B=0$$
$$A-B+C=0$$
$$A-C=3$$
$$\therefore A=1,\ B=-1,\ C=-2$$
$$\therefore \frac{3}{x^3-1}=\frac{1}{x-1}-\frac{x+2}{x^2+x+1}$$

6. $\dfrac{x^2-x-3}{x(x^2-4)}$.

Assume $$\frac{x^2-x-3}{x(x^2-4)}\equiv\frac{A}{x}+\frac{B}{x+2}+\frac{C}{x-2}$$

then $$x^2-x-3\equiv(A+B+C)x^2+(-2B+2C)x-4A$$
$$\therefore A+B+C=1$$
$$-2B+2C=-1$$
$$-4A=-3$$
$$\therefore A=\tfrac{3}{4},\ B=\tfrac{3}{8},\ C=-\tfrac{1}{8}$$
$$\therefore \frac{x^2-x-3}{x(x^2-4)}=\frac{3}{4x}+\frac{3}{8(x+2)}-\frac{1}{8(x-2)}.$$

7. $\dfrac{3x^2-4}{x^2(x+5)}$.

Assume $$\frac{3x^2-4}{x^2(x+5)}\equiv\frac{A}{x^2}+\frac{B}{x}+\frac{C}{x+5}$$

then $$3x^2-4\equiv(B+C)x^2+(A+5B)x+5A$$
$$\therefore B+C=3$$
$$A+5B=0$$
$$5A=-4$$
$$\therefore A=-\tfrac{4}{5},\ B=\tfrac{4}{25},\ C=\tfrac{71}{25}$$
$$\therefore \frac{3x^2-4}{x^2(x+5)}=-\frac{4}{5x^2}+\frac{4}{25x}+\frac{71}{25(x+5)}.$$

8. $\dfrac{7x^2-x}{(x-1)^2(x+2)}$.

Assume $$\frac{7x^2-x}{(x-1)^2(x+2)}\equiv\frac{A}{(x-1)^2}+\frac{B}{x-1}+\frac{C}{x+2}$$

then $$7x^2-x\equiv(B+C)x^2+(A+B-2C)x+2A-2B+C$$
$$\therefore B+C=7$$
$$A+B-2C=-1$$
$$2A-2B+C=0$$
$$\therefore A=2,\ B=\tfrac{11}{3},\ C=\tfrac{10}{3}$$
$$\therefore \frac{7x^2-x}{(x-1)^2(x+2)}\equiv\frac{2}{(x-1)^2}+\frac{11}{3(x-1)}+\frac{10}{3(x+2)}.$$

9. $\dfrac{2x^2-7x+1}{x^3-1}.$

Assume $$\frac{2x^2-7x+1}{x^3-1}\equiv\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}$$

then $$2x^2-7x+1\equiv(A+B)x^2+(A-B+C)x+A-C$$

$$\therefore\ A+B=2$$

$$A-B+C=-7$$

$$A-C=1$$

$$\therefore\ A=-\tfrac{4}{3},\ B=\tfrac{10}{3},\ C=-\tfrac{7}{3}$$

$$\therefore\ \frac{2x^2-7x+1}{x^3-1}=\frac{10x-7}{3(x^2+x+1)}-\frac{4}{3(x-1)}.$$

10. $\dfrac{7x-1}{6x^2-5x+1}.$

Assume $$\frac{7x-1}{6x^2-5x+1}\equiv\frac{7x-1}{(3x-1)(2x-1)}\equiv\frac{A}{3x-1}+\frac{B}{2x-1}$$

then $$7x-1=(2A+3B)x-A-B$$

$$\therefore\ 2A+3B=7$$

$$A+B=1$$

$$\therefore\ A=-4,\ B=5$$

$$\therefore\ \frac{7x-1}{6x^2-5x+1}\equiv\frac{5}{2x-1}-\frac{4}{3x-1}.$$

11. $$\frac{13x+46}{12x^2-11x-15}\equiv\frac{13x+46}{(4x+3)(3x-5)}\equiv\frac{A}{4x+3}+\frac{B}{3x-5}.$$

$$13x+46\equiv(3A+4B)x-5A+3B$$

$$\therefore\ 3A+4B=13$$

$$5A-3B=-46$$

$$A=-5,\ B=7$$

$$\frac{13x+46}{12x^2-11x-15}\equiv\frac{7}{3x-5}-\frac{5}{4x+3}$$

12. $$\frac{2x^2-11x+5}{x^3-x^2-11x+15}\equiv\frac{2x^2-11x+5}{(x-3)(x^2+2x-5)}.$$

Assume

$$\frac{2x^2-11x+5}{(x-3)(x^2+2x-5)}=\frac{A}{x-3}+\frac{Bx+C}{x^2+2x-5}$$

then $$2x^2-11x+5\equiv(A+B)x^2+(2A-3B+C)x-5A-3C$$

$$\therefore\ A+B=2$$

$$2A - 3B + C = -11$$
$$-5A - 3C = 5$$
$$\therefore A = -1,\ B = 3,\ C = 0$$
$$\therefore \frac{2x^2 - 11x + 5}{x^3 - x^2 - 11x + 15} = \frac{3x}{x^2 + 2x - 5} - \frac{1}{x - 3}$$

Exercise 61.

Expand to four terms in ascending powers of x.

1. $\dfrac{1}{1 - 2x}$.

Divide 1 by $1 - 2x$.

Then $$\frac{1}{1 - 2x} = 1 + 2x + 4x^2 + 8x^3 + \cdots\cdots$$

2. $\dfrac{1}{2 - 3x}$.

Divide 1 by $2 - 3x$.

Then $$\frac{1}{2 - 3x} = \tfrac{1}{2} + \tfrac{3}{4}x + \tfrac{9}{8}x^2 + \tfrac{27}{16}x^3 + \cdots\cdots$$

3. $\dfrac{1 + x}{2 + 3x}$.

Divide $1 + x$ by $2 + 3x$.

Then $$\frac{1 + x}{2 + 3x} = \tfrac{1}{2} - \tfrac{1}{4}x + \tfrac{3}{8}x^2 - \tfrac{9}{16}x^3 + \cdots\cdots$$

4. $\dfrac{1 - x}{1 + x + x^2}$.

Let $$\frac{1 - x}{1 + x + x^2} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots\cdots$$

$$\therefore 1 - x = A + (B + A)x + (C + B + A)x^2$$
$$+ (D + C + B)x^3 + (E + B + C)x^4 + \cdots\cdots$$

$$\begin{aligned} \therefore A &= 1 & A &= 1 \\ A + B &= -1 & B &= -2 \\ C + B + A &= 0 & C &= 1 \\ D + C + B &= 0 & D &= 1 \\ E + D + C &= 0 & E &= -2 \end{aligned}$$

. . . .

$$\therefore \frac{1 - x}{1 + x + x^2} = 1 - 2x + x^2 + x^3 - 2x^4 + x^5 + \cdots\cdots$$

5. $\dfrac{5-2x}{1+x-x^2}$.

Let $$\frac{5-2x}{1+x-x^2}=A+Bx+Cx^2+Dx^3+Ex^4+\cdots\cdots$$

$$\begin{aligned}\therefore\ 5-2x=A+(A+B)x+(-A+B+C)x^2&+(-B+C+D)x^3\\&+(-C+D+E)x^4+\cdots\cdots\end{aligned}$$

$$\begin{array}{rl} \therefore\ A=5 & A=5\\ A+B=-2 & B=-7\\ -A+B+C=0 & C=12\\ -B+C+D=0 & D=-19\\ -C+D+E=0 & E=31\\ \cdot\quad\cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot \end{array}$$

$$\therefore\ \frac{5-2x}{1+x-x^2}=5-7x+12x^2-19x^3+31x^4+\cdots\cdots$$

6. $\dfrac{4x-6x^2}{1-2x+3x^2}$.

Let $$\frac{4x-6x^2}{1-2x+3x^2}=A+Bx+Cx^2+Dx^3+Ex^4+\cdots\cdots$$

$$\begin{aligned}\therefore\ 4x-6x^2=A+(-2A+B)x&+(3A-2B+C)x^2\\&+(3B-2C+D)x^3+(3C-2D+E)x^4+\cdots\cdots\end{aligned}$$

$$\begin{array}{rl} \therefore\ A=0 & A=0\\ -2A+B=4 & B=4\\ 3A-2B+C=-6 & C=2\\ 3B-2C+D=0 & D=-8\\ 3C-2D+E=0 & E=-22\\ \cdot\quad\cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot\quad\cdot \end{array}$$

$$\therefore\ \frac{4x-6x^2}{1-2x+3x^2}=4x+2x^2-8x^3-22x^4+\cdots\cdots$$

7. $\dfrac{x(x-1)}{(x+1)(x^2+1)}\equiv\dfrac{A}{1+x}+\dfrac{Bx+C}{1+x^2}$.

$$\therefore\ x^2-x\equiv(A+B)x^2+(B+C)x+A+C$$

$$\begin{aligned}\therefore\ A+B&=1\\ B+C&=-1\\ A+C&=0\end{aligned}$$

$$\therefore\ A=+1,\ B=0,\ C=-1$$

$$\therefore\ \frac{x(x-1)}{(x+1)(x^2+1)}=\frac{1}{1+x}-\frac{1}{1+x^2}$$

$$\frac{1}{1+x}=1-x+x^2-x^3+x^4-x^5+\cdots\cdots$$

$$\frac{1}{1+x^2}=1-x^2+x^4-x^6+x^8-x^{10}+\cdots\cdots$$

$$\therefore\ \frac{x(x-1)}{(x+1)(x^2+1)}=-x+2x^2-x^3-x^5+\cdots\cdots$$

8. $\dfrac{x^2-x+1}{x^2(x^2-1)}\equiv\dfrac{A}{x^2}+\dfrac{B}{x}+\dfrac{C}{x+1}+\dfrac{D}{x-1}.$

$$\therefore\ x^2-x+1\equiv(B+C+D)x^3+(A-C+D)x^2-Bx-A$$

$$\therefore\ B+C+D=0$$
$$A-C+D=0$$
$$B=1$$
$$A=-1$$
$$\therefore\ A=-1,\ B=1,\ C=-\tfrac{3}{2},\ D=\tfrac{1}{2}$$

$$\therefore\ \frac{x^2-x+1}{x^2(x^2-1)}=-\frac{1}{x^2}+\frac{1}{x}-\frac{3}{2(x+1)}+\frac{1}{2(x-1)}$$

$$\frac{3}{2(x+1)}=\frac{3}{2}\frac{1}{1+x}=\tfrac{3}{2}(1-x+x^2-x^3+x^4-\cdots\cdots)$$

$$\frac{1}{2(x-1)}=-\frac{1}{2}\frac{1}{1-x}=-\tfrac{1}{2}(1+x+x^2+x^3+x^4+\cdots\cdots)$$

$$\therefore\ \frac{x^2-x+1}{x^2(x^2-1)}=-\frac{1}{x^2}+\frac{1}{x}-2+x-2x^2+x^3-2x^4-\cdots\cdots)$$

9. $\dfrac{2x^2-1}{x(x^3+1)}\equiv\dfrac{A}{x}+\dfrac{B}{x+1}+\dfrac{Cx+D}{x^2-x+1}.$

$$\therefore\ 2x^2-1\equiv(A+B+C)x^3+(-B+D+C)x^2+(B+D)x+A$$

$$\therefore\ A+B+C=0$$
$$-B+C+D=2$$
$$B+D=0$$
$$A=-1$$
$$\therefore\ A=-1,\ B=-\tfrac{1}{3},\ C=\tfrac{4}{3},\ D=\tfrac{1}{3}$$

$$\therefore\ \frac{2x^2-1}{x(x^3+1)}=-\frac{1}{x}-\frac{1}{3(x+1)}+\frac{4x+1}{3(x^2-x+1)}$$

$$\frac{1}{3(x+1)}=\frac{1}{3(1+x)}=\tfrac{1}{3}(1-x+x^2-x^3+x^4-x^5+\cdots\cdots)$$

$$\frac{4x+1}{3(x^2-x+1)}=\frac{1}{3}\cdot\frac{1+4x}{1-x+x^2}$$

Let $$\frac{1+4x}{1-x+x^2} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots\cdots$$

$$\therefore 1 + 4x = A + (B-A)x + (C-B+A)x^2 + (D-C+B)x^3 + (E-D+C)x^4 + \cdots\cdots$$

$$\begin{array}{rl} \therefore A = 1 & A = 1 \\ B - A = 4 & B = 5 \\ C - B + A = 0 & C = 4 \\ D - C + B = 0 & D = -1 \\ E - D + C = 0 & E = -5 \\ \cdot\quad\cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot \end{array}$$

$$\therefore \frac{1+4x}{1-x+x^2} = 1 + 5x + 4x^2 - x^3 - 5x^4 + \cdots\cdots$$

$$\therefore \frac{2x^2-1}{x(x^3+1)} = -\frac{1}{x} + 2x + x^2 - 2x^4 + \cdots\cdots$$

10. $$\frac{4}{2+x} = \frac{4}{x+2}$$

$$= \frac{4}{x} - \frac{8}{x^2} + \frac{16}{x^3} - \frac{32}{x^4} + \cdots\cdots$$

11. $$\frac{2-x}{3+x} = -1 + \frac{5}{x+3}$$

$$= -1 + \frac{5}{x} - \frac{15}{x^2} + \frac{45}{x^3} - \frac{135}{x^4} + \cdots\cdots$$

12. $$\frac{5-2x}{1+3x-x^2} \equiv \frac{2x-5}{x^2-3x-1}.$$

Let $$\frac{2x-5}{x^2-3x-1} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x^4} + \cdots\cdots$$

$$\therefore 2x - 5 = \begin{array}{r|l|l|l} Ax + B & -3B & \frac{1}{x} + D & \frac{1}{x^2} + \cdots\cdots \\ -3A & +C & -3C & \\ & -A & -B & \end{array}$$

$$\begin{array}{lll} \therefore A = 2, & B - 3A = -5, & B = 3A - 5 = 1 \\ & C - 3B - A = 0, & C = 3B + A = 5 \\ & D - 3C - B = 0, & D = 3C + B = 16 \end{array}$$

$$\therefore \frac{5-2x}{1+3x-x^2} = \frac{2}{x} + \frac{1}{x^2} + \frac{5}{x^3} + \frac{16}{x^4} + \cdots\cdots$$

13. $$\frac{x^2-x+1}{x(x-2)} \equiv \frac{x^2-x+1}{x^2-2x} \equiv 1 + \frac{x+1}{x(x-2)}$$

$$= 1 + \frac{A}{x} + \frac{B}{x-2}$$

$$\therefore\ x+1 \equiv (A+B)x - 2A$$

$$\therefore\ A+B=1 \qquad -2A=1$$

$$A = -\tfrac{1}{2},\ B = \tfrac{3}{2}$$

$$\therefore\ \frac{x^2-x+1}{x(x-2)} \equiv 1 - \frac{1}{2x} + \frac{3}{2(x-2)}.$$

$$\frac{1}{x-2} = \frac{1}{x} + \frac{2}{x^2} + \frac{4}{x^3} + \frac{8}{x^4} + \cdots\cdots$$

$$\therefore\ \frac{x^2-x+1}{x(x-2)} = 1 - \frac{1}{2x} + \frac{3}{2x} + \frac{6}{2x^2} + \frac{12}{2x^3} + \frac{24}{2x^4} + \cdots\cdots$$

$$= 1 + \frac{1}{x} + \frac{3}{x^2} + \frac{6}{x^3} + \frac{12}{x^4} + \cdots\cdots$$

14. $$\frac{3x-2}{x(x-1)^2} \equiv \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$\therefore\ 3x-2 \equiv (A+B)x^2 + (-2A-B+C)x + A$$

$$\therefore\ A+B=0$$

$$-2A-2B+C=3$$

$$A=-2$$

$$\therefore\ A=-2,\ B=2,\ C=3$$

$$\therefore\ \frac{3x-2}{x(x-1)^2} \equiv -\frac{2}{x} + \frac{2}{x-1} + \frac{3}{(x-1)^2}$$

$$\frac{1}{x-1} = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \cdots\cdots$$

$$\frac{1}{(x-1)^2} = \left(\frac{1}{x-1}\right)^2 = \frac{1}{x^2} + \frac{2}{x^3} + \frac{3}{x^4} + \frac{4}{x^5} + \cdots\cdots$$

$$\therefore\ \frac{3x-2}{x(x-1)^2} \equiv -\frac{2}{x} + \frac{2}{x} + \frac{2}{x^2} + \frac{2}{x^3} + \frac{2}{x^4} + \frac{2}{x^5} + \cdots\cdots$$

$$+ \frac{3}{x^2} + \frac{6}{x^3} + \frac{9}{x^4} + \frac{12}{x^5} + \cdots\cdots$$

$$\equiv \frac{5}{x^2} + \frac{8}{x^3} + \frac{11}{x^4} + \frac{14}{x^5} + \cdots\cdots$$

15. $$\frac{x^2-x+1}{(x-1)(x^2+1)} = \frac{(x^2-x+1)(x+1)}{(x-1)(x+1)(x^2+1)} = \frac{x^3+1}{x^4-1}.$$

$$\frac{1}{x^4-1} = \frac{1}{x^4} + \frac{1}{x^8} + \frac{1}{x^{12}} + \frac{1}{x^{16}} + \cdots\cdots$$

$$\frac{x^3+1}{x^4-1}=\frac{x^3+1}{x^4}+\frac{x^3+1}{x^8}+\frac{x^3+1}{x^{12}}+\cdots\cdots$$
$$=\frac{1}{x}+\frac{1}{x^4}+\frac{1}{x^5}+\frac{1}{x^8}+\frac{1}{x^9}+\frac{1}{x^{12}}+\cdots\cdots$$

Revert:

16. $y=x-2x^2+3x^3-4x^4+\cdots\cdots$

Here $a=1,\ b=-2,\ c=3,\ d=-4$

$\therefore\ A=1,\ B=-2,\ C=5,\ D=14$

$\therefore\ x=y+2y^2+5y^3+14y^4+\cdots\cdots$

17. $y=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots\cdots$

Here $a=1,\ b=0,\ c=-\frac{1}{3},\ d=0,\ e=\frac{1}{5}$

$\therefore\ A=1,\ B=0,\ C=\frac{1}{3},\ D=0,\ E=\frac{2}{15}$

$\therefore\ x=y+\frac{y^3}{3}+\frac{2y^5}{15}+\cdots\cdots$

18. $y=x+\frac{x^2}{1\cdot2}+\frac{x^3}{2\cdot3}+\frac{x^4}{3\cdot4}+\cdots\cdots$

Here $a=1,\ b=\frac{1}{2},\ c=\frac{1}{6},\ d=\frac{1}{12}$

$\therefore\ A=1,\ B=-\frac{1}{2},\ C=\frac{1}{3},\ D=-\frac{7}{24}$

$\therefore\ x=y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{7y^4}{24}+\cdots\cdots$

Exercise 62.

Find the identical relation and generating function of:

1. $1+2x+7x^2+23x^3+76x^4+\cdots\cdots$

Notice that $7=3\times2+1$

$23=3\times7+2$

$76=3\times23+7$

$\therefore$ The identical relation is $u_n=3xu_{n-1}+x^2u_{n-2}$.

But
$$\begin{aligned}s&=1+2x+7x^2+23x^3+76x^4+\cdots\cdots\\-3xs&=\quad\ -3x-6x^2-21x^3-69x^4-\cdots\cdots\\-x^2s&=\qquad\qquad\ -x^2-2x^3-7x^4-\cdots\cdots\end{aligned}$$

$\therefore\ (1-3x-x^2)s=1-x$

$$\therefore\ s=\frac{1-x}{1-3x-x^2}$$

2. $3 + 2x + 3x^2 + 7x^3 + 18x^4 + \cdots\cdots$

Notice that

$$\begin{aligned} 3 &= 3 \times 2 - 3 \\ 7 &= 3 \times 3 - 2 \\ 18 &= 3 \times 7 - 3 \end{aligned}$$

$\therefore$ The identical relation is $u_n = 3xu_{n-1} - x^2u_{n-2}$.

But

$$\begin{aligned} s &= 3 + 2x + 3x^2 + 7x^3 + 18x^4 + \cdots\cdots \\ -3xs &= \quad -9x - 6x^2 - 9x^3 - 21x^4 - \cdots\cdots \\ x^2s &= \qquad\qquad 3x^2 + 2x^3 + 3x^4 + \cdots\cdots \end{aligned}$$

$$\therefore (1 - 3x + x^2)s = 3 - 7x$$

$$\therefore s = \frac{3 - 7x}{1 - 3x + x^2}$$

Find the generating function and the general term of:

3. $2 + 3x + 5x^2 + 9x^3 + 17x^4 + 33x^5 + \cdots\cdots$

Suppose $\quad u_n = pxu_{n-1} + qx^2u_{n-2}$.

Then

$$\begin{aligned} 5 &= 3p + 2q \\ 9 &= 5p + 3q \\ \therefore p &= 3,\ q = -2 \end{aligned}$$

$\therefore$ The identical relation is $u_n = 3xu_{n-1} - 2x^2u_{n-2}$.

$$\begin{aligned} s &= 2 + 3x + 5x^2 + 9x^3 + 17x^4 + 33x^5 + \cdots\cdots \\ -3xs &= \quad -6x - 9x^2 - 15x^3 - 27x^4 - 51x^5 - \cdots\cdots \\ 2x^2s &= \qquad\qquad 4x^2 + 6x^3 + 10x^4 + 18x^5 + \cdots\cdots \end{aligned}$$

$$\therefore (1 - 3x + 2x^2)s = 2 - 3x$$

$$\begin{aligned} \therefore s &= \frac{2 - 3x}{1 - 3x + 2x^2} \\ &= \frac{1}{1 - 2x} + \frac{1}{1 - x} \end{aligned}$$

$$\frac{1}{1 - 2x} = 1 + 2x + 2^2x^2 + \cdots\cdots + 2^nx^n + \cdots\cdots$$

$$\frac{1}{1 - x} = 1 + x + x^2 + \cdots\cdots + x^n + \cdots\cdots$$

$\therefore s = 1 + (2 + 1)x + (2^2 + 1)x^2 + \cdots\cdots + (2^n + 1)x^n + \cdots\cdots$

$\therefore$ The general term is $(2^n + 1)x^n$.

4. $7 - 6x + 9x^2 + 27x^3 + 54x^4 + 189x^5 + \cdots\cdots$

Suppose $\quad u_n = pxu_{n-1} + qx^2u_{n-2} + rx^3u_{n-3}$

Then

$$\begin{aligned} 27 &= 9p - 6q + 7r \\ 54 &= 27p + 9q - 6r \\ 189 &= 54p + 27q + 9q \\ \therefore p &= 2,\ q = 2,\ r = 3 \end{aligned}$$

$\therefore$ The identical relation is

$$u_n = 2xu_{n-1} + 2x^2u_{n-2} + 3x^3u_{n-3}$$

$$\begin{array}{rl} s = & 7 - 6x + 9x^2 + 27x^3 + 54x^4 + 189x^5 + \cdots\cdots \\ -2xs = & - 14x + 12x^2 - 18x^3 - 54x^4 - 108x^5 - \cdots\cdots \\ -2x^2s = & - 14x^2 + 12x^3 - 18x^4 - 54x^5 - \cdots\cdots \\ -3x^3s = & - 21x^3 + 18x^4 - 27x^5 - \cdots\cdots \\ \hline (1 - 2x - 2x^2 - 3x^3)s = & 7 - 20x + 7x^2 \end{array}$$

$$\therefore s = \frac{7 - 20x + 7x^2}{1 - 2x - 2x^2 - 3x^3}$$

$$\frac{7 - 20x + 7x^2}{1 - 2x - 2x^2 - 3x^3} \equiv \frac{7 - 20x + 7x^2}{(1 - 3x)(1 + x + x^2)}$$

$$\equiv \frac{10}{13(1 - 3x)} + \frac{27(3 - x)}{13(1 + x + x^2)}$$

But $$\frac{1}{1 - 3x} = 1 + 3x + 3^2x^2 + 3^3x^3 + \cdots\cdots + 3^nx^n + \cdots\cdots$$

$$\frac{3 - x}{1 + x + x^2} = \frac{(1 - x)(3 - x)}{(1 + x + x^2)(1 - x)} = \frac{3 - 4x + x^2}{1 - x^3}$$

and $$\frac{1}{1 - x^3} = 1 + x^3 + x^6 + x^9 + \cdots\cdots + x^{3n} + \cdots\cdots$$

$$\therefore \frac{3 - x}{1 - x^3} = 3 - 4x + x^2 + 3x^3 - 4x^4 + x^5 + 3x^6 - 4x^7 + x^8 + \cdots\cdots$$

$$\therefore \frac{7 - 20x + 7x^2}{1 - 2x - 2x^2 - 3x^3} = \tfrac{10}{13}(1 + 3x + 3^2x^2 + 3^3x^3 + \cdots\cdots)$$
$$+ \tfrac{27}{13}(3 - 4x + x^2 + 3x^3 - 4x^4 + x^5 + \cdots\cdots)$$

$$\begin{aligned} \therefore u_{3n} &= \tfrac{10}{13}3^{3n-1} + \tfrac{27}{13} &&= \tfrac{27}{13}(10 \times 3^{3n-4} + 1) \\ u_{3n+1} &= \tfrac{10}{13}3^{3n} + \tfrac{27}{13} \times 3 &&= \tfrac{81}{13}(10 \times 3^{3n-4} + 1) \\ u_{3n+2} &= \tfrac{10}{13}3^{3n+1} + \tfrac{27}{13} \times (-4) &&= \tfrac{27}{13}(10 \times 3^{3n-2} - 4) \end{aligned}$$

5. $1 + 5x + 9x^2 + 13x^3 + 17x^4 + 21x^5 + \cdots\cdots$

Suppose $$u_n = pxn_{n-1} + qx^2n_{n-2}$$

Then $$9 = 5p + q$$
$$13 = 9p + 5p$$
$$\therefore p = 2,\ q = -1$$

$\therefore$ The identical relation is $u_n = 2u_{n-1} - u_{n-2}$.

$$\begin{array}{rl} s = & 1 + 5x + 9x^2 + 13x^3 + 17x^4 + \cdots\cdots \\ -2xs = & - 2x - 10x^2 - 18x^3 - 26x^4 + \cdots\cdots \\ x^2s = & x^2 + 5x^3 + 9x^4 + \cdots\cdots \\ \hline (1 - 2x + x^2)s = & 1 + 3x \end{array}$$

$$s=\frac{1+3x}{1-2x+x^2}=\frac{1+3x}{(1-x)^2}$$

Also $1+5x+9x^2+13x^3+\cdots\cdots$

$=1+(4+1)x+(2\times4+1)x^2+(3\times4+1)x^3+\cdots\cdots+(4n+1)x^n+\cdots\cdots$

∴ The general term is $(4n+1)x_n$.

6. $1+x-7x^3+33x^4-130x^5+499x^6+\cdots\cdots$

Suppose $u_n=pu_{n-1}+qu_{n-2}+ru_{n-3}$.

Then
$$\begin{aligned}-7&=q+r\\ 33&=-7p+r\\ -130&=33p-7q\\ \therefore\ p&=-5,\ q=-5,\ r=-2\end{aligned}$$

The identical relation is $u_n=-5xu_{n-1}-5x^2u_{n-2}-2x^3u_{n-3}$.

$$\begin{aligned}s&=1+x-7x^3+33x^4-130x^5+499x^6+\cdots\cdots\\ 5xs&=5x+5x^2-35x^4+165x^5-650x^6-\cdots\cdots\\ 5x^2s&=5x^2+5x^3-35x^5+165x^6+\cdots\cdots\\ 2x^3s&=2x^3+2x^4-14x^6\end{aligned}$$

$(1+5x+5x^2+2x^3)s=1+6x+10x^2$

$$\therefore\ s=\frac{1+6x+10x^2}{1+5x+5x^2+2x^3}$$

7. $3+6x+14x^2+36x^3+98x^4+276x^5+\cdots\cdots$

Suppose $u_n=pu_{n-1}+qu_{n-2}+ru_{n-3}$

Then
$$\begin{aligned}36&=14p+6q+3r\\ 98&=36p+14q+6r\\ 276&=98p+36q+14r\\ \therefore\ p&=6,\ r=-11,\ q=6\end{aligned}$$

∴ The identical relation is

$$u_n=6u_{n-1}-11u_{n-2}+6u_{n-3}$$

$$\begin{aligned}s&=3+6x+14x^2+36x^3+98x^4+276x^5+\cdots\cdots\\ -6xs&=-18x-36x^2-84x^3-216x^4-588x^5-\cdots\cdots\\ +11x^2s&=33x^2+66x^3+154x^4+396x^5+\cdots\cdots\\ -6x^3s&=-18x^3-36x^4-84x^5\end{aligned}$$

$(1-6x+11x^2-6x^3)s=3-12x+11x^2$

$$\therefore\ s=\frac{3-12x+11x^2}{1-6x+11x^2-6x^3}=\frac{3-12x+11x^2}{(1-x)(1-2x)(1-3x)}$$

$$=\frac{1}{1-x}+\frac{1}{1-2x}+\frac{1}{1-3x}$$

But $$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots\cdots + x^n + \cdots\cdots$$

$$\frac{1}{1-2x} = 1 + 2x + 2^2x^2 + 2^3x^3 + \cdots\cdots + 2^nx^n + \cdots\cdots$$

$$\frac{1}{1-3x} = 1 + 3x + 3^2x^2 + 3^3x^3 + \cdots\cdots + 3^nx^n + \cdots\cdots$$

$$\therefore\ s = 3 + (1 + 2 + 3)x + (1 + 2^2 + 3^2)x^2 + \cdots\cdots$$
$$+ (1 + 2^n + 3^n)x^n + \cdots\cdots$$

$\therefore$ The general term is $(1 + 2^n + 3^n)x^n$.

Find the sum of n terms of:

8. $2 + 5 + 10 + 17 + 26 + 37 + 50 + \cdots\cdots$

First differences	3		5		7		9		11		.....
Second differences,		2		2		2		2		.....	
Third differences,			0		0		0		.....		

$$\therefore\ a = 2,\ b = 3,\ c = 2,\ d = 0$$

$$\therefore\ s_n = na + \frac{n(n-1)}{2}b + \frac{n(n-1)(n-2)}{6}$$

$$= 2n + \frac{3(n^2-n)}{2} + \frac{n(n-1)(n-2)}{3}$$

$$= \frac{2n^3 + 3n^2 + 7n}{6}$$

9. $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + \cdots\cdots$

$$= 1 + 8 + 27 + 64 + 125 + \cdots\cdots$$

First differences,	7		19		37		61		91		.....
Second differences,		12		18		29		30		.....	
Third differences,			6		6		6				
Fourth differences,				0		0					

$$\therefore\ a = 1,\ b = 7,\ c = 12,\ d = 6,\ e = 0$$

$$\therefore\ Sn = n + \frac{7n(n-1)}{2} + \frac{12n(n-1)(n-2)}{6}$$
$$+ \frac{6n(n-1)(n-2)(n-3)}{24}$$

$$= \frac{6n^4 + 12n^3 + 6n^2}{24}$$

$$= \frac{n^2(n+1)^2}{4}$$

Exercise 63.

1. Show that the infinite series

$$\frac{1}{1\times 2}-\frac{1}{2\times 2^2}+\frac{1}{3\times 2^3}-\frac{1}{4\times 2^4}+\cdots\cdots$$

is convergent, and find its sum.

The series may be obtained from the series

$$\log_e(1+y)=y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{4}+\cdots\cdots$$

by putting $y=\frac{1}{2}$

$$\therefore \log_e 2=\frac{1}{1\times 2}-\frac{1}{2.2^2}+\frac{1}{3\times 2^3}-\frac{1}{4\times 2^4}+\cdots\cdots$$

The series is convergent. For the terms are alternately positive and negative and are continually decreasing, and the limit of the nth term is 0.

2. Find the limit which $\sqrt[n]{1+nx}$ approaches as n approaches 0 as a limit.

Let $$m=\frac{1}{n}$$

Then $$\sqrt[n]{1+nx}=(1+ny)^{\frac{1}{n}}$$
$$=\left(1+\frac{x}{m}\right)^m$$

As n approaches 0, m increases indefinitely.

But $$\underset{m\text{ infinite}}{\text{limit}}\left(1+\frac{x}{m}\right)^m=e^x$$
$$\therefore \underset{n\doteq 0}{\text{limit}}\sqrt[n]{1+nx}=e^x$$

3. Prove that $\frac{1}{e}=2\left(\frac{1}{\lfloor\underline{3}}+\frac{2}{\lfloor\underline{5}}+\frac{3}{\lfloor\underline{7}}+\cdots\cdots\right)$

Consider the series

$$e^x=1+x+\frac{x^2}{\lfloor\underline{2}}+\frac{x^3}{\lfloor\underline{3}}+\cdots\cdots$$

put $x=-1$

$$e^{-1}=1-1+\frac{1}{\lfloor\underline{2}}-\frac{1}{\lfloor\underline{3}}+\cdots\cdots$$

$$\therefore \frac{1}{e}=\frac{1}{\underline{|2}}-\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}-\frac{1}{\underline{|5}}+\frac{1}{\underline{|6}}-\frac{1}{\underline{|7}}+\cdots\cdots$$

$$=\frac{3}{\underline{|3}}-\frac{1}{\underline{|3}}+\frac{5}{\underline{|5}}-\frac{1}{\underline{|5}}+\frac{7}{\underline{|7}}-\frac{1}{\underline{|7}}+\cdots\cdots$$

$$=\frac{2}{\underline{|3}}+\frac{4}{\underline{|5}}+\frac{6}{\underline{|7}}+\cdots\cdots$$

$$=2\left(\frac{1}{\underline{|3}}+\frac{2}{\underline{|5}}+\frac{3}{\underline{|7}}+\cdots\cdots\right)$$

4. Calculate to four places, log 4, log 5, log 6, log 7.

$$\log_e 4=2\log_e 2=1.3863$$

$$\log_e 5=\log_e 4+\frac{2}{9}+\frac{2}{3\times 9^3}+\frac{2}{5\times 9^5}+\cdots\cdots$$

$$=1.6094$$

$$\log_e 6=\log_e 5+\frac{2}{11}+\frac{2}{3\times 11^3}+\frac{2}{5\times 11^5}+\cdots\cdots$$

$$=1.7918=\log_e 2+\log_e 3$$

$$\log_e 7=\log_e 6+\frac{2}{13}+\frac{2}{3\times 13^3}+\frac{2}{5\times 13^5}$$

$$=1.9459$$

5. Find to four places the moduli of the systems of which the bases are: 2, 3, 4, 5, 6, 7.

$$\log_2 e=\frac{1}{\log^e 2}=\frac{1}{0.693147}=1.4427$$

$$\log_3 e=\frac{1}{\log_e 3}=\frac{1}{1.098612}=0.9102$$

$$\log_4 e=\frac{1}{\log_e 4}=\frac{1}{2\log_e 2}=0.7213$$

$$\log_5 e=\frac{1}{\log_e 5}=\frac{1}{1.609432}=0.6213$$

$$\log_6 e=\frac{1}{\log_e 6}=\frac{1}{1.791759}=0.5581$$

$$\log_7 e=\frac{1}{\log_e 7}=\frac{1}{1.9459}=0.5139$$

6. Show that

$$\log_e\left(\frac{8}{e}\right)=\frac{5}{1\times2\times3}+\frac{7}{3\times4\times5}+\frac{9}{5\times6\times7}+\cdots\cdots$$

$$\frac{5}{1\times2\times3}+\frac{7}{3\times4\times5}+\frac{9}{5\times6\times7}+\cdots\cdots$$

$$=\frac{3+2}{1\times2\times3}+\frac{5+2}{3\times4\times5}+\frac{7+2}{5\times6\times7}+\cdots\cdots$$

$$=\frac{1}{1\times2}+\frac{1}{3\times4}+\frac{1}{5\times6}+\cdots\cdots$$

$$+2\left(\frac{1}{1\times2\times3}+\frac{1}{3\times4\times5}+\frac{1}{5\times6\times7}+\cdots\cdots\right)$$

$$=(1-\tfrac{1}{2})+(\tfrac{1}{3}-\tfrac{1}{4})+(\tfrac{1}{5}-\tfrac{1}{6})+\cdots\cdots$$

$$+2\left(\frac{1}{1\times2\times3}+\frac{1}{3\times4\times5}+\frac{1}{5\times6\times7}+\cdots\cdots\right)$$

$$=\log2+2\left(\frac{3-2}{1\times2\times3}+\frac{5-4}{3\times4\times5}+\frac{7-6}{5\times6\times7}+\cdots\cdots\right)$$

$$=\log2+2\left(\frac{1}{1\times2}+\frac{1}{3\times4}+\frac{1}{5\times6}+\cdots\cdots\right)$$

$$-2\left(\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\cdots\cdots\right)$$

$$=3\log2-\left(\frac{2}{1\times3}+\frac{2}{3\times5}+\frac{2}{5\times7}+\cdots\cdots\right)$$

$$=\log8-[(1-\tfrac{1}{3})+(\tfrac{1}{3}-\tfrac{1}{5})+(\tfrac{1}{5}-\tfrac{1}{7})+\cdots\cdots]$$

$$=\log8-1$$

$$=\log8-\log e$$

$$=\log\left(\frac{8}{e}\right)$$

7. Show that

$$\log_e a-\log_e b=\frac{a-b}{a}+\frac{1}{2}\left(\frac{a-b}{a}\right)^2+\frac{1}{3}\left(\frac{a-b}{a}\right)^3+\cdots\cdots$$

$$\log_e a-\log_e b=\log\frac{a}{b}=-\log\frac{b}{a}=-\log\left[1-\left(1-\frac{b}{a}\right)\right]$$

$$=-\log\left(1+\frac{b-a}{a}\right)$$

But $\log\left(1+\frac{b-a}{a}\right)=\frac{b-a}{a}-\frac{1}{2}\left(\frac{b-a}{a}\right)^2+\frac{1}{3}\left(\frac{b-a}{a}\right)^3+\cdots\cdots$

$$\therefore\ \log_e a-\log_e b=\frac{a-b}{a}+\frac{1}{2}\left(\frac{a-b}{a}\right)^2+\frac{1}{3}\left(\frac{a-b}{a}\right)^3+\cdots\cdots$$

8. Show that, if x is positive,

$$x+\frac{1}{x}-\frac{1}{2}\left(x^2+\frac{1}{x^2}\right)+\frac{1}{3}\left(x^3+\frac{1}{x^3}\right)-\cdots\cdots=\log_e\left(2+x+\frac{1}{x}\right).$$

$$\log\left(2+x+\frac{1}{x}\right)=\log\frac{x^2+2x+1}{x}=\log\left[(x+1)\frac{x+1}{x}\right]$$

$$=\log(x+1)+\log\left(1+\frac{1}{x}\right)$$

But $\log(1+x)=x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+\cdots\cdots$

and $\quad\log\left(1+\dfrac{1}{x}\right)=\dfrac{1}{x}-\dfrac{1}{2x^2}+\dfrac{1}{3x^3}-\dfrac{1}{4x^4}+\cdots\cdots$

$$\therefore\ \log\left(2+x+\frac{1}{x}\right)=x+\frac{1}{x}-\frac{1}{2}\left(x^2+\frac{1}{x^2}\right)+\frac{1}{3}\left(x^3+\frac{1}{x^3}\right)$$

$$-\frac{1}{4}\left(x^4+\frac{1}{x^4}\right)+\cdots\cdots$$

The series is, however, convergent only when $x=1$.

9. Show that $1+\dfrac{2^3}{\lfloor\!\underline{2}}+\dfrac{3^3}{\lfloor\!\underline{3}}+\dfrac{4^3}{\lfloor\!\underline{4}}\cdots\cdots=5e.$

$$1+\frac{2^3}{\lfloor\!\underline{2}}+\frac{3^3}{\lfloor\!\underline{3}}+\frac{4^3}{\lfloor\!\underline{4}}+\frac{5^3}{\lfloor\!\underline{5}}+\frac{6^3}{\lfloor\!\underline{6}}$$

$$=1+2^2+\frac{3^2}{\lfloor\!\underline{2}}+\frac{4^2}{\lfloor\!\underline{3}}+\frac{5^2}{\lfloor\!\underline{4}}+\frac{6^2}{\lfloor\!\underline{5}}$$

$$=1+1+3+\frac{8+1}{\lfloor\!\underline{2}}+\frac{15+1}{\lfloor\!\underline{3}}+\frac{24+1}{\lfloor\!\underline{4}}+\frac{35+1}{\lfloor\!\underline{5}}+\cdots\cdots$$

$$=e+3+\frac{8}{\lfloor\!\underline{2}}+\frac{15}{\lfloor\!\underline{3}}+\frac{24}{\lfloor\!\underline{4}}+\frac{35}{\lfloor\!\underline{5}}+\cdots\cdots$$

$$=e+3+4+\frac{5}{\lfloor\!\underline{2}}+\frac{6}{\lfloor\!\underline{3}}+\frac{7}{\lfloor\!\underline{4}}+\cdots\cdots$$

$$=e+3e+1+\frac{2}{\lfloor\!\underline{2}}+\frac{3}{\lfloor\!\underline{3}}+\frac{4}{\lfloor\!\underline{4}}+\cdots\cdots$$

$$=4e+1+1+\frac{1}{\lfloor\!\underline{2}}+\frac{1}{\lfloor\!\underline{3}}+\cdots\cdots$$

$$=5e$$

10. Show that $e^{x\sqrt{-1}}=X+Y\sqrt{-1}$ where

$$X=1-\frac{x^2}{\lfloor\!\underline{2}}+\frac{x^4}{\lfloor\!\underline{4}}-\frac{x^6}{\lfloor\!\underline{6}}+\cdots\cdots,\quad Y=x-\frac{x^3}{\lfloor\!\underline{3}}+\frac{x^5}{\lfloor\!\underline{5}}-\frac{x^7}{\lfloor\!\underline{7}}+\cdots\cdots$$

$$e^{x\sqrt{-1}} = 1 + x\sqrt{-1} + \frac{(x\sqrt{-1})^2}{\underline{|2}} + \frac{(x\sqrt{-1})^3}{\underline{|3}} + \frac{(x\sqrt{-1})^4}{\underline{|4}}$$

$$+ \frac{(x\sqrt{-1})^5}{\underline{|5}} + \frac{(x\sqrt{-1})^6}{\underline{|6}} + \cdots\cdots$$

$$= 1 + x\sqrt{-1} - \frac{x^2}{\underline{|2}} - \frac{x^3\sqrt{-1}}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \frac{x^5\sqrt{-1}}{\underline{|5}} - \frac{x^6}{\underline{|6}} + \cdots\cdots$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \cdots\cdots$$

$$+ \sqrt{-1}\left(x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \cdots\cdots\right)$$

11. Expand $\frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2}$ in ascending powers of x.

$$e^{x\sqrt{-1}} = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} - \cdots\cdots + \sqrt{-1}\left(x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \cdots\cdots\right)$$

$$e^{-x\sqrt{-1}} = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} - \cdots\cdots + \sqrt{-1}\left(-x + \frac{x^3}{\underline{|3}} - \frac{x^5}{\underline{|5}} - \cdots\cdots\right)$$

$$\therefore \frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2} = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \cdots\cdots$$

12. Expand $\frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}}$ in ascending powers of x.

$$e^{x\sqrt{-1}} = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} - \cdots\cdots + \sqrt{-1}\left(x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \cdots\cdots\right)$$

$$e^{-x\sqrt{-1}} = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} - \cdots\cdots + \sqrt{-1}\left(-x + \frac{x^3}{\underline{|3}} - \frac{x^5}{\underline{|5}} + \cdots\cdots\right)$$

$$\therefore \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}} = \frac{\sqrt{-1}}{\sqrt{-1}}\left(x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \cdots\cdots -\right)$$

$$= x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \cdots\cdots -$$

Exercise 64.

Prove the following relations by expanding:

1. $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \equiv \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \equiv - \begin{vmatrix} a_2 & a_1 \\ b_2 & b_1 \end{vmatrix} \equiv \begin{vmatrix} b_2 & b_1 \\ a_2 & a_1 \end{vmatrix}.$

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 \qquad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

$$-\begin{vmatrix} a_2 & a_1 \\ b_2 & b_1 \end{vmatrix} \begin{aligned} &= -(a_2b_1 - a_1b_2) \\ &= a_1b_2 - a_2b_1 \end{aligned} \qquad \begin{vmatrix} b_2 & b_1 \\ a_2 & a_1 \end{vmatrix} \begin{aligned} &= b_2a_1 - a_2b_1 \\ &= a_1b_2 - a_2b_1 \end{aligned}$$

2. $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \equiv \begin{vmatrix} a_3 & a_2 & a_1 \\ c_3 & c_2 & c_1 \\ b_3 & b_2 & b_1 \end{vmatrix} \equiv - \begin{vmatrix} b_1 & c_1 & a_1 \\ b_3 & c_3 & a_3 \\ b_2 & c_2 & a_2 \end{vmatrix}.$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1$$

$$\begin{vmatrix} a_3 & a_2 & a_1 \\ c_3 & c_2 & c_1 \\ b_3 & b_2 & b_1 \end{vmatrix} \begin{aligned} &= a_3c_2b_1 + a_2b_3c_1 + a_1c_3b_2 - a_3c_1b_2 - a_2c_3b_1 - a_1c_2b_3 \\ &= a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1 \end{aligned}$$

$$-\begin{vmatrix} b_1 & c_1 & a_1 \\ b_3 & c_3 & a_3 \\ b_2 & c_2 & a_2 \end{vmatrix} \begin{aligned} &= -(b_1c_3a_2 + b_2c_1a_3 + c_2b_3a_1 - c_2b_1a_3 - a_2b_3c_1 - a_1b_2c_3) \\ &= a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1 \end{aligned}$$

3. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 4 \\ 3 & 4 & 5 \end{vmatrix}.$

In the expansion of

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \text{ put } a_1 = 1,\ a_2 = 2,\ a_3 = 3,\ b_1 = 2, \text{ etc.}$$

$$\therefore \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 4 \\ 3 & 4 & 5 \end{vmatrix} \begin{aligned} &= 1 \times 4 \times 5 + 2 \times 4 \times 3 + 3 \times 2 \times 4 - 1 \times 4 \times 4 \\ &\qquad - 2 \times 2 \times 5 - 3 \times 4 \times 3 \\ &= -4 \end{aligned}$$

4. $\begin{vmatrix} 3 & 2 & 4 \\ 7 & 6 & 1 \\ 5 & 3 & 8 \end{vmatrix} \begin{aligned} &= 3 \times 6 \times 8 + 2 \times 1 \times 5 + 4 \times 7 \times 3 - 3 \times 3 \times 1 \\ &\qquad - 2 \times 7 \times 8 - 4 \times 6 \times 5 \\ &= -3 \end{aligned}$

5. $\begin{vmatrix} 4 & 5 & 2 \\ -1 & 2 & -3 \\ 6 & -4 & 5 \end{vmatrix} \begin{aligned} &= 4 \times 2 \times 5 - 5 \times 3 \times 6 + 2 \times 1 \times 4 - 4 \times 4 \times 3 \\ &\qquad + 5 \times 1 \times 5 - 2 \times 2 \times 6 \\ &= -89 \end{aligned}$

6. Count the inversions in the series:

5 4 1 3 2. 7 5 1 4 3 6 2. *d a c e b.*
4 1 5 2 3. 6 5 4 2 1 3 7. *c e b d a.*

5 4 1 3 2 has 8 inversions, 54, 51, 53, 52, 41, 43, 42, 32.
4 1 5 2 3 has 5 inversions, 41, 42, 43, 52, 53.
7514362 has 13 inversions, 75, 71, 74, 73, 76, 72, 51, 54, 53, 52, 43, 42, 62.
6542137 has 13 inversions, 65, 64, 62, 61, 53, 54, 52, 51, 53, 42, 41, 43, 21.
d a c e b has 5 inversions, *da, dc, db, cb, eb.*
c e b d a has 7 inversions, *cb, ca, eb, ed, ea, ba, da.*

7. In the determinant $|a_1b_2c_3d_4e_5|$ find the signs of the following terms.

$a_1b_4c_5d_3e_2$: $a_5b_1c_3d_4e_2$. $e_1c_4a_2b_5d_3$.
$a_2b_5c_3d_1e_4$. $b_4c_5a_1e_3d_2$. $c_1a_5b_3e_4d_2$.

$a_1b_4c_5d_3e_2$ has 5 subscript inversions, 43, 42, 53, 52, 32.
∴ Its sign is $-$.

$a_2b_5c_3d_1e_4$ has 5 subscript inversions, 21, 53, 51, 54, 31.
∴ Its sign is $-$.

$a_5b_1c_3d_4e_2$ has 6 subscript inversions, 51, 53, 54, 52, 32, 42.
∴ Its sign is $+$.

$$b_4c_5a_1e_3d_2 = a_1b_4c_5d_2e_3$$

$a_1b_4c_5d_2e_3$ has 4 subscript inversions, 42, 43, 52, 53.
∴ Its sign is $+$.

$$e_1c_4a_2b_5d_3 = a_2b_5c_4d_3e_1$$

$a_2b_5c_4d_3e_1$ has 7 subscript inversions, 21, 54, 53, 51, 43, 41, 31.
∴ Its sign is $-$.

$$c_1a_5b_3e_4d_2 = a_5b_3c_1d_2e_4$$

$a_5b_3c_1d_2e_4$ has 6 subscript inversions, 53, 51, 52, 54, 31, 32.
∴ Its sign is $+$

8. Write, with their proper signs, all the terms of the determinant $|a_1b_2c_3d_4|$.

$$\begin{aligned}&a_1b_2c_3d_4 - a_1b_2c_4d_3 - a_1b_3c_2d_4 + a_1b_3c_4d_2 + a_1b_4c_2d_3 - a_1b_4c_3d_2\\ &- a_2b_1c_3d_4 + a_2b_1c_4d_3 + a_2b_3c_1d_4 - a_2b_3c_4d_1 - a_2b_4c_1d_3 + a_2b_4c_3d_1\\ &+ a_3b_1c_2d_4 - a_3b_1c_4d_2 - a_3b_2c_1d_4 + a_3b_2c_4d_1 + a_3b_4c_1d_2 - a_3b_4c_2d_1\\ &- a_4b_1c_2d_3 + a_4b_1c_3d_2 + a_4b_2c_1d_3 - a_4b_2c_3d_1 - a_4b_3c_1d_2 + a_4b_3c_2d_1\end{aligned}$$

9. Write, with their proper signs, all the terms of the determinant $|a_1b_2c_3d_4e_5|$ which contain both a_1 and b_4; all the terms which contain both b_3 and e_5.

(1) $a_1b_4(c_2d_3e_5 - c_2d_5e_3 - c_3d_2e_5 + c_3d_5e_2 + c_5d_2e_3 - c_5d_3e_2)$

(2) $b_3e_5(-a_1c_2d_4 + a_1c_4d_2 + a_2c_1d_4 - a_2c_4d_1 - a_4c_1d_2 + a_4c_2d_1)$

10. $\begin{vmatrix} a & b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & a & a & b \\ 0 & b & b & a \end{vmatrix}.$

Since a_3, a_4, b_3, b_4, c_1, and d_1 are here all 0, the formula of Example 8 reduces to

$$\begin{aligned} a_1b_2c_3d_4 - a_1b_2c_4d_3 - a_2b_1c_3d_4 + a_2b_1c_4d_3 &= (a_1b_2 - a_2b_1)(c_3d_4 - c_4d_3) \\ &= (a^2 - b^2)(a^2 - b^2) \\ &= (a^2 - b^2)^2. \end{aligned}$$

11. $\begin{vmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ a & a & b & b \\ b & b & a & a \end{vmatrix}.$

The formula of Example 8 reduces to

$$-a_4b_3c_1d_2 + a_4b_3c_2d_1 = -a^2b^2 + a^2b^2 = 0.$$

12. $\begin{vmatrix} a & b & c & 0 \\ c & a & b & 0 \\ b & c & a & 0 \\ a & b & c & 1 \end{vmatrix}.$

The formula of Example 8 reduces to

$$\begin{aligned} a_1b_2c_3d_4 - a_1b_3c_2d_4 - a_2b_1c_3d_4 + a_2b_3c_1d_4 + a_3b_1c_2d_4 - a_3b_2c_1d_4 & \\ = a^3 - abc - bca + b^3 + c^3 - cab & \\ = a^3 + b^3 + c^3 - 3abc & \end{aligned}$$

Exercise 65.

Show that:

1. $\begin{vmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{vmatrix} \equiv 2abc.$

$$\begin{vmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{vmatrix} = abc + abc = 2abc.$$

2. $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} \equiv 4abc.$

$$\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} \begin{aligned} &= (b+c)(c+a)(a+b) + abc + abc - (b+c)bc \\ &\qquad - ab(a+b) - ac(c+a) \\ &= 4abc \end{aligned}$$

3. $$\begin{vmatrix} 1 & a^2 & a^3 & a^4 \\ 1 & b^2 & b^3 & b^4 \\ 1 & c^2 & c^3 & c^4 \\ 1 & d^2 & d^3 & d^4 \end{vmatrix} \equiv \begin{vmatrix} bcd & a & a^2 & a^3 \\ cda & b & b^2 & b^3 \\ dab & c & c^2 & c^3 \\ abc & d & d^2 & d^3 \end{vmatrix}.$$

$$\begin{vmatrix} 1 & a^2 & a^3 & a^4 \\ 1 & b^2 & b^3 & b^4 \\ 1 & c^2 & c^3 & c^4 \\ 1 & d^2 & d^3 & d^4 \end{vmatrix} = abcd \begin{vmatrix} \frac{1}{a} & a & a^2 & a^3 \\ \frac{1}{b} & b & b^2 & b^3 \\ \frac{1}{c} & c & c^2 & c^3 \\ \frac{1}{d} & d & d^2 & d^3 \end{vmatrix} = \begin{vmatrix} bcd & a & a^2 & a^3 \\ acd & b & b^2 & b^3 \\ abd & c & c^2 & c^3 \\ abc & d & d^2 & d^3 \end{vmatrix}$$

Divide the first row by a, the second by b, the third by c, and the fourth by d. Then multiply the first column by $abcd$.

4. $$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 \\ 1 & c^2 & 0 & a^2 \\ 1 & b^2 & a^2 & 0 \end{vmatrix} \equiv \begin{vmatrix} 0 & a & b & c \\ a & 0 & c & b \\ b & c & 0 & a \\ c & b & a & 0 \end{vmatrix}.$$

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 \\ 1 & c^2 & 0 & a^2 \\ 1 & b^2 & a^2 & 0 \end{vmatrix} = \frac{1}{abc} \begin{vmatrix} 0 & a & b & c \\ 1 & 0 & bc^2 & cb^2 \\ 1 & ac^2 & 0 & ca^2 \\ 1 & ab^2 & ba^2 & 0 \end{vmatrix} = abc \begin{vmatrix} 0 & a & b & c \\ \frac{1}{bc} & 0 & c & b \\ \frac{1}{ac} & c & 0 & a \\ \frac{1}{ab} & b & a & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & a & b & c \\ a & 0 & c & b \\ b & c & 0 & a \\ c & b & a & 0 \end{vmatrix}$$

Multiply the second column by a, the third by c, and the fourth by d. Then divide the second row by bc, the third by ac, and the fourth by ab. Then multiply the first column by abc.

Find the value of:

5. $$\begin{vmatrix} 20 & 15 & 25 \\ 17 & 12 & 22 \\ 19 & 20 & 16 \end{vmatrix} = \tfrac{1}{2} \begin{vmatrix} 0 & 15 & 25 \\ 0 & 12 & 22 \\ 2 & 20 & 16 \end{vmatrix} = \begin{vmatrix} 0 & 15 & 25 \\ 0 & 12 & 22 \\ 1 & 20 & 16 \end{vmatrix} = \begin{vmatrix} 0 & 15 & 10 \\ 0 & 12 & 10 \\ 1 & 20 & -4 \end{vmatrix}$$

$$= 15 \times 10 - 12 \times 10 = 30$$

Multiply the first column by 2, and subtract from it the sum of the second and third columns.

Take out the factor 2, and subtract the second column from the third.

6. $\begin{vmatrix} 3 & 23 & 13 \\ 7 & 53 & 30 \\ 9 & 70 & 39 \end{vmatrix} = \begin{vmatrix} 3 & 23 & 13 \\ 1 & 27 & 4 \\ 0 & 1 & 0 \end{vmatrix} = -3 \times 4 + 13 = 1$

Subtract twice the first row from the second, and three times the first row from the third.

7. $\begin{vmatrix} 22 & 29 & 27 \\ 25 & 23 & 30 \\ 28 & 26 & 24 \end{vmatrix} = 2 \times 3 \begin{vmatrix} 22 & 29 & 9 \\ 25 & 23 & 10 \\ 14 & 13 & 4 \end{vmatrix} = 2 \times 3 \begin{vmatrix} 4 & 2 & 9 \\ 5 & -7 & 10 \\ 6 & 1 & 4 \end{vmatrix}$

$$= 2 \times 3 \begin{vmatrix} -8 & 2 & 1 \\ 47 & -7 & 38 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= 2 \times 3 (8 \times 38 + 47) = 2106$$

Take out the factor 3 from the third column and the factor 2 from the third row.

Then subtract twice the third column from the first, and then twice the third column from the second.

Then subtract six times the second column from the first and four times the second column from the third.

8. $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$.

$(a-b)(b-c)(c-a)$ is evidently a factor.

There is also another factor of the first degree, which is therefore of the form

$$la + mb + nc$$

In the product $(a-b)(b-c)(c-a)(la+mb+nc)$ the coefficients of a^3, b^3, and c^3 are

$$-l(b-c), \quad -m(c-a), \quad -n(a-b)$$

But in the expansion of the determinant the coefficients of a^3, b^3, and c^3 are

$$c-b, \qquad a-c, \qquad b-a$$

$$\therefore\ l = 1, \qquad m = 1, \qquad n = 1$$

$$\therefore \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$$

9. $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}$.

$(a-b)(b-c)(c-a)$ is a factor.

There is also another factor of the second degree.

$$\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} a-c & a^2-c^2 & bc-ab \\ b-c & b^2-c^2 & ca-ab \\ c & c^2 & ab \end{vmatrix}$$

$$= (a-c)(b-c)\begin{vmatrix} 1 & a+c & -b \\ 1 & b+c & -a \\ c & c^2 & ab \end{vmatrix}$$

$$= (a-c)(b-c)\begin{vmatrix} 0 & a-b & a-b \\ 1 & b+c & -a \\ c & c^2 & ab \end{vmatrix}$$

$$= (a-b)(b-c)(a-c)\begin{vmatrix} 0 & 1 & 1 \\ 1 & b+c & -a \\ c & c^2 & ab \end{vmatrix}$$

$$= (a-b)(b-c)(a-c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & a+b+c & -a \\ c & c^2-ab & ab \end{vmatrix}$$

$$= (a-b)(b-c)(a-c)[c^2-ab-(a+b+c)c]$$

$$= (a-b)(b-c)(a-c)(-ab-ac-bc)$$

$$= (a-b)(b-c)(c-a)(ab+bc+ac)$$

Subtract the third row from the first and second.

Remove the factors $a-c$ and $b-c$ from the first and second rows.

Subtract the second row from the first.

Remove the factor $a-b$ from the first row.

Subtract the third column from the second.

10. $\begin{vmatrix} a^3 & bc & 1 \\ b^3 & ca & 1 \\ c^3 & ab & 1 \end{vmatrix}$. $(a-b)(b-c)(c-a)$ is a factor.

There is also another factor of the second degree.

$$\begin{vmatrix} a^3 & bc & 1 \\ b^3 & ac & 1 \\ c^3 & ab & 1 \end{vmatrix} = \begin{vmatrix} a^3-c^3 & bc-ab & 0 \\ b^3-c^3 & ac-ab & 0 \\ c^3 & ab & 1 \end{vmatrix}$$

$$= (a-c)(b-c)\begin{vmatrix} a^2+ac+c^2 & -b & 0 \\ b^2+bc+c^2 & -a & 0 \\ c^3 & ab & 1 \end{vmatrix}$$

$$= (a-c)(b-c)\begin{vmatrix} a^2-b^2+ac-bc & a-b & 0 \\ b^2+bc+c^2 & -a & 0 \\ c^3 & ab & 1 \end{vmatrix}$$

$$= (a-b)(b-c)(a-c)\begin{vmatrix} a+b+c & 1 & 0 \\ b^2+bc+c^2 & -a & 0 \\ c^3 & ab & 1 \end{vmatrix}$$

$$= (a-b)(b-c)(a-c)[-(a+b+c)a-(b^2+bc+c^2)]$$

$$= (a-b)(b-c)(c-a)(a^2+b^2+c^2+ab+ac+bc)$$

Subtract the third row from the first and second.

Remove the factors $a-c$ and $b-c$ from the first and second rows.

Subtract the second row from the first.

Remove the factor $a-b$ from the first row.

11. $$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \begin{vmatrix} a+b+c & bc \\ a+b+c & ca \\ a+b+c & ab \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix} \begin{aligned} &= (a+b+c)(cb-a^2-b^2+ac+ab-c^2) \\ &= -(a+b+c)(a^2+b^2+c^2-ab-ac-bc) \end{aligned}$$

Add the second and third rows to the first. Remove the factor $a+b+c$ from the first column.

12. $$\begin{vmatrix} 1 & a & a^2 & a^4 \\ 1 & b & b^2 & b^4 \\ 1 & c & c^2 & c^4 \\ 1 & d & d^2 & d^4 \end{vmatrix}.$$

$(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$ is a factor

There is also another factor of the first degree.

$$\begin{vmatrix} 1 & a & a^2 & a^4 \\ 1 & b & b^2 & b^4 \\ 1 & c & c^2 & c^4 \\ 1 & d & d^2 & d^4 \end{vmatrix} = \begin{vmatrix} 0 & a-d & a^2-d^2 & a^4-d^4 \\ 0 & b-d & b^2-d^2 & b^4-d^4 \\ 0 & c-d & c^2-d^2 & c^4-d^4 \\ 1 & d & d^2 & d^4 \end{vmatrix} = (a-d)(b-d)(c-d)\begin{vmatrix} 0 & 1 & a+d & a^3+a^2d+ad^2+d^3 \\ 0 & 1 & b+d & b^3+b^2d+bd^2+d^3 \\ 0 & 1 & c+d & c^3+c^2d+cd^2+d^3 \\ 1 & d & d^2 & d^4 \end{vmatrix}$$

$$= (a-d)(b-d)(c-d)\begin{vmatrix} 0 & 0 & a-b & a^3-c^3+a^2d-c^2d+ad^2-cd^2 \\ 0 & 0 & b-c & b^3-c^3+b^2d-c^2d+bd^2-cd^2 \\ 0 & 1 & c+d & c^3+c^2d+cd^2+d^3 \\ 1 & d & d^2 & d^4 \end{vmatrix}$$

$$= (a-d)(b-d)(c-d)(a-c)(b-c)\begin{vmatrix} 0 & 0 & 1 & a^2+ac+c^2+ad+cd+d^2 \\ 0 & 0 & 1 & b^2+bc+c^2+bd+cd+d^2 \\ 0 & 1 & c+d & c^3+c^2d+cd^2+d^3 \\ 1 & d & d^2 & d^4 \end{vmatrix}$$

$$= (a-d)(b-d)(c-d)(a-c)(b-c)\begin{vmatrix} 0 & 0 & 0 & a^2-b^2+ac-bc+ad-bd \\ 0 & 0 & 1 & b^2+bc+c^2+bd+cd+d^2 \\ 0 & 1 & c+d & c^3+c^2d+cd^2+d^3 \\ 1 & d & d^2 & d^4 \end{vmatrix}$$

$$= (a-d)(b-d)(c-d)(a-c)(b-c)(a-b)\begin{vmatrix} 0 & 0 & 0 & a+b+c+d \\ 0 & 0 & 1 & b^2+bc+c^2+bd+cd+d^2 \\ 0 & 1 & c+d & c^3+c^2d+cd^2+d^3 \\ 1 & d & d^2 & d^4 \end{vmatrix}$$

The formula of Example 8, Exercise 64, reduces in this case to

$$a_4b_3c_2d_1 = (a+b+c+d)$$

$$\therefore \begin{vmatrix} 1 & a & a^2 & a^4 \\ 1 & b & b^2 & b^4 \\ 1 & c & c^2 & c^4 \\ 1 & d & d^2 & d^4 \end{vmatrix}$$

$$= (a-d)(b-d)(c-d)(a-c)(b-c)(a-b)(a+b+c+d)$$

$$= (a-b)(a-c)(a-d)(b-c)(b-d)(c-d)(a+b+c+d)$$

Subtract the fourth row from the first, second, and third.

Remove the factors $a-d$, $b-d$, and $c-d$ from the first, second, and third rows.

Subtract the third row from the first and second.

Remove the factors $a-c$ and $b-c$ from the first and second rows.

Subtract the second row from the first.

Remove the factor $a-b$ from the first row.

13. $\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix}$

$$= \begin{vmatrix} a+b+c+d & a+b+c+d & a+b+c+d & a+b+c+d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix}$$

$$= (a+b+c+d)\begin{vmatrix} 1 & 1 & 1 & 1 \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix}$$

$$= (a+b+c+d)\begin{vmatrix} 0 & 1 & 1 & 1 \\ a+b-c-d & a & d & c \\ c+d-a-b & d & a & b \\ c+d-a-b & c & b & a \end{vmatrix}$$

$$= (a+b+c+d)(a+b-c-d)\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & a & d & c \\ -1 & d & a & b \\ -1 & c & b & a \end{vmatrix}$$

This result was obtained by the following operations:

Add the second, third, and fourth rows to the first.

Remove the factor $a+b+c+d$ from the first row.

Add the second column to the first, and subtract the third and fourth columns from the result.

Remove the factor $a+b-c-d$ from the first column.

It appears that $a+b+c+d$ and $a+b-c-d$ are factors of the determinant. Similarly we have:

$$\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix} = \begin{vmatrix} a+c-b-d & b & c & d \\ b+d-a-c & a & d & c \\ a+c-b-d & d & a & b \\ b+d-a-c & c & b & a \end{vmatrix}$$

$$= (a+c-b-d)\begin{vmatrix} 1 & b & c & d \\ -1 & a & d & c \\ 1 & d & a & b \\ -1 & c & b & a \end{vmatrix}$$

Add the third column to the first, and subtract the second and fourth columns from the result.

Remove the factor $a+c-b-d$ from the first column.

$\therefore$ $a+c-b-d$ is a factor of the determinant.

Also

$$\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix} = \begin{vmatrix} a+d-b-c & b & c & d \\ b+c-a-d & a & d & c \\ b+c-a-d & d & a & b \\ a+d-b-c & c & b & a \end{vmatrix}$$

$$= (a+d-b-c)\begin{vmatrix} 1 & b & c & d \\ -1 & a & d & c \\ -1 & d & a & b \\ 1 & c & b & a \end{vmatrix}$$

Add the fourth column to the first and subtract the second and third from the result.

Remove the factor $a+d-b-c$ from the first column.

$\therefore$ $a+d-b-c$ is a factor of the determinant.

$\therefore$ $(a+b+c+d)(a+b-c-d)(a-b-c+d)(a-b+c-d)$ is a factor of the determinant.

But the determinant is of the fourth degree in a, b, c, and d.

Hence the determinant is equal to the product multiplied by some number.

In the expansion of the product the coefficient of a^4 is 1.

In the expansion of the determinant the coefficient of a^4 is 1.

$$\therefore \begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix} = (a+b+c+d)(a+b-c-d)(a-b-c+d)(a-b+c-d)$$

14. If all the elements on one side of a diagonal term are zeros, show that the expansion reduces to this term.

Suppose the determinant written in the form

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \cdots\cdots \\ 0 & b_2 & b_3 & b_4 \cdots\cdots \\ 0 & 0 & c_3 & c_4 \cdots\cdots \\ 0 & 0 & 0 & d_4 \cdots\cdots \end{vmatrix}$$

In expanding the determinant we must take one element from each row and one from each column.

From the first column we can only take a_1. From the second column we must take either a_2 or b_2. But we cannot take a_2, since we already have a_1, from the first row. Hence we must take b_2. Similarly in the third column we cannot take a_3 or b_3, and there remains only c_3, etc.

Hence the only term is $a_1b_2c_3d_4 \cdots\cdots$.

15. $$\begin{vmatrix} a^2-bc & a & 1 \\ b^2-ca & b & 1 \\ c^2-ab & c & 1 \end{vmatrix} \equiv \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} - \begin{vmatrix} bc & a & 1 \\ ca & b & 1 \\ ab & c & 1 \end{vmatrix}.$$

In the second determinant multiply the first row by a, the second by b, and the third by c; remove the factor abc from the first column, and transpose the columns.

Then $$\begin{vmatrix} bc & a & 1 \\ ca & b & 1 \\ ab & c & 1 \end{vmatrix} \equiv \frac{1}{abc}\begin{vmatrix} abc & a^2 & a \\ abc & b^2 & b \\ abc & c^2 & c \end{vmatrix} \equiv \begin{vmatrix} 1 & a^2 & a \\ 1 & b^2 & b \\ 1 & c^2 & c \end{vmatrix} \equiv \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}$$

$$\therefore \begin{vmatrix} a^2-bc \\ b^2-ca \\ c^2-ab \end{vmatrix} \equiv \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} - \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} \equiv 0$$

16. $$\begin{vmatrix} a+2b & a+4b & a+6b \\ a+3b & a+5b & a+7b \\ a+4b & a+6b & a+8b \end{vmatrix} \equiv \begin{vmatrix} a+2b & a+4b & a+6b \\ b & b & b \\ b & b & b \end{vmatrix} \equiv 0$$

since two rows are identical.

Subtract the first row from the second, and the second from the third.

17. $$\begin{vmatrix} b^2+c^2 & ba & ca \\ ab & c^2+a^2 & cb \\ ac & bc & a^2+b^2 \end{vmatrix}$$

$$\equiv \frac{1}{abc}\begin{vmatrix} 0 & ba & ca \\ -2bc^2 & c^2+a^2 & cb \\ -2b^2c & bc & a^2+b^2 \end{vmatrix} \equiv -\frac{2bc}{a}\begin{vmatrix} 0 & ba & ca \\ c & c^2+a^2 & cb \\ b & bc & a^2+b^2 \end{vmatrix}$$

$$\equiv -2c\begin{vmatrix} 0 & b & c \\ 0 & a^2b & -a^2c \\ b & bc & a^2+b^2 \end{vmatrix} \equiv -2a^2b^2c\begin{vmatrix} 0 & 1 & c \\ 0 & 1 & -c \\ 1 & c & a^2+b^2 \end{vmatrix} \equiv 4a^2b^2c^2$$

Multiply the first column by a, and subtract from it b times the second column and c times the third.

Remove the factors $-2bc$ and a from the first column and the first row.

Multiply the second row by b, and subtract from it c times the third row.

18. $$\begin{vmatrix} (a+b)^2 & c^2 & c^2 \\ a^2 & (b+c)^2 & a^2 \\ b^2 & b^2 & (c+a)^2 \end{vmatrix}$$

$$\equiv \begin{vmatrix} (a+b)^2-c^2 & 0 & c^2 \\ 0 & (b+c)^2-a^2 & a^2 \\ b^2-(c+a)^2 & b^2-(c+a)^2 & (c+a)^2 \end{vmatrix}$$

$$\equiv (a+b+c)^2 \begin{vmatrix} a+b-c & 0 & c^2 \\ 0 & b+c-a & a^2 \\ b-c-a & b-c-a & (c+a)^2 \end{vmatrix}$$

$$\equiv (a+b+c)^2 \begin{vmatrix} a+b-c & 0 & c^2 \\ 0 & b+c-a & a^2 \\ -2a & -2c & 2ac \end{vmatrix}$$

$$\equiv (a+b+c)^2\{(a+b-c)[(b+c-a)2ac+2a^2c]+2ac^2(b+c-a)\}$$

$$\equiv 2abc\,(a+b+c)^3$$

Subtract the third column from the first and second.

Remove the factor $a+b+c$ from the first and second columns.

Subtract the sum of the first and second rows from the third row.

19. $$\begin{vmatrix} 1+x & 2 & 3 & 4 \\ 1 & 2+x & 3 & 4 \\ 1 & 2 & 3+x & 4 \\ 1 & 2 & 3 & 4+x \end{vmatrix} \equiv \begin{vmatrix} x & 0 & 0 & -x \\ 0 & x & 0 & -x \\ 0 & 0 & x & -x \\ 1 & 2 & 3 & 4+x \end{vmatrix}$$

$$\equiv x^3 \begin{vmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & 2 & 3 & 4+x \end{vmatrix} \equiv x^3 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 2 & 3 & 10+x \end{vmatrix} \equiv x^3(10+x)$$

Subtract the fourth row from each of the others.

Remove the factor x from the first, second, and third rows.

Add the sum of the first three columns to the fourth.

Then all the elements to the right of the diagonals are 0.

20. $$\begin{vmatrix} a^2+1 & ba & ca & da \\ ab & b^2+1 & cb & db \\ ac & bc & c^2+1 & dc \\ ad & bd & cd & d^2+1 \end{vmatrix} \equiv \frac{1}{d^3} \begin{vmatrix} d & 0 & 0 & -a \\ 0 & d & 0 & -b \\ 0 & 0 & d & -c \\ ad & bd & cd & d^2+1 \end{vmatrix}$$

$$\equiv \frac{1}{d^3} \begin{vmatrix} d & 0 & 0 & -a \\ 0 & d & 0 & -b \\ 0 & 0 & d & -c \\ 0 & 0 & 0 & a^2+b^2+c^2+d^2+1 \end{vmatrix} \equiv a^2+b^2+c^2+d^2+1$$

Multiply the first row by d and subtract from it a times the fourth row.

Multiply the second row by d and subtract from it b times the fourth row.

Multiply the third row by d and subtract from it c times the fourth row.

Then multiply the first row by a, the second by b, and the third by c, and subtract their sum from the fourth row.

Then all the elements to the left of the diagonal are 0.

Exercise 66.

1. In the determinant $|a_1\, b_2\, c_3\, d_4|$ write the co-factors of a_3, b_2, b_4, c_1, c_4, d_2, d_3.

Write out the determinant $\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$

Then $A_3 = \begin{vmatrix} b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \\ d_1 & d_2 & d_4 \end{vmatrix}$ $B_2 = \begin{vmatrix} a_1 & a_3 & a_4 \\ c_1 & c_3 & c_4 \\ d_1 & d_3 & d_4 \end{vmatrix}$ $B_4 = \begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$

$C_1 = \begin{vmatrix} a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \\ d_2 & d_3 & d_4 \end{vmatrix}$ $C_4 = -\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$ $D_2 = \begin{vmatrix} a_1 & a_3 & a_4 \\ b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \end{vmatrix}$

$D_3 = -\begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \end{vmatrix}$

2. Express as a single determinant

$$\begin{vmatrix} e & f & g \\ f & h & k \\ g & k & l \end{vmatrix} + \begin{vmatrix} b & e & g \\ c & f & k \\ d & g & l \end{vmatrix} + \begin{vmatrix} b & g & f \\ c & k & h \\ d & l & k \end{vmatrix} + \begin{vmatrix} b & f & e \\ c & h & f \\ d & k & g \end{vmatrix}.$$

The sum may be written:

$$\begin{vmatrix} e & f & g \\ f & h & k \\ g & k & l \end{vmatrix} + \begin{vmatrix} b & e & g \\ c & f & k \\ d & g & l \end{vmatrix} - \begin{vmatrix} b & f & g \\ c & h & k \\ d & k & l \end{vmatrix} - \begin{vmatrix} b & e & f \\ c & f & h \\ d & g & k \end{vmatrix}$$

$$= \begin{vmatrix} e & f & g \\ f & h & k \\ g & k & l \end{vmatrix} - \begin{vmatrix} b & f & g \\ c & h & k \\ d & k & l \end{vmatrix} + \begin{vmatrix} b & e & g \\ c & f & k \\ d & g & l \end{vmatrix} - \begin{vmatrix} b & e & f \\ c & f & h \\ d & g & k \end{vmatrix}$$

These four determinants are evidently the co-factors of the first row of the determinant:

$\begin{vmatrix} 1 & 1 & 1 & 1 \\ b & e & f & g \\ c & f & h & k \\ d & g & k & l \end{vmatrix}$, which is therefore the required sum.

3. Write all the terms of the following determinant which contain a:

$$\begin{vmatrix} a & 0 & b & c & b \\ a & b & c & b & 0 \\ 0 & c & b & c & 0 \\ 0 & 0 & 0 & b & c \\ b & 0 & 0 & c & b \end{vmatrix}.$$

The only terms containing a come from the products of the first and second elements in the first column, with their respective co-factors.

$\therefore$ These terms are:

$$a\begin{vmatrix} b & c & b & 0 \\ c & b & c & 0 \\ 0 & 0 & b & c \\ 0 & 0 & c & b \end{vmatrix} - a\begin{vmatrix} 0 & b & c & b \\ c & b & c & 0 \\ 0 & 0 & b & c \\ 0 & 0 & c & b \end{vmatrix}$$

$$= ab\begin{vmatrix} b & c & 0 \\ 0 & b & c \\ 0 & c & b \end{vmatrix} - ac\begin{vmatrix} c & b & 0 \\ 0 & b & c \\ 0 & c & b \end{vmatrix} + ac\begin{vmatrix} b & c & b \\ 0 & b & c \\ 0 & c & b \end{vmatrix}$$

$$= ab^2(b^2 - c^2) - ac^2(b^2 - c^2) + abc(b^2 - c^2)$$

$$= a(b^2 + bc - c^2)(b^2 - c^2)$$

$$= ab^4 + ab^3c - 2ab^2c^2 - abc^3 + ac^4$$

Expand:

4. $\begin{vmatrix} a & b & b & a \\ b & a & a & b \\ a & a & b & b \\ 0 & a & b & b \end{vmatrix}.$

Subtract the fourth row from the third.

The result is $\begin{vmatrix} a & b & b & a \\ b & a & a & b \\ a & 0 & 0 & 0 \\ 0 & a & b & b \end{vmatrix} = a\begin{vmatrix} b & b & a \\ a & a & b \\ a & b & b \end{vmatrix}$

Subtract the second row from the first, and the third from the second.

The result is $a\begin{vmatrix} b-a & b-a & a-b \\ 0 & a-b & 0 \\ a & b & b \end{vmatrix} = a(a-b)^2\begin{vmatrix} -1 & -1 & 1 \\ 0 & 1 & 0 \\ a & b & b \end{vmatrix}$

$$= a(a-b)^2\begin{vmatrix} -1 & 1 \\ a & b \end{vmatrix} = -a(a-b)^2(a+b)$$

5. $\begin{vmatrix} 0 & d & d & d \\ a & 0 & a & a \\ b & b & 0 & b \\ c & c & c & 0 \end{vmatrix}.$

Multiply the first column by 2, and subtract from it the sum of the other three columns.

The result is $\frac{1}{2}\begin{vmatrix} -3d & d & d & d \\ 0 & 0 & a & a \\ 0 & b & 0 & b \\ 0 & c & c & 0 \end{vmatrix} = \frac{1}{2}abcd\begin{vmatrix} -3 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{vmatrix}$

$$= -\tfrac{3}{2}abcd\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \tfrac{3}{2}abcd\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} - \tfrac{3}{2}abcd\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= -3abcd$$

6. $\begin{vmatrix} 1 & a & a & a \\ 1 & b & a & a \\ 1 & a & b & a \\ 1 & a & a & b \end{vmatrix}.$

Subtract the second row from the first.

The result is $\begin{vmatrix} 0 & a-b & 0 & 0 \\ 1 & b & a & a \\ 1 & a & b & a \\ 1 & a & a & b \end{vmatrix} = -(a-b)\begin{vmatrix} 1 & a & a \\ 1 & b & a \\ 1 & a & b \end{vmatrix}$

Again, subtract the second row from the first.

The result is $-(a-b)\begin{vmatrix} 0 & a-b & 0 \\ 1 & b & a \\ 1 & a & b \end{vmatrix} = (a-b)^2\begin{vmatrix} 1 & a \\ 1 & b \end{vmatrix}$

$$= -(a-b)^3 = (b-a)^3$$

7. $\begin{vmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{vmatrix}.$

Subtract the fourth row from each of the others.

The result is $\begin{vmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 2 & 2 & 2 & 3 \end{vmatrix}$

Add the sum of the first three columns to the fourth.

The result is $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 1 & 9 \end{vmatrix} = 9$

8. $\begin{vmatrix} 3 & 2 & 1 & 4 \\ 15 & 29 & 2 & 14 \\ 16 & 19 & 3 & 17 \\ 33 & 39 & 8 & 38 \end{vmatrix}.$

Subtract twice the first row from the second, 3 times the first row from the third, and 8 times the first row from the fourth.

The result is $\begin{vmatrix}3&2&1&4\\9&25&0&6\\7&15&0&5\\9&23&0&6\end{vmatrix}=\begin{vmatrix}9&25&6\\7&13&5\\9&23&6\end{vmatrix}$

Subtract the third row from the first.

The result is $\begin{vmatrix}0&2&0\\7&13&5\\9&23&6\end{vmatrix}=-2\begin{vmatrix}7&5\\9&6\end{vmatrix}=6$

9. $\begin{vmatrix}2&1&3&4\\7&4&5&9\\3&3&6&2\\1&7&7&5\end{vmatrix}.$

Subtract twice the second column from the first, 3 times the second column from the third, and 4 times the second column from the fourth.

The result is $\begin{vmatrix}0&1&0&0\\-1&4&-7&-7\\-3&3&-3&-10\\-13&7&-14&-23\end{vmatrix}=\begin{vmatrix}1&7&7\\3&3&10\\13&14&23\end{vmatrix}.$

Subtract 7 times the first column from the second and third:

The result is $\begin{vmatrix}1&0&0\\3&-18&-11\\13&-77&-68\end{vmatrix}=\begin{vmatrix}18&11\\77&68\end{vmatrix}=\begin{vmatrix}7&11\\9&68\end{vmatrix}=377$

10. $\left.\begin{aligned}3x-4y+2z&=1\\2x+3y-3z&=-1\\5x-5y+4z&=7\end{aligned}\right\}.$

The solutions are:

$$x=\frac{\begin{vmatrix}1&-4&2\\-1&3&-3\\7&-5&4\end{vmatrix}}{\begin{vmatrix}3&-4&2\\2&3&-3\\5&-5&4\end{vmatrix}}\quad y=\frac{\begin{vmatrix}3&1&2\\2&-1&-3\\5&7&4\end{vmatrix}}{\begin{vmatrix}3&-4&2\\2&3&-3\\5&-5&4\end{vmatrix}}\quad z=\frac{\begin{vmatrix}3&-4&1\\2&3&-1\\5&-5&7\end{vmatrix}}{\begin{vmatrix}3&-4&2\\2&3&-3\\5&-5&4\end{vmatrix}}.$$

But $\begin{vmatrix}1&-4&2\\-1&3&-3\\7&-5&4\end{vmatrix}=\begin{vmatrix}1&0&0\\-1&-1&-1\\7&23&-10\end{vmatrix}=33$

$\begin{vmatrix}3&1&2\\2&-1&-3\\5&7&4\end{vmatrix}=\begin{vmatrix}0&1&0\\5&-1&-1\\-16&7&-10\end{vmatrix}=66$

$\begin{vmatrix}3&-4&1\\2&3&-1\\5&-5&7\end{vmatrix}=\begin{vmatrix}0&0&1\\5&-1&-1\\-16&23&7\end{vmatrix}=99.$

$$\begin{vmatrix} 3 & -4 & 2 \\ 2 & 3 & -3 \\ 5 & -5 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2 \\ 5 & -8 & -8 \\ 1 & 8 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 5 & -3 & -13 \\ 1 & 3 & 2 \end{vmatrix} = 33$$

$$\therefore\ x = \tfrac{33}{33} = 1, \qquad y = \tfrac{66}{33} = 2, \qquad z = \tfrac{99}{33} = 3.$$

11. $\left.\begin{aligned} 4x - 7y + z &= 16 \\ 3x + y - 2z &= 10 \\ 5x - 6y - 3z &= 10 \end{aligned}\right\}.$

The solutions are:

$$x = \frac{\begin{vmatrix} 16 & -7 & 1 \\ 10 & 1 & -2 \\ 10 & -6 & -3 \end{vmatrix}}{\begin{vmatrix} 4 & -7 & 1 \\ 3 & 1 & -2 \\ 5 & -6 & -3 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} 4 & 16 & 1 \\ 3 & 10 & -2 \\ 5 & 10 & -3 \end{vmatrix}}{\begin{vmatrix} 4 & -7 & 1 \\ 3 & 1 & -2 \\ 5 & -6 & -3 \end{vmatrix}} \qquad z = \frac{\begin{vmatrix} 4 & -7 & 16 \\ 3 & 1 & 10 \\ 5 & -6 & 10 \end{vmatrix}}{\begin{vmatrix} 4 & -7 & 1 \\ 3 & 1 & -2 \\ 5 & -6 & -3 \end{vmatrix}}$$

But $\begin{vmatrix} 16 & -7 & 1 \\ 10 & 1 & -2 \\ 10 & -6 & -3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 42 & -13 & -2 \\ 58 & -27 & -3 \end{vmatrix} = -380$

$$\begin{vmatrix} 4 & 16 & 1 \\ 3 & 10 & -2 \\ 5 & 10 & -3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 11 & 42 & -2 \\ 17 & 58 & -3 \end{vmatrix} = -\ 76$$

$$\begin{vmatrix} 4 & -7 & 16 \\ 3 & 1 & 10 \\ 5 & -6 & 10 \end{vmatrix} = \begin{vmatrix} 25 & 0 & 86 \\ 3 & 1 & 10 \\ 23 & 0 & 70 \end{vmatrix} = -228$$

$$\begin{vmatrix} 4 & -7 & 1 \\ 3 & 1 & -2 \\ 5 & -6 & -3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 11 & -13 & -2 \\ 17 & -27 & -3 \end{vmatrix} = -\ 76$$

$$\therefore\ x = \frac{-380}{-76} = 5, \qquad y = \frac{-76}{-76} = 1, \qquad z = \frac{-228}{-76} = 3.$$

12. $\left.\begin{aligned} 4x + 7y + 3z - 3w &= 6 \\ 2x - y - 4z + 3w &= 13 \\ 3x + 2y - 7z - 4w &= 2 \\ 5x - 3y + z + 5w &= 13 \end{aligned}\right\}.$

The solutions are:

$$Rx = \begin{vmatrix} 6 & 7 & 3 & -3 \\ 13 & -1 & -4 & 3 \\ 2 & 2 & -7 & -4 \\ 13 & -3 & 1 & 5 \end{vmatrix} \qquad Ry = \begin{vmatrix} 4 & 6 & 3 & -3 \\ 2 & 13 & -4 & 3 \\ 3 & 2 & -7 & -4 \\ 5 & 13 & 1 & 5 \end{vmatrix}$$

$$Rz = \begin{vmatrix} 4 & 7 & 6 & -3 \\ 2 & -1 & 13 & 3 \\ 3 & 2 & 2 & -4 \\ 5 & -3 & 13 & 5 \end{vmatrix} \qquad Rw = \begin{vmatrix} 4 & 7 & 3 & 6 \\ 2 & -1 & -4 & 13 \\ 3 & 2 & -7 & 2 \\ 5 & -3 & 1 & 13 \end{vmatrix}$$

where
$$R=\begin{vmatrix}4&7&3&-3\\2&-1&-4&3\\2&2&-7&-4\\5&-3&1&5\end{vmatrix}$$

We find

$$\begin{vmatrix}6&7&3&-3\\13&-1&-4&3\\2&2&-7&-4\\13&-3&1&5\end{vmatrix}=-1330 \qquad \begin{vmatrix}4&6&3&-3\\2&13&-4&3\\3&2&-7&-4\\5&13&1&5\end{vmatrix}=-2660$$

$$\begin{vmatrix}4&7&6&-3\\2&-1&13&3\\3&2&2&-4\\5&-3&13&5\end{vmatrix}=+1330 \qquad \begin{vmatrix}4&7&3&6\\2&-1&-4&13\\3&2&-7&2\\5&-3&1&13\end{vmatrix}=-3990$$

$$R=\begin{vmatrix}4&7&3&-3\\2&-1&-4&3\\2&2&-7&-4\\5&-3&1&5\end{vmatrix}=-1330$$

$$\therefore\ x=\frac{-1330}{-1330}=1 \qquad z=\frac{1330}{-1330}=-1$$

$$y=\frac{-2660}{-1330}=2 \qquad w=\frac{-3990}{-1330}=3$$

13.
$$\left.\begin{array}{r}3x+2y+4z-\ w=13\\5x+\ y-\ z+2w=\ 9\\2x+3y-7z+3w=14\\4x-4y+3z-5w=\ 4\end{array}\right\}.$$

The solutions are

$$x=\frac{\begin{vmatrix}13&2&4&-1\\9&1&-1&2\\14&3&-7&3\\4&-4&3&-5\end{vmatrix}}{\begin{vmatrix}3&2&4&-1\\5&1&-1&2\\2&3&-7&3\\4&-4&3&-5\end{vmatrix}} \qquad y=\frac{\begin{vmatrix}3&13&4&-1\\5&9&-1&2\\2&14&-7&3\\4&4&3&-5\end{vmatrix}}{\begin{vmatrix}3&2&4&-1\\5&1&-1&2\\2&3&-7&3\\4&-4&3&-5\end{vmatrix}}$$

$$z=\frac{\begin{vmatrix}3&2&13&-1\\5&1&9&2\\2&3&14&3\\4&-4&4&-5\end{vmatrix}}{\begin{vmatrix}3&2&4&-1\\5&1&-1&2\\2&3&-7&3\\4&-4&3&-5\end{vmatrix}} \qquad w=\frac{\begin{vmatrix}3&2&4&13\\5&1&-1&9\\2&3&-7&14\\4&-4&3&4\end{vmatrix}}{\begin{vmatrix}3&2&4&-1\\5&1&-1&2\\2&3&-7&3\\4&-4&3&-5\end{vmatrix}}$$

$$\text{But}\begin{vmatrix} 13 & 2 & 4 & -1 \\ 9 & 1 & -1 & 2 \\ 14 & 3 & -7 & 3 \\ 4 & -4 & 3 & -5 \end{vmatrix} = -1276 \qquad \begin{vmatrix} 3 & 13 & 4 & -1 \\ 5 & 9 & -1 & 2 \\ 2 & 14 & -7 & 3 \\ 4 & 4 & 3 & -5 \end{vmatrix} = -2552$$

$$\begin{vmatrix} 3 & 2 & 13 & -1 \\ 5 & 1 & 9 & 2 \\ 2 & 3 & 14 & 3 \\ 4 & -4 & 4 & -5 \end{vmatrix} = 638 \qquad \begin{vmatrix} 3 & 2 & 4 & 13 \\ 5 & 1 & -1 & 9 \\ 2 & 3 & -7 & 14 \\ 4 & -4 & 3 & 4 \end{vmatrix} = 1914$$

$$\begin{vmatrix} 3 & 2 & 4 & -1 \\ 5 & 1 & -1 & 2 \\ 2 & 3 & -7 & 3 \\ 4 & -4 & 3 & -5 \end{vmatrix} = -638$$

$$\therefore x = \frac{-1276}{-638} = 2 \qquad z = \frac{638}{-638} = -1$$

$$y = \frac{-2552}{-638} = 4 \qquad w = \frac{1914}{-638} = -3$$

14. **Eliminate y from the equations**

$$\left.\begin{aligned} x^2 + 2xy + 3x + 4y + 1 &= 0 \\ 4x + 3y + 1 &= 0 \end{aligned}\right\}.$$

Write the equations in the form:

$$\begin{aligned} (2x+4)\,y + x^2 + 3x + 1 &= 0 \\ 3y + 4x + 1 \qquad &= 0 \end{aligned}$$

The result is
$$\begin{vmatrix} 2x+4 & x^2+3x+1 \\ 3 & 4x+1 \end{vmatrix} = 0$$

This reduces to

$$(2x+4)(4x+1) - 3(x^2+3x+1) = 0$$

or
$$5x^2 + 9x + 1 = 0$$

15. **Eliminate m from the equations**

$$\left.\begin{aligned} m^2x - 2mx^2 + 1 &= 0 \\ m + x^2 - 3mx &= 0 \end{aligned}\right\}.$$

Multiply the second equation by m.

Then we have
$$\begin{aligned} xm^2 - 2x^2m + 1 &= 0 \\ (1-3x)m^2 + x^2m \qquad &= 0 \\ (1-3x)m + x^2 &= 0 \end{aligned}$$

The result is
$$\begin{vmatrix} x & -2x^2 & 1 \\ 1-3x & x^2 & 0 \\ 0 & 1-3x & x^2 \end{vmatrix} = 0$$

This reduces to

$$x^5 - (1-3x)(-2x^4 - 1 + 3x) = 0$$

or
$$-5x^5 + 2x^4 + 9x^2 - 6x + 1 = 0$$

16. Eliminate x from the equations

$$\left.\begin{array}{r} ax^2+bx+c=0 \\ x^3=1 \end{array}\right\}.$$

Multiply the first equation by x and by x^2, and the second by x.
Then we have

$$\begin{array}{rr} ax^2+bx+c &= 0 \\ ax^3+bx^2+cx\quad &= 0 \\ ax^4+bx^3+cx^2\qquad &= 0 \\ x^3\qquad -1 &= 0 \\ x^4\qquad\quad -x\; &= 0 \end{array}$$

The result of the elimination is $\begin{vmatrix} 0 & 0 & a & b & c \\ 0 & a & b & c & 0 \\ a & b & c & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 \end{vmatrix}=0$

To reduce this determinant, multiply the fifth row by a and subtract from the third row.

The result is $\begin{vmatrix} 0 & 0 & a & b & c \\ 0 & a & b & c & 0 \\ 0 & b & c & a & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & -1 & 0 \end{vmatrix}=\begin{vmatrix} 0 & a & b & c \\ a & b & c & 0 \\ b & c & a & 0 \\ 1 & 0 & 0 & -1 \end{vmatrix}$

Add the first column to the fourth.

The result is $\begin{vmatrix} 0 & a & b & c \\ a & b & c & a \\ b & c & a & b \\ 1 & 0 & 0 & 0 \end{vmatrix}=-\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$\therefore$ the result of the elimination is $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}=0$

or $$a^3+b^3+c^3-3abc=0$$

17. Eliminate x from the equations

$$\left.\begin{array}{r} ax^2+bx+c=0 \\ a'x^2+b'x+c'=0 \end{array}\right\}.$$

Multiply each equation by x.
Then we have

$$\begin{array}{rr} ax^2+bx+c &= 0 \\ ax^3+bx^2+cx\quad &= 0 \\ a'x^2+b'x+c' &= 0 \\ a'x^3+b'x^2+c'x\quad &= 0 \end{array}$$

The result of the elimination is

$$\begin{vmatrix} 0 & a & b & c \\ a & b & c & 0 \\ 0 & a' & b' & c' \\ a' & b' & c' & 0 \end{vmatrix} = 0$$

or $(ab' - a'b)(bc' - b'c) - (ac' - a'c)^2 = 0$

18. Eliminate x from the equations

$$\left.\begin{aligned} ax^2 + bx + c &= 0 \\ x^3 + qx + r &= 0 \end{aligned}\right\}. \qquad \begin{matrix}(1)\\(2)\end{matrix}$$

Multiply (1) by x, (2) by a, and subtract,

$$bx^2 + (c - aq)\,x - ar = 0 \qquad (3)$$

Multiply (3) by a, (1) by b,

$$\begin{aligned} abx^2 + (ac - a^2q)\,x - a^2r &= 0 \\ abx^2 + b^2x + bc &= 0 \end{aligned}$$

Subtract, $(ac - b^2 - a^2q)x - a^2r - bc = 0 \qquad (4)$

Eliminate x from (1) and (4),

$$\begin{vmatrix} ac - b^2 - a^2q & -(a^2r + bc) & 0 \\ 0 & -b^2 - a^2q & -(a^2r + bc) \\ a & b & c \end{vmatrix} = 0$$

$$a\,(a^2r + bc)^2 + b\,(ac - b^2 - a^2q)(a^2r + bc) + c\,(ac - b^2 - a^2q)^2 = 0$$

19. Are the following equations consistent?

$$\left.\begin{aligned} 4x^2 + 3x + 2 &= 0 \\ 2x^2 + x + 1 &= 0 \end{aligned}\right\}.$$

In the determinant of Example 17 put $a = 4$, $b = 3$, $c = 2$, $a' = 2$, $b' = 1$, $c' = 1$.

The determinant becomes $\begin{vmatrix} 0 & 4 & 3 & 2 \\ 4 & 3 & 2 & 0 \\ 0 & 2 & 1 & 1 \\ 2 & 1 & 1 & 0 \end{vmatrix}$

Subtract twice the fourth row from the second.

The result is $\begin{vmatrix} 0 & 4 & 3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 2 & 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 3 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 0 \end{vmatrix} = 2\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 2$

∴ Since the determinant is not 0, the two equations are inconsistent.

20. Are the following equations consistent?

$$\left.\begin{aligned} 3x^2+4xy+4x+1&=0\\ x-3y-7&=0\\ 2x-y-4&=0\end{aligned}\right\}.$$

Multiply the second and third equations by x.

We have then

$$\begin{aligned} 3x^2+4xy+4x+1\qquad&=0\\ x-3y-7&=0\\ x^2-3xy-7x\qquad&=0\\ 2x-y-4&=0\\ 2x^2-xy-4x\qquad&=0\end{aligned}$$

Eliminate x^2, xy, x, and y from these five equations.

The resulting determinant is $\begin{vmatrix} 3 & 4 & 4 & 0 & 1\\ 0 & 0 & 1 & -3 & -7\\ 1 & -3 & -7 & 0 & 0\\ 0 & 0 & 2 & -1 & -4\\ 2 & -1 & -4 & 0 & 0\end{vmatrix}$

$$=\begin{vmatrix} 0 & 13 & 25 & 0 & 1\\ 0 & 0 & 1 & -3 & -7\\ 1 & -3 & -7 & 0 & 0\\ 0 & 0 & 2 & -1 & -4\\ 0 & 5 & 10 & 0 & 0\end{vmatrix}=\begin{vmatrix} 13 & 25 & 0 & 1\\ 0 & 1 & -3 & -7\\ 0 & 2 & -1 & -4\\ 5 & 10 & 0 & 0\end{vmatrix}$$

$$=\begin{vmatrix} 13 & -1 & 0 & 1\\ 0 & 1 & -3 & -7\\ 0 & 2 & -1 & -4\\ 5 & 0 & 0 & 0\end{vmatrix}=5\begin{vmatrix} -1 & 0 & 1\\ 1 & -3 & -7\\ 2 & -1 & -4\end{vmatrix}=5\begin{vmatrix} 0 & 0 & 1\\ -6 & -3 & -7\\ -2 & -1 & -4\end{vmatrix}=0$$

$\therefore$ The equations are consistent.

21. If ω is one of the imaginary cube roots of 1, show that:

$$\begin{vmatrix} 1 & -\omega & \omega^2\\ -\omega & \omega^2 & 1\\ \omega^2 & 1 & -\omega\end{vmatrix}=-4.\qquad \begin{vmatrix} 1 & \omega & \omega^2 & 1\\ \omega & \omega^2 & 1 & 1\\ \omega^2 & 1 & 1 & \omega\\ 1 & 1 & \omega & \omega^2\end{vmatrix}=3\sqrt{-3}.$$

(1) Multiply the first column by ω and add to the second, and multiply the second by ω and add to the third, remembering that $\omega^3=1$.

$$\therefore\begin{vmatrix} 1 & -\omega & \omega^2\\ -\omega & \omega^2 & 1\\ \omega^2 & 1 & -\omega\end{vmatrix}=\begin{vmatrix} 1 & 0 & 0\\ -\omega & 0 & 2\\ \omega^2 & 2 & 0\end{vmatrix}=-4$$

(2) Multiply the three columns each by ω, and subtract each from the following.

$$\therefore \begin{vmatrix} 1 & \omega & \omega^2 & 1 \\ \omega & \omega^2 & 1 & 1 \\ \omega^2 & 1 & 1 & \omega \\ 1 & 1 & \omega & \omega^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ \omega & 0 & 0 & 1-\omega \\ \omega^2 & 0 & 1-\omega & 0 \\ 1 & 1-\omega & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1-\omega \\ 0 & 1-\omega & 0 \\ 1-\omega & 0 & 0 \end{vmatrix} = -(1-\omega)^3$$

$$= \omega^3 - 3\omega^2 + 3\omega - 1$$

$$= 3(\omega - \omega^2)$$

But $\omega = -\frac{1}{2} + \frac{1}{2}\sqrt{-3}$

$$\omega^2 = -\frac{1}{2} - \frac{1}{2}\sqrt{-3}$$

$$\therefore 3(\omega - \omega^2) = 3\sqrt{-3}.$$

22. Show that in any determinant there are two terms which have all but two elements alike; and that these two terms have different signs.

Consider any determinant $\begin{vmatrix} a_1 & a_2 & \cdots\cdots \\ b_2 & b_2 & \cdots\cdots \\ \cdot & \cdot & \cdot \quad \cdot \\ \cdot & \cdot & \cdot \quad \cdot \end{vmatrix}$

and let any term be $a_i b_j c_k \cdots\cdots g_l h_k \cdots\cdots$

Then consider any two elements, $g_l h_k$, in this term.

If we interchange the subscripts of these two elements, we have $a_i b_j c_k \cdots\cdots g_k h_l \cdots\cdots$

But this is also a term of the determinant, and these two terms have all their elements alike except g and h.

The second term will have the opposite sign from the first one. For in interchanging the two indices we either add one inversion or subtract one from those contained in the first term. Thus, if l is greater than k, then lk gives no inversion, but lk gives an inversion; and if k is greater than l, kl gives an inversion, while lk gives no inversion.

Exercise 67.

1. Show that $\begin{vmatrix} a & b & 0 \\ c & 0 & c \\ 0 & b & a \end{vmatrix} \times \begin{vmatrix} 0 & a & b \\ c & 0 & c \\ b & a & 0 \end{vmatrix} \equiv -4a^2b^2c^2.$

$$\begin{vmatrix} a & b & 0 \\ c & 0 & c \\ 0 & b & a \end{vmatrix} \times \begin{vmatrix} 0 & a & b \\ c & 0 & c \\ b & a & 0 \end{vmatrix} = \begin{vmatrix} ab & bc & 2ab \\ ac & 2c^2 & ac \\ 2ab & cb & ab \end{vmatrix} = b^2c \begin{vmatrix} a & c & 2b \\ a & 2c & a \\ 2a & c & a \end{vmatrix}$$

$$= a^2b^2c^2 \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{vmatrix} = -4a^2b^2c^2$$

2. Express as a single determinant

$$\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} \times \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}.$$

$$\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} \times \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} = \begin{vmatrix} c^2+b^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & cb & b^2+a^2 \end{vmatrix}$$

3. Express as a single determinant

$$\begin{vmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{vmatrix} \times \begin{vmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix},$$

and thence resolve the first determinant into its simplest factors.

$$\begin{vmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{vmatrix} \times \begin{vmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -a+b & -a+b & -a+b \\ 0 & 0 & -b+c & -b+c \\ 0 & 0 & 0 & -c+d \\ 2a & a+b & a+b & a+b+c-d \end{vmatrix}$$

$$= (b-a)(c-b)(d-c) \begin{vmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 2a & a+b & a+b & a+b+c-d \end{vmatrix}$$

$$= -2a(b-a)(c-b)(d-c) \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= -2a(b-a)(c-b)(d-c)$$

$$= 2a(a-b)(b-c)(c-d)$$

But
$$\begin{vmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 2 & 1 & -1 \end{vmatrix}$$

$$= -\begin{vmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = -2$$

$\therefore$ The first determinant $= -a(a-b)(b-c)(c-d)$.

4. Express as a single determinant

$$\begin{vmatrix} a+bi & -c+di \\ c+di & a-bi \end{vmatrix} \times \begin{vmatrix} \alpha+\beta i & -\gamma+\delta i \\ \gamma+\delta i & \alpha-\beta i \end{vmatrix},$$

where $i=\sqrt{-1}$; and thence prove Euler's theorem, viz.: *the product of two sums of four squares can itself be expressed as the sum of four squares.*

$$\begin{vmatrix} a+bi & -c+di \\ c+di & a-bi \end{vmatrix} = a^2+b^2+c^2+d^2$$

$$\begin{vmatrix} \alpha+\beta i & -\gamma+\delta i \\ \gamma+\delta i & \alpha+\beta i \end{vmatrix} = \alpha^2+\beta^2+\gamma^2+\delta^2$$

$$\begin{vmatrix} a+bi & -c+di \\ c+di & a-bi \end{vmatrix} \times \begin{vmatrix} \alpha+\beta i & -\gamma+\delta i \\ \gamma+\delta i & \alpha-\beta i \end{vmatrix} = \begin{vmatrix} A+Bi & -C+Di \\ C+Di & A-Bi \end{vmatrix};$$

$$= A^2+B^2+C^2+D^2$$

where $\begin{cases} A = a\alpha - b\beta + c\gamma - d\delta & B = a\beta + b\alpha - c\delta - d\gamma \\ C = a\gamma - b\delta - c\alpha + d\beta & D = a\delta + b\gamma + c\beta + d\alpha \end{cases}$

5. Show that $\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2.$

$$\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \times \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} A_1a_1 + A_2a_2 + A_3a_3, & A_1b_1 + A_2b_2 + A_3b_3, & A_1c_1 + A_2c_2 + A_3c_3 \\ B_1a_1 + B_2a_2 + B_3a_3, & B_1b_1 + B_2b_2 + B_3b_3, & B_1c_1 + B_2c_2 + B_3c_3 \\ C_1a_1 + C_2a_2 + C_3a_2, & C_1b_1 + C_2b_2 + C_3b_3, & C_1c_1 + C_2c_2 + C_3c_3 \end{vmatrix}$$

But all the terms in the product except those in the diagonal vanish, and each of them is equal to Δ.

$$\text{Hence the product} = \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} = \Delta^3.$$

$$\therefore \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \frac{\Delta^3}{\Delta} = \Delta^2$$

Exercise 68.

Find the quotient and remainder obtained by dividing each of the following quantics by the divison opposite it.

1. $x^4 - 3x^3 - x^2 + 2x - 1 \qquad x-2.$

$$\begin{array}{rrrrr|l} 1 & -3 & -1 & +2 & -1 & \underline{2} \\ & 2 & -2 & -6 & -8 & \\ \hline 1 & -1 & -3 & -4 & -9 & \end{array}$$

$\therefore$ The quotient $= x^3 - x^2 - 3x - 4.$
The remainder $= -9.$

2. $x^4 - 3x^3 + 2x - 7$ $\qquad x - 3.$

$$\begin{array}{rrrrr|l}
1 & +0 & -3 & +2 & -7 & \underline{3} \\
 & 3 & +9 & +18 & +60 & \\
\hline
1 & +3 & +6 & +20 & +53 &
\end{array}$$

$\therefore$ The quotient $= x^3 + 3x^2 + 6x + 20.$
The remainder $= 53.$

3. $2x^4 + 3x^3 - 8x^2 - 7x - 10$ $\qquad x - 2.$

$$\begin{array}{rrrrr|l}
2 & +3 & -8 & -7 & -10 & \underline{2} \\
 & 4 & +14 & +12 & +10 & \\
\hline
2 & +7 & +6 & +5 & +0 &
\end{array}$$

$\therefore$ The quotient $= 2x^3 + 7x^2 + 6x + 5.$
The remainder $= 0.$

4. $3x^4 + 2x^2 - 6x + 50$ $\qquad x + 3.$

$$\begin{array}{rrrrr|l}
3 & +0 & +2 & -6 & +50 & \underline{-3} \\
 & -9 & +27 & -87 & +279 & \\
\hline
3 & -9 & +29 & -93 & +329 &
\end{array}$$

$\therefore$ The quotient $= 3x^3 - 9x^2 + 29x - 93.$
The remainder $= 329.$

5. $ax^2 + 3bx^2 + 3cx + d$ $\qquad x + h.$

$$\begin{array}{llll|l}
a & +3b & +3c & +d & \underline{-h} \\
 & -ah & -3bh + ah^2 & -3ch + 3bh^2 - ah^3 & \\
\hline
a & +3b - ah & +3c - 3bh + ah^2 & +d - 3ch + 3bh^2 - ax^3 &
\end{array}$$

$\therefore$ The quotient $= ax^2 + (3b - ah)x + 3c - 3bh + ah^2.$
The remainder $= -ah^3 + 3bh^2 - 3ch + d.$

Are the following numbers roots of the equations opposite them (§ 434)?

6. (3) $\qquad x^4 + x^2 - 6x + 2 = 0.$

$$\begin{array}{rrrrr|l}
1 & +0 & +1 & -6 & +2 & \underline{3} \\
 & 3 & +9 & +30 & +72 & \\
\hline
1 & +3 & +10 & +24 & +74 &
\end{array}$$

The remainder $= 74.$ $\therefore$ 3 is not a root.

7. (-7) $$x^4+7x^3+21x+147=0.$$

$$\begin{array}{rrrrr|l}
1 & +7 & +0 & +21 & +147 & \underline{-7} \\
 & -7 & 0 & +0 & -147 & \\
\hline
1 & +0 & +0 & +21 & +0 &
\end{array}$$

The remainder $=0$. $\therefore -7$ is a root.

8. (0.3) $$x^4-2.3x^3+3.6x^2+4.9x+1.2=0.$$

$$\begin{array}{rrrrr|l}
1 & -2.3 & +3.6 & +4.9 & +1.2 & \underline{0.3} \\
 & 0.3 & -0.6 & +0.9 & +1.74 & \\
\hline
1 & -2 & +3 & +5.8 & +2.94 &
\end{array}$$

The remainder $=2.94$. $\therefore 0.3$ is not a root.

Find the value of the following expressions when for x we put the number in parenthesis:

9. $3x^3+2x^2-6x+1$ (-3).

$$\begin{array}{rrrr|l}
3 & +2 & -6 & +1 & \underline{-3} \\
 & -9 & +21 & -45 & \\
\hline
3 & -7 & +15 & -44 &
\end{array}$$

$\therefore$ The required value is -44.

10. $2x^4+6x^2-9x-5$ (6).

$$\begin{array}{rrrrr|l}
2 & +0 & +6 & -9 & -5 & \underline{6} \\
 & 12 & +72 & +468 & +2754 & \\
\hline
2 & +12 & +78 & +459 & +2749 &
\end{array}$$

$\therefore$ The required value is 2749.

11. $x^5+7x^3-2x^2-49$ (-4).

$$\begin{array}{rrrrrr|l}
1 & +0 & +7 & -2 & +0 & -49 & \underline{-4} \\
 & -4 & +16 & -92 & +376 & -1504 & \\
\hline
1 & -4 & +23 & -94 & +376 & -1553 &
\end{array}$$

$\therefore$ The required value is -1553.

12. $x^4+6x^3-7x^2-3x+1$ (-0.2).

$$\begin{array}{rrrrr|l}
1 & +6 & -7 & -3 & +1 & \underline{-0.2} \\
 & -0.2 & -1.16 & +1.632 & +0.2736 & \\
\hline
1 & +5.8 & -8.16 & -1.368 & +1.2736 &
\end{array}$$

$\therefore$ The required value is 1.2736.

Exercise 69.

Solve the equations:

1. $x^3 - 7x^2 + 16x - 12 = 0.$

Try 2,
$$\begin{array}{rrrr|l} 1 & -7 & +16 & -12 & \underline{2} \\ & 2 & -10 & +12 & \\ \hline 1 & -5 & +6 & +0 & \end{array}$$

∴ 2 is a root.

The quotient $= x^2 - 5x + 6 = (x-2)(x-3)$

∴ The three roots are 2, 2, and 3.

2. $x^3 + 9x^2 + 2x - 48 = 0.$

Try 2,
$$\begin{array}{rrrr|l} 1 & +9 & +2 & -48 & \underline{2} \\ & 2 & +22 & +48 & \\ \hline 1 & +11 & +24 & +0 & \end{array}$$

∴ 2 is a root.

The quotient $= x^2 + 11x + 24 = (x+3)(x+8).$

∴ The three roots are 2, -3, and -8.

3. $x^3 - 4x^2 - 8x + 8 = 0.$

Try -2,
$$\begin{array}{rrrr|l} 1 & -4 & -8 & +8 & \underline{-2} \\ & -2 & +12 & -8 & \\ \hline 1 & -6 & +4 & +0 & \end{array}$$

∴ -2 is a root.

The quotient $= x^2 - 6x + 4$

Solve $\quad x^2 - 6x + 4 = 0$

$$x = 3 \pm \sqrt{5}$$

∴ The three roots are -2, $3 + \sqrt{5}$, and $3 - \sqrt{5}$.

4. $x^3 - 5x^2 - 2x + 24 = 0.$

Try 3,
$$\begin{array}{rrrr|l} 1 & -5 & -2 & +24 & \underline{3} \\ & 3 & -6 & -24 & \\ \hline 1 & -2 & -8 & +0 & \end{array}$$

∴ 3 is a root.

The quotient $= x^2 - 2x - 8 = (x-4)(x+2)$

∴ The three roots are 3, 4, and -2.

5. $x^3 + 2x^2 + 4x + 3 = 0.$

Try -1,
$$\begin{array}{rrrr|l} 1 & +2 & +4 & +3 & \underline{-1} \\ & -1 & -1 & -3 & \\ \hline 1 & +1 & +3 & -0 & \end{array}$$

$\therefore -1$ is a root.

The quotient $= x^2 + x + 3$

Solve $$x^2 + x + 3 = 0$$

$$x = \frac{-1 \pm \sqrt{-11}}{2}$$

$\therefore$ The three roots are -1, $\frac{-1 + \sqrt{-11}}{2}$, and $\frac{-1 - \sqrt{-11}}{2}$.

6. $x^3 - 6x^2 + 6x + 99 = 0$.

Try -3,
$$\begin{array}{rrrr|l} 1 & -6 & +6 & +99 & \underline{-3} \\ & -3 & +27 & -99 & \\ \hline 1 & -9 & +33 & +0 & \end{array}$$

$\therefore -3$ is a root.

The quotient $= x^2 - 9x + 33$

Solve $$x^2 - 9x + 33 = 0$$

$$x = \frac{9 \pm \sqrt{-51}}{2}$$

$\therefore$ The three roots are -3, $\frac{9 + \sqrt{-51}}{2}$, and $\frac{9 - \sqrt{-51}}{2}$.

7. $6x^3 - 29x^2 + 14x + 24 = 0$.

Try 4,
$$\begin{array}{rrrr|l} 6 & -29 & +14 & +24 & \underline{4} \\ & 24 & -20 & -24 & \\ \hline 6 & -5 & -6 & +0 & \end{array}$$

$\therefore$ 4 is a root.

The quotient $= 6x^2 - 5x - 6 = (3x + 2)(2x - 3)$

$\therefore$ The three roots are 4, $-\frac{2}{3}$, and $\frac{3}{2}$.

8. $2x^3 + 3x^2 - 13x - 12 = 0$.

Try -3,
$$\begin{array}{rrrr|l} 2 & +3 & -13 & -12 & \underline{-3} \\ & -6 & +9 & +12 & \\ \hline 2 & -3 & -4 & 0 & \end{array}$$

$\therefore -3$ is a root.

The quotient $= 2x^2 - 3x - 4$.

Solve $$2x^2 - 3x - 4 = 0$$

$$x = \frac{3 \pm \sqrt{41}}{4}$$

$\therefore$ The three roots are -3, $\frac{3 + \sqrt{41}}{4}$, and $\frac{3 - \sqrt{41}}{4}$.

9. $x^4 - 15x^2 - 10x + 24 = 0.$

$$\begin{array}{lrrrrrl}
\text{Try } 1, & 1 & +0 & -15 & -10 & +24 & \underline{|1} \\
 & & 1 & +1 & -14 & -24 & \\
\cline{2-6}
 & 1 & +1 & -14 & -24 & +0 &
\end{array}$$

$\therefore$ 1 is a root.

The quotient $= x^3 + x^2 - 14x - 24.$

$$\begin{array}{lrrrrl}
\text{Try } -2, & 1 & +1 & -14 & -24 & \underline{|-2} \\
 & & -2 & +2 & +24 & \\
\cline{2-5}
 & 1 & -1 & -12 & +0 &
\end{array}$$

$\therefore -2$ is a root.

The quotient $= x^2 - x - 12 = (x-4)(x+3).$

$\therefore$ The four roots are 1, -2, 4, and -3.

10. $x^4 + 5x^3 - 5x^2 - 45x - 36 = 0.$

$$\begin{array}{lrrrrrl}
\text{Try } -1, & 1 & +5 & -5 & -45 & -36 & \underline{|-1} \\
 & & -1 & -4 & +9 & +36 & \\
\cline{2-6}
\text{Try } 3, & 1 & +4 & -9 & -36 & +0 & \underline{|3} \\
 & & 3 & +21 & +36 & & \\
\cline{2-6}
 & 1 & +7 & +12 & +0 & &
\end{array}$$

$\therefore -1$ and 3 are roots.

The quotient $= x^2 + 7x + 12 = (x+3)(x+4).$

$\therefore$ The four roots are -1, 3, -3, and -4.

11. $x^4 + 4x^3 - 29x^2 - 156x + 180 = 0.$

$$\begin{array}{lrrrrrl}
\text{Try } 1, & 1 & +4 & -29 & -156 & +180 & \underline{|1} \\
 & & 1 & +5 & -24 & -180 & \\
\cline{2-6}
 & 1 & +5 & -24 & -180 & +0 &
\end{array}$$

$\therefore$ 1 is a root.

There are no other commensurable roots.

12. $x^4 - 5x^3 - 2x^2 + 12x + 8 = 0.$

$$\begin{array}{lrrrrrl}
\text{Try } -1, & 1 & -5 & -2 & +12 & +8 & \underline{|-1} \\
 & & -1 & +6 & -4 & -8 & \\
\cline{2-6}
\text{Try } 2, & 1 & -6 & +4 & +8 & +0 & \underline{|2} \\
 & & 2 & -8 & -8 & & \\
\cline{2-5}
 & 1 & -4 & -4 & +0 & &
\end{array}$$

$\therefore -1$ and 2 are roots.

The quotient $= x^2 - 4x - 4$.

Solve
$$x^2 - 4x - 4 = 0$$
$$x = 2 \pm 2\sqrt{2}$$

$\therefore$ The four roots are -1, 2, $2 + 2\sqrt{2}$, and $2 - 2\sqrt{2}$.

13. $6x^4 - 5x^3 - 30x^2 + 20x + 24 = 0$.

Try 2,

Try -2,

$$\begin{array}{rrrrr|r}
6 & -5 & -30 & +20 & +24 & \underline{2} \\
 & 12 & +14 & -32 & -24 & \\
\hline
6 & +7 & -16 & -12 & +0 & \underline{-2} \\
 & -12 & +10 & +12 & & \\
\hline
6 & -5 & -6 & +0 & &
\end{array}$$

$\therefore$ 2 and -2 are roots.

The quotient $= 6x^2 - 5x - 6 = (2x - 3)(3x + 2)$

$\therefore$ The four roots are 2, -2, $\frac{3}{2}$, and $-\frac{2}{3}$.

14. $4x^4 + 8x^3 - 23x^2 - 7x + 78 = 0$.

Try -2,

Try -3,

$$\begin{array}{rrrrr|r}
4 & +8 & -23 & -7 & +78 & \underline{-2} \\
 & -8 & +0 & +46 & -78 & \\
\hline
4 & +0 & -23 & +39 & +0 & \underline{-3} \\
 & -12 & +36 & -39 & & \\
\hline
4 & -12 & +13 & +0 & &
\end{array}$$

$\therefore -2$ and -3 are roots.

The quotient $= 4x^2 - 12x + 13$.

Solve
$$4x^2 - 12x + 13 = 0$$
$$x = \tfrac{3}{2} \pm \tfrac{1}{2}\sqrt{-4}$$

$\therefore$ The four roots are -2, -3, $\frac{3 + \sqrt{-4}}{2}$, and $\frac{3 - \sqrt{-4}}{2}$.

Form the equations which have the following roots:

15. 2, 6, -7.
$$(x - 2)(x - 6)(x + 7) = 0$$
or
$$x^3 - x^2 - 44x + 84 = 0$$

16. 2, 4, -3.
$$(x - 2)(x - 4)(x + 3) = 0$$
or
$$x^3 - 3x^2 - 10x + 24 = 0$$

17. $2, 0, -2.$

$$(x-2)\,x\,(x+2)=0$$

or $$x^3-4x=0$$

18. $2, 1, -2, -1.$

$$(x-2)(x-1)(x+2)(x+1)=0$$

or $$x^4-5x^2+4=0$$

19. $5, 3+\sqrt{-1}, 3-\sqrt{-1}.$

$$(x-5)(x-3-\sqrt{-1})(x-3+\sqrt{-1})=0$$

or $$x^3-11x^2+40x-50=0$$

20. $2, \frac{1}{2}, 2, -\frac{1}{2}.$

$$(x-2)(x-\tfrac{1}{2})(x-2)(x+\tfrac{1}{2})=0$$

or $$(x-2)(x-2)(2x-1)(2x+1)=0$$

or $$4x^4-16x^3+15x^2+4x-4=0$$

21. $2, 3, -2, -3, -6.$

$$(x-2)(x-3)(x+2)(x+3)(x+6)=0$$

or $$x^5+6x^4-13x^3-78x^2+36x+216=0$$

22. $\frac{1}{3}, \frac{2}{3}, -\frac{1}{2}, -\frac{3}{2}.$

$$(x-\tfrac{1}{3})(x-\tfrac{2}{3})(x+\tfrac{1}{2})(x+\tfrac{3}{2})=0$$

or $$x^4+x^3-\tfrac{37}{36}x^2-\tfrac{11}{36}x+\tfrac{1}{6}=0$$

or $$36x^4+36x^3-37x^2-11x+6=0$$

23. $3+\sqrt{2}, 3-\sqrt{2}, 2+\sqrt{3}, 2-\sqrt{3}.$

$$(x-3-\sqrt{2})(x-3+\sqrt{2})(x-2-\sqrt{3})(x-2+\sqrt{3})=0$$

or $$x^4-10x^3+32x^2-34x+7=0$$

24. $0.2, 0.125, -0.4.$

$$(x-0.2)(x-0.125)(x+0.4)=0$$

or $$x^3+0.075x^2-0.105x+0.01=0$$

$$200x^3+15x^2-21x+10=0$$

25. $0.3, -0.2, -\frac{1}{20}, -\frac{6}{5}.$

$$(x-0.3)(x+0.2)(x+\tfrac{1}{20})(x+\tfrac{6}{5})=0$$

or $$(10x-1)(5x+1)(20x+1)(5x+6)=0$$

or $$5000x^4+5750x^3-625x^2-405x-18=0$$

26. $2+\sqrt{-1}, 2-\sqrt{-1}, 1+2\sqrt{-1}, 1-2\sqrt{-1}.$

$$(x-2-\sqrt{-1})(x-2+\sqrt{-1})(x-1-2\sqrt{-1})(x-1+2\sqrt{-1})=0$$

or $$x^4-6x^3+18x^2-30x+25=0$$

Exercise 70.

1. **Form the equations of which the roots are:**

$$2, 4, -3; \qquad 3, -2, -4.$$

$$2 + 4 - 3 = 3$$
$$3 \times 4 + 2 \times (-3) + 4 \times (-3) = -10$$
$$2 \times 4 \times (-3) = -24$$

∴ The required equation is

$$x^3 - 3x^2 - 10x + 24 = 0$$

The roots of the second equation are the negatives of the roots of the first equations.

∴ The second equation is

$$x^2 + 3x^2 - 10x - 24 = 0$$

In examples 2–10,
$$\alpha + \beta + \gamma = 5$$
$$\alpha\beta + \beta\gamma + \gamma\alpha = 4$$
$$\alpha\beta\gamma = 3$$

If α, β, γ are the roots of $x^3 - 5x^2 + 4x - 3 = 0$, find the value of:

2.
$$\begin{aligned} \Sigma\alpha^2 &= \alpha^2 + \beta^2 + \gamma^2 \\ &= (\alpha + \beta + \gamma)^2 - 2\alpha\beta - 2\alpha\gamma - 2\beta\gamma \\ &= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) \\ &= 25 - 8 \\ &= 17 \end{aligned}$$

3.
$$\begin{aligned} \Sigma\alpha^2\beta &= \alpha^2\beta + \alpha^2\gamma + \beta^2\alpha + \beta^2\gamma + \gamma^2\alpha + \gamma^2\beta \\ &= (\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha + \beta + \gamma) - 3\alpha\beta\gamma \\ &= 4 \times 5 - 3 \times 3 \\ &= 11 \end{aligned}$$

4.
$$\begin{aligned} \Sigma\alpha^3 &= \alpha^3 + \beta^3 + \gamma^3 \\ &= (\alpha + \beta + \gamma)^3 - 3(\alpha^2\beta + \alpha^2\gamma + \beta^2\alpha + \beta^2\gamma + \gamma^2\alpha + \gamma^2\beta) \\ &\qquad - 6\alpha\beta\gamma \\ &= (\alpha + \beta + \gamma)^3 - 3(\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha + \beta + \gamma) + 3\alpha\beta\gamma \\ &= 5^3 - 3 \times 4 \times 5 + 3 \times 3 \\ &= 74 \end{aligned}$$

5.
$$\begin{aligned} \Sigma\alpha^2\beta\gamma &= \alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2 \\ &= (\alpha + \beta + \gamma)\alpha\beta\gamma \\ &= 5 \times 3 \\ &= 15 \end{aligned}$$

6.
$$\begin{aligned}\Sigma\alpha^2\beta^2 &= \alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2\\ &= (\alpha\beta+\beta\gamma+\gamma\alpha)^2-2\,(\alpha\beta^2\gamma+\alpha^2\beta\gamma+\beta\gamma^2\alpha)\\ &= (\alpha\beta+\beta\gamma+\gamma\alpha)^2-2\,\alpha\beta\gamma\,(\alpha+\beta+\gamma)\\ &= 16-2\times 3\times 5\\ &= -14\end{aligned}$$

7.
$$\begin{aligned}\Sigma\alpha^3\beta &= \alpha^3\beta+\alpha\beta^3+\alpha^3\gamma+\alpha\gamma^3+\beta^3\gamma+\beta\gamma^3\\ &= (\alpha^2+\beta^2+\gamma^2)(\alpha\beta+\beta\gamma+\gamma\alpha)-\alpha\beta\gamma(\alpha+\beta+\gamma)\\ &= [(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)](\alpha\beta+\beta\gamma+\gamma\alpha)\\ &\qquad -\alpha\beta\gamma(\alpha+\beta+\gamma)\\ &= (25-2\times 4)\times 4-3\times 5\\ &= 53\end{aligned}$$

8.
$$\begin{aligned}\Sigma\alpha^4 &= \alpha^4+\beta^4+\gamma^4\\ &= (\alpha^2+\beta^2+\gamma^2)^2-2(\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2)\\ &= [(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)]^2-2[(\alpha\beta+\beta\gamma+\gamma\alpha)^2\\ &\qquad -2\,\alpha\beta\gamma(\alpha+\beta+\gamma)]\\ &= (25-2\times 4)^2-2(16-2\times 3\times 5)\\ &= 317\end{aligned}$$

9.
$$\begin{aligned}\Sigma\alpha^3\beta\gamma &= \alpha^3\beta\gamma+\alpha\beta^3\gamma+\alpha\beta\gamma^3\\ &= \alpha\beta\gamma(\alpha^2+\beta^2+\gamma^2)\\ &= \alpha\beta\gamma[(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)]\\ &= 3(25-2\times 4)\\ &= 51\end{aligned}$$

10.
$$\begin{aligned}\Sigma\alpha^2\beta^2\gamma &= \alpha^2\beta^2\gamma+\alpha^2\beta\gamma^2+\alpha\beta^2\gamma^2\\ &= \alpha\beta\gamma(\alpha\beta+\alpha\gamma+\beta\gamma)\\ &= 3\times 4\\ &= 12\end{aligned}$$

In examples 11–15,
$$\begin{aligned}\alpha+\beta+\gamma &= -p\\ \alpha\gamma+\beta\gamma+\gamma\alpha &= q\\ \alpha\beta\gamma &= -r\end{aligned}$$

If α, β, γ are the roots of $x^3+px^2+qx+r=0$, find in terms of the coefficients the values of:

11.
$$\begin{aligned}\Sigma\alpha^2 &= \alpha^2+\beta^2+\gamma^2\\ &= (\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)\\ &= p^2-2q\end{aligned}$$

12. $$\begin{aligned}\Sigma\alpha^2\beta &= \alpha^2\beta + \alpha^2\gamma + \alpha\beta^2 + \beta^2\gamma + \alpha\gamma^2 + \beta\gamma^2\\ &= (\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha + \beta + \gamma) - 3\,\alpha\beta\gamma\\ &= -pq + 3r\end{aligned}$$

13. $$\begin{aligned}\Sigma\alpha^3 &= \alpha^3 + \beta^3 + \gamma^3\\ &= (\alpha + \beta + \gamma)^3 - 3(\alpha^2\beta + \alpha^2\gamma + \alpha\beta^2 + \beta^2\gamma + \alpha\gamma^2 + \beta\gamma^2)\\ &\qquad - 6\,\alpha\beta\gamma\\ &= (\alpha + \beta + \gamma)^3 - 3(\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha + \beta + \gamma) + 3\,\alpha\beta\gamma\\ &= -p^3 + 3\,pq - 3\,r\end{aligned}$$

14. $$\begin{aligned}\Sigma\alpha^2\beta^2 &= \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2\\ &= (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\,\alpha\beta\gamma(\alpha + \beta + \gamma)\\ &= q^2 - 2\,pr\end{aligned}$$

15. $$\begin{aligned}\Sigma\alpha^4 &= \alpha^4 + \beta^4 + \gamma^4\\ &= (\alpha^2 + \beta^2 + \gamma^2)^2 - 2(\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2)\\ &= [(\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)]^2 - 2[(\alpha\beta + \beta\gamma + \gamma\alpha)^2\\ &\qquad - 2\,\alpha\beta\gamma(\alpha + \beta + \gamma)]\\ &= (p^2 - 2\,q)^2 - 2(q^2 - 2\,pr)\\ &= p^4 - 4\,p^2q + 2\,q^2 + 4\,pr\end{aligned}$$

16. $$\begin{aligned}&(\beta + \gamma)(\gamma + \alpha)(\alpha + \beta)\\ &= (\alpha + \beta + \gamma - \alpha)(\alpha + \beta + \gamma - \beta)(\alpha + \beta + \gamma - \gamma)\\ &= (-p - \alpha)(-p - \beta)(-p - \gamma)\\ &= -(p + \alpha)(p + \beta)(p + \gamma)\\ &= -[p^3 + (\alpha + \beta + \gamma)p^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)p + \alpha\beta\gamma]\\ &= -(p^3 - p^3 + pq - r)\\ &= -pq + r\end{aligned}$$

17. $$\begin{aligned}&\frac{\beta\gamma}{\alpha} + \frac{\gamma\alpha}{\beta} + \frac{\alpha\beta}{\gamma}\\ &= \frac{\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2}{\alpha\beta\gamma}\\ &= \frac{(\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\,\alpha\beta\gamma(\alpha + \beta + \gamma)}{\alpha\beta\gamma}\\ &= \frac{q^2 - 2\,pr}{-r}\\ &= \frac{2\,pr - q^2}{r}\end{aligned}$$

18. $$\frac{\beta^2+\gamma^2}{\beta\gamma}+\frac{\gamma^2+\alpha^2}{\gamma\alpha}+\frac{\alpha^2+\beta^2}{\alpha\beta}$$

$$=\frac{\alpha\beta^2+\alpha\gamma^2+\beta\gamma^2+\beta\alpha^2+\gamma\alpha^2+\gamma\beta^2}{\alpha\beta\gamma}$$

$$=\frac{(\alpha\beta+\beta\gamma+\gamma\alpha)(\alpha+\beta+\gamma)-3\,\alpha\beta\gamma}{\alpha\beta\gamma}$$

$$=\frac{-pq+3r}{-r}$$

$$=\frac{pq-3r}{r}$$

19. $$\frac{\beta^2+\gamma^2}{\beta+\gamma}+\frac{\gamma^2+\alpha^2}{\gamma+\alpha}+\frac{\alpha^2+\beta^2}{\alpha+\beta}$$

$$=\frac{(\beta+\gamma)^2-2\,\beta\gamma}{\beta+\gamma}+\frac{(\gamma+\alpha)^2-2\,\gamma\alpha}{\gamma+\alpha}+\frac{(\alpha+\beta)^2-2\,\alpha\beta}{\alpha+\beta}$$

$$=2(\alpha+\beta+\gamma)-2\left(\frac{\beta\gamma}{\beta+\gamma}+\frac{\gamma\alpha}{\gamma+\alpha}+\frac{\alpha\beta}{\alpha+\beta}\right)$$

$$=-2p-2\left(\frac{\begin{array}{r}\alpha^2\beta\gamma+\beta^2\gamma\alpha+\gamma^2\alpha\beta+\beta^2\gamma^2+\beta^2\gamma\alpha+\gamma^2\alpha\beta+\gamma\alpha^2\beta+\gamma^2\alpha^2\\ +\gamma^2\alpha\beta+\alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha^2\beta^2\end{array}}{(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)}\right)$$

$$=-2p-2\left(\frac{3\,\alpha\beta\gamma(\alpha+\beta+\gamma)+\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2}{(\alpha+\beta+\gamma-\gamma)(\alpha+\beta+\gamma-\alpha)(\alpha+\beta+\gamma-\beta)}\right)$$

$$=-2p-2\left(\frac{3pr+(\alpha\beta+\beta\gamma+\gamma\alpha)^2-2\,\alpha\beta\gamma(\alpha+\beta+\gamma)}{(-p-\gamma)(-p-\alpha)(-p-\beta)}\right)$$

$$=-2p+2\left(\frac{3pr+q^2-2pr}{p^3+(\alpha+\beta+\gamma)p^2+(\alpha\beta+\beta\gamma+\gamma\alpha)p+\alpha\beta\gamma}\right)$$

$$=-2p+2\left(\frac{pr+q^2}{p^3-p^3+pq-r}\right)$$

$$=-2p+2\,\frac{pr+q^2}{pq-r}$$

$$=\frac{-2p^2q+4pr+2q^2}{pq-r}$$

In the equation $x^3+px^2+qx+r=0$, find the condition that:

20. One root is the negative of one of the other two roots.

Let $\alpha=-\beta$

Then $p=-\alpha-\beta-\gamma=-\gamma$

$q=\alpha\beta+\beta\gamma+\gamma\alpha=-\beta^2$

$r=-\alpha\beta\gamma=\beta^2\gamma$

Eliminating β and γ from these three equations, we have

$$pq = r$$

This condition is sufficient. For if we put pq for r in the equation, we have

$$x^3 + px^2 + qx + pq = 0$$

or

$$(x^2 + q)(x + p) = 0$$

$$\therefore\ x = \pm\sqrt{q}, \text{ or } -p$$

21. One root is double another.

Let $\alpha = 2\beta$

Then
$$p = -\alpha - \beta - \gamma = -3\beta - \gamma \qquad (1)$$
$$q = \alpha\beta + \beta\gamma + \gamma\alpha = 2\beta^2 + 3\beta\gamma \qquad (2)$$
$$r = -\alpha\beta\gamma = -2\beta^2\gamma \qquad (3)$$

Eliminate β and γ from these three equations.

From (1), $\gamma = -p - 3\beta$

Substitute the value of γ in (2) and (3),

$$q = 2\beta^2 - 3\beta p - 9\beta^2$$
$$= -7\beta^2 - 3\beta p \qquad (4)$$
$$r = 2\beta^2 p + 6\beta^3 \qquad (5)$$

(4) is $\quad 7\beta^2 + 3p\beta + q = 0$

$\beta \times (4)$ is $\quad 7\beta^3 + 3p\beta^2 + q\beta = 0$

$\beta^2 \times (4)$ is $\quad 7\beta^4 + 3p\beta^3 + q\beta^2 = 0$

(5) is $\quad 6\beta^3 + 2p\beta^2 - r = 0$

$\beta \times (5)$ is $\quad 6\beta^4 + 2p\beta^3 - r\beta = 0$

From these 5 equations eliminate β^4, β^3, β^2, and β.

The result is:

$$\begin{vmatrix} 0 & 0 & 7 & 3p & q \\ 0 & 7 & 3p & q & 0 \\ 7 & 3p & q & 0 & 0 \\ 0 & 6 & 2p & 0 & -r \\ 6 & 2p & 0 & -r & 0 \end{vmatrix} = 0$$

Or, $\begin{vmatrix} 0 & 0 & 7 & 3p & q \\ 0 & 7 & 3p & q & 0 \\ 7 & 3p & q & 0 & 0 \\ 0 & 6 & 2p & 0 & -r \\ 0 & -4p & -6q & -7r & 0 \end{vmatrix} = 0$ Or, $\begin{vmatrix} 0 & 7 & 3p & q \\ 7 & 3p & q & 0 \\ 6 & 2p & 0 & -r \\ -4p & -6q & -7r & 0 \end{vmatrix} = 0$

Or, $\begin{vmatrix} 0 & 7 & 3p & q \\ 7 & 3p & q & 0 \\ 6q & 2pq + 7r & 3pr & 0 \\ -4p & -6q & -7r & 0 \end{vmatrix} = 0$ Or, $\begin{vmatrix} 7 & 3p & q \\ 6q & 2pq + 7r & 3pr \\ -4p & -6q & -7r \end{vmatrix} = 0$

This reduces to

$$7(-14pqr-4qr^2+18pqr)-6q(-21pr-6q^2)-4p(9p^2r-2pq^2-7qr)=0$$

Or, $$182pqr-343r^2+36q^3-36p^3r+8p^2q^2=0$$

Or, $$36p^3r-8p^2q^2-182pqr-36q^3+343r^2=0$$

which is the required condition.

22. The three roots are in arithmetical progression.

Let β be the middle root.

Then $$\beta=\frac{\alpha+\gamma}{2}$$

$$\therefore\ r=-\alpha-\beta-\gamma=-\frac{3(\alpha+\gamma)}{2} \qquad (1)$$

$$q=\alpha\beta+\beta\gamma+\gamma\alpha=\frac{(\alpha+\gamma)^2}{2}+\alpha\gamma \qquad (2)$$

$$r=-\alpha\beta\gamma=-\frac{(\alpha+\gamma)\alpha\gamma}{2} \qquad (3)$$

Eliminate α and γ from these three equations.

From (1), $$\alpha+\gamma=-\frac{2p}{3}$$

Substitute value of $\alpha+\gamma$ in (3),

$$\therefore\ r=\frac{p\alpha\gamma}{3}$$

$$\therefore\ \alpha\gamma=\frac{3r}{p}$$

Substitute value of $\alpha+\gamma$ and $\alpha\gamma$ in (2),

$$\therefore\ q=\frac{\left(-\frac{2p}{3}\right)^2}{2}+\frac{3r}{p}$$

$$q=\frac{2p^2}{9}+\frac{3r}{p}$$

$$\therefore\ 9pq-2p^3-27r=0$$

which is the required condition.

23. The three roots are in harmonical progression.

Let β be the middle root.

Then $$\frac{2}{\beta}=\frac{1}{\alpha}+\frac{1}{\gamma}$$

$$\beta=\frac{2\alpha\gamma}{\alpha+\gamma}$$

$$\alpha\beta+\beta\gamma=2\alpha\gamma$$

$$\therefore\ p = -\alpha - \beta - \gamma = -(\alpha + \gamma) - \frac{2\alpha\gamma}{\alpha + \gamma} \qquad (1)$$

$$q = \alpha\beta + \beta\gamma + \gamma\alpha = 3\alpha\gamma \qquad (2)$$

$$r = -\alpha\beta\gamma = -\frac{2\alpha^2\gamma^2}{\alpha + \gamma} \qquad (3)$$

Eliminate α and γ from these three equations.

From (2), $\alpha\gamma = \frac{q}{3}$

Substitute value of $\alpha\gamma$ in (3),

$$\therefore\ r = -\frac{2q^2}{9(\alpha + \gamma)}$$

$$\therefore\ \alpha + \gamma = -\frac{2q^2}{9r}$$

Substitute values of $\alpha\gamma$ and $\alpha + \gamma$ in (1),

$$\therefore\ p = \frac{2q^2}{9r} + \frac{\frac{2q}{3}}{\frac{2q^2}{9r}}$$

$$= \frac{2q^2}{9r} + \frac{3r}{q}$$

$$\therefore\ 9pqr - 2q^3 - 27r^2 = 0$$

which is the required condition.

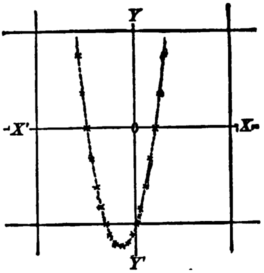
Exercise 71.

Construct the graphs of the following functions:

1. $x^2 + 3x - 10$.

Put $y = x^2 + 3x - 10$.

$x =$	$y =$	$x =$	$y =$
0	-10	$-\frac{1}{2}$	$-11\frac{1}{4}$
$\frac{1}{2}$	$-8\frac{1}{4}$	-1	-12
1	-6	$-\frac{3}{2}$	$-12\frac{1}{4}$
$\frac{3}{2}$	$-3\frac{1}{4}$	-2	-12
2	0	$-\frac{5}{2}$	$-11\frac{1}{4}$
$\frac{5}{2}$	$3\frac{3}{4}$	-3	-10
3	8	$-\frac{7}{2}$	$-8\frac{1}{4}$
		-4	-6
		$-\frac{9}{2}$	$-3\frac{1}{4}$
		-5	0
		$-\frac{11}{2}$	$3\frac{3}{4}$
		-6	8



The curve crosses the axis $X'X$ at the points for which

$$x^2+3x-10=0$$

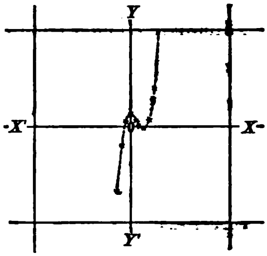
or $$(x+5)(x-2)=0$$

$$\therefore\ x=2, \text{ or } -5$$

2. x^3-2x^2+1.

Put $y=x^3-2x^2+1$.

$x=$	$y=$	$x=$	$y=$
0	1	$-\frac{1}{2}$	$\frac{5}{8}$
$\frac{1}{2}$	$\frac{5}{8}$	-1	-2
1	0	$-\frac{3}{2}$	$-6\frac{7}{8}$
$\frac{3}{2}$	$-\frac{1}{8}$	-2	-17
2	1		
$\frac{5}{2}$	$4\frac{1}{8}$		
3	10		



The curve crosses the axis $X'X$ at the points for which

$$x^3-2x^2+1=0$$

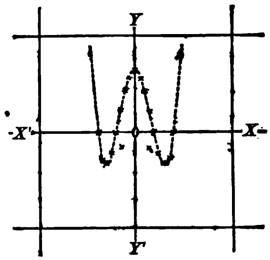
or $$(x-1)(x^2-x-1)=0$$

$$\therefore\ x=1, \text{ or } \frac{1\pm\sqrt{5}}{2}$$

3. x^4-20x^2+64.

Put $y=x^4-20x^2+64$.

$x=$	$y=$
0	64
$\pm\frac{1}{2}$	$59\frac{1}{16}$
± 1	45
$\pm\frac{3}{2}$	$24\frac{1}{16}$
± 2	0
$\pm\frac{5}{2}$	$-41\frac{15}{16}$
± 3	-35
$\pm\frac{7}{2}$	$-30\frac{15}{16}$
± 4	0
$\pm\frac{9}{2}$	$69\frac{1}{16}$
± 5	189



The curve cuts the axis $X'X$ in the points for which

$$x^4-20x^2+64=0$$

or $$(x^2-16)(x^2-4)=0$$

$$\therefore\ x=\pm 2, \text{ or } \pm 4$$

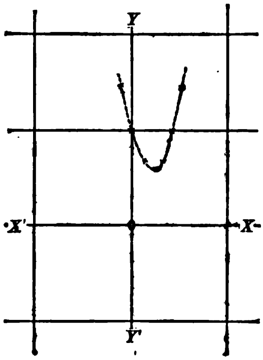
The curve is evidently symmetrical with respect to the axis $Y'Y$.

In the figure the curve has been shortened vertically. Each division on the $Y'Y$ axis represents 10 units.

4. $x^2 - 4x + 10$.

Let $$y = x^2 - 4x + 10$$

$x =$	$y =$	$x =$	$y =$
0	10	-1	15
1	7	-2	22
2	6		
3	7		
4	10		
5	15		



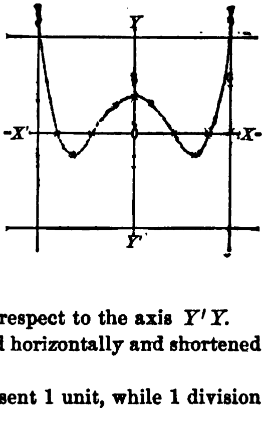
The curve does not cut the axis $X'X$.

For $x^2 - 4x + 10 = (x-2)^2 + 6$,
and this is never negative.

5. $x^4 - 5x^2 + 4$.

Let $$y = x^4 - 5x^2 + 4$$

$x =$	$y =$
0	4
$\pm \frac{1}{2}$	$3\frac{13}{16}$
± 1	0
$\pm \frac{3}{2}$	$-2\frac{3}{16}$
± 2	0
$\pm \frac{5}{2}$	$11\frac{13}{16}$



The curve cuts the axis $X'X$ in the points for which

$$x^4 - 5x^2 + 4 = 0$$

or $$(x^2 - 1)(x^2 - 4) = 0$$

$$\therefore\ x = \pm 1, \text{ or } \pm 4$$

The curve is evidently symmetrical with respect to the axis $Y'Y$.

In the figure the curve has been magnified horizontally and shortened vertically.

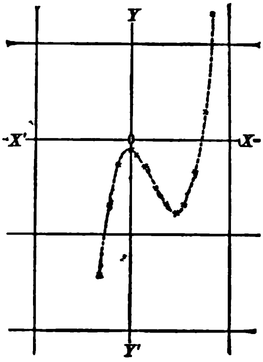
Thus 4 divisions of the $X'X$ axis represent 1 unit, while 1 division of the $Y'Y$ axis represents 10 units.

6. $x^3 - 4x^2 + x - 1$.

Let $$y = x^3 - 4x^2 + x - 1$$

$x =$	$y =$	$x =$	$y =$
0	-1	$-\frac{1}{2}$	$-2\frac{5}{8}$
$\frac{1}{2}$	$-1\frac{3}{8}$	-1	-7
1	-3	$-\frac{3}{2}$	$-14\frac{7}{8}$

$x =$	$y =$	$x =$	$y =$
$\frac{3}{2}$	$-5\frac{1}{8}$	$\frac{7}{2}$	$-3\frac{5}{8}$
2	-7	4	3
$\frac{5}{2}$	$-7\frac{7}{8}$	$\frac{9}{2}$	$13\frac{5}{8}$
3	-7		



The curve cuts the axis $X'X$ in the points for which

$$x^3 - 4x^2 + x - 1 = 0$$

From the figure it appears that two of these points are imaginary, while the third lies between $\frac{7}{2}$ and 4.

The figure is magnified horizontally. Two divisions of the $X'X$ axis represent 1 unit.

Exercise 72.

Find the derivatives with respect to x of:

1. x^2.	2. x^3.	3. $\frac{1}{x}$.	4. x^{-2}.	5. x^4.	6. $\frac{1}{x^3}$.	7. x^{-4}.
$2x$.	$3x^2$.	$-\frac{1}{x^2}$.	$-2x^{-3}$.	$4x^3$.	$-\frac{3}{x^4}$.	$-4x^{-5}$.

8. $x^2 + x$.

$$\frac{(x+h)^2 + x + h - x^2 - x}{h} = \frac{2hx + h^2 + h}{h}$$

$$= 2x + h + 1$$

Put $\qquad h = 0$

Then the derivative $= 2x + 1$.

9. $x^3 + 2x^2$.

$$\frac{(x+h)^3 + 2(x+h)^2 - x^3 - 2x^2}{h} = \frac{3hx^2 + 3h^2x + h^3 + 4hx + h^2}{h}$$

$$= 3x^2 + 3hx + h^2 + 4x + h$$

Put $\qquad h = 0$

Then the derivative $= 3x^2 + 4x$.

10. $(x + a)^2$.

$$\frac{(x+h+a)^2 - (x+a)^2}{h} = \frac{2h(x+a) + h^2}{h}$$

$$= 2(x+a) + h$$

Put $\qquad h = 0$

Then the derivative $= 2(x + a)$.

11. $\dfrac{1}{x^2-3}$.

$$\left(\frac{1}{(x+h)^2-3}-\frac{1}{x^2-3}\right)\div h=\frac{x^2-3-(x+h)^2+3}{h[(x+h)^2-3](x^2-3)}$$

$$=\frac{-2hx+h^2}{h[(x+h)^2-3](x^2-3)}$$

$$=-\frac{2x+h}{[(x+h)^2-3](x^2-3)}$$

Put $h=0$

Then the derivative $=-\dfrac{2x}{(x^2-3)^2}$.

12. $(x+1)^{-2}$.

$$\left(\frac{1}{(x+h+1)^2}-\frac{1}{(x+1)^2}\right)\div h=\frac{(x+1)^2-(x+h+1)^2}{h(x+h+1)^2(x+1)^2}$$

$$=\frac{-2h(x+1)-h^2}{h(x+h+1)^2(x+1)^2}$$

$$=\frac{-2(x+1)+h}{(x+h+1)^2(x+1)^2}$$

Put $h=0$

Then the derivative $=-\dfrac{2(x+1)}{(x+1)^4}=-\dfrac{2}{(x+1)^3}$.

Exercise 73.

Write the derivatives with respect to x of:

1. x^2+4.
   $2x$.

2. x^3+3x^2-1.
   $3x^2+6x$.

3. x^4+x^2+2.
   $4x^3+2x$.

4. $x^5-3x^4+x^3$.
   $5x^4-12x^3+3x^2$.

5. $4x^4+6x^3+2$.
   $16x^3+18x^2$.

6. $6x^5-7x^2+7x$.
   $30x^4-14x+7$.

7. $3x^5+4x^4+x^3-x^2-6x+5$.
   $15x^4+16x^3+3x^2-2x-6$.

8. $4x^5-2x^4-x^3+6x^2-7$.
   $20x^4-8x^3-3x^2+12x$.

9. $(x-2)(x+3)$.

$$(x-2)+(x+3)=2x+1.$$

10. $(x-1)(x-2)(x-3)$.

$$(x-1)(x-2)+(x-1)(x-3)+(x-2)(x-3)=3x^2-12x+11.$$

11. $(x-3)^2(x+4)$.

$$(x-3)^2+2(x-3)(x+4)=(x-3)(3x+5).$$

12. $(x-4)^2(x-2)(x+1)$.

$$(x-4)^2(x-2)+(x-4)^2(x+1)+2(x-4)(x-2)(x+1)$$
$$=(x-4)(4x^2-11x).$$

13. $(x-\alpha)^2(x-\beta)^2$.

$$2(x-\alpha)^2(x-\beta)+2(x-\alpha)(x-\beta)^2=2(x-\alpha)(x-\beta)(2x-\alpha-\beta).$$

14. $(x-\alpha)(x-\beta)(x-\gamma)$.

$$(x-\alpha)(x-\beta)+(x-\alpha)(x-\gamma)+(x-\beta)(x-\gamma)$$
$$=3x^2-2(\alpha+\beta+\gamma)x+\alpha\beta+\alpha\gamma+\beta\gamma.$$

15. $(x-2)(x-3)(x+5)(x+4)$.

$$(x-2)(x-3)(x+5)+(x-2)(x-3)(x+4)+(x-2)(x+5)(x+4)$$
$$+(x-3)(x+5)(x+4)=4x^3+12x^2-38x-46.$$

16. $(x^2+2)(x^2-4x+8)$.

$$(x^2+2)(2x-4)+2x(x^2-4x+8)=4x^3-12x^2+20x-8.$$

Exercise 74.

Write the successive derivatives with respect to x of:

1. x^3-4x^2+2.

$fx \quad =x^3-4x^2+2$

$f'x \quad =3x^2-8x$

$f''x \quad =6x-8$

$f'''x=6$

$f^{\text{iv}}x \quad =0$

2. x^3+4x^2-5x.

$fx \quad =x^3+4x^2-5x$

$f'x \quad =3x^2+8x-5$

$f''x \quad =6x+8$

$f'''x=6$

$f^{\text{iv}}x \quad =0$

3. $2x^3+2x^2-4x+1$.

$fx \quad =2x^3+2x^2-4x+1$

$f'x \quad =6x^2+4x-4$

$f''x \quad =12x+4$

$f'''x=12$

$f^{\text{iv}}x \quad =0$

4. $3x^4+3x^3-x^2+x$.

$fx \quad =3x^4+3x^3-x^2+x$

$f'x \quad =12x^3+9x^2-2x+1$

$f''x \quad =36x^2+18x-2$

$f'''x=72x+18$

$f^{\text{iv}}x \quad =72$

$f^{\text{v}}x \quad =0$

5. $ax^3 + 3bx^2 + 3cx + d.$

$$\begin{aligned} fx &= ax^3 + 3bx^2 + 3cx + d \\ f'x &= 3ax^2 + 6bx + 3c \\ f''x &= 6ax + 6b \\ f'''x &= 6a \\ f^{\text{iv}}x &= 0 \end{aligned}$$

6. $ax^4 + 4bx^3 + 6cx^2 + 4dx + e.$

$$\begin{aligned} fx &= ax^4 + 4bx^3 + 6cx^2 + 4dx + e \\ f'x &= 4ax^3 + 12bx^2 + 12cx + 4d \\ f''x &= 12ax^2 + 24bx + 12c \\ f'''x &= 24ax + 24b \\ f^{\text{iv}}x &= 24a \\ f^{\text{v}} &= 0 \end{aligned}$$

7. $(x-\alpha)^2(x-\beta).$

$$\begin{aligned} fx &= (x-\alpha)^2(x-\beta) \\ f'x &= (x-\alpha)^2 + 2(x-\alpha)(x-\beta) \\ &= (x-\alpha)(3x-\alpha-2\beta) \\ f''x &= 3(x-\alpha) + 3x - \alpha - 2\beta \\ &= 6x - 4\alpha - 2\beta \\ f'''x &= 6 \\ f^{\text{iv}}x &= 0 \end{aligned}$$

8. $(x-\alpha)(x-\beta)(x-\gamma).$

$$\begin{aligned} fx &= (x-\alpha)(x-\beta)(x-\gamma) \\ f'x &= (x-\alpha)(x-\beta) + (x-\alpha)(x-\gamma) + (x-\beta)(x-\gamma) \\ f''x &= (x-\alpha)+(x-\beta)+(x-\alpha)+(x-\gamma)+(x-\beta)+(x-\gamma) \\ &= 6x - 2(\alpha+\beta+\gamma) \\ f'''x &= 6 \\ f^{\text{iv}}x &= 0 \end{aligned}$$

9. $(x-\alpha)^2(x-\beta)^2.$

$$\begin{aligned} fx &= (x-\alpha)^2(x-\beta)^2 \\ f'x &= 2(x-\alpha)^2(x-\beta) + 2(x-\alpha)(x-\beta)^2 \\ f''x &= 2(x-\alpha)^2 + 4(x-\alpha)(x-\beta) + 4(x-\alpha)(x-\beta) + 2(x-\beta)^2 \\ &= 2(x-\alpha)^2 + 8(x-\alpha)(x-\beta) + 2(x-\beta)^2 \\ f'''x &= 4(x-\alpha) + 8(x-\alpha) + 8(x-\beta) + 4(x-\beta) \\ &= 12(2x-\alpha-\beta) \\ f^{\text{iv}}x &= 24 \\ f^{\text{v}}x &= 0 \end{aligned}$$

10. $x^3 - x^2 + 1 \qquad 2.$

$f\,(x) = x^3 - x^2 + 1$

$f'\,(x) = 3x^2 - 2x$

$\therefore f'\,(2) = 8$

$\therefore f\,(x)$ is increasing.

11. $x^4 - x^2 + 6x - 1 \qquad 4.$

$f\,(x) = x^4 - x^2 + 6x - 1$

$f'\,(x) = 4x^3 - 2x + 6$

$\therefore f\,(4) = 254$

$\therefore f\,(x)$ is increasing.

12. $2x^4 + 3x^2 - 6x \qquad 1.$

$f\,(x) \;= 2x^4 + 3x^2 - 6x$

$f'\,(x) \;= 8x^3 + 6x - 6$

$\therefore f'\,(1) \;= 8$

$\therefore f\,(x)$ is increasing.

13. $4x^4 - 3x^2 + 4x - 6 \quad -3.$

$f\,(x) \;= 4x^4 - 3x^2 + 4x - 6$

$f'\,(x) \;= 16x^3 - 6x + 4$

$\therefore f'\,(-3) = -410$

$\therefore f\,(x)$ is decreasing.

Exercise 75.

The following equations have multiple roots. Find all the roots of each equation:

1. $x^3 - 8x^2 + 13x - 6 = 0.$

$$f\,(x) = x^3 - 8x^2 + 13x - 6$$
$$f'\,(x) = 3x^2 - 16x + 13$$

Put
$$f'x = 0$$
$$3x^2 - 16x + 13 = 0$$
$$(x-1)(3x-13) = 0$$
$$\therefore\ x = 1 \text{ or } \tfrac{13}{3}$$

Substitute $x = 1$ in $f(x)$,

$$\begin{array}{rrrr} 1 & -8 & +13 & -6 \\ & 1 & -7 & +6 \\ \hline 1 & -7 & +6 & +0 \end{array}$$

1 is therefore a double root of $f(x) = 0$, and there can be no other double root.

We find
$$fx = (x-1)^2(x-6)$$

$\therefore$ The three roots are 1, 1, and 6.

2. $x^3 - 7x^2 + 16x - 12 = 0.$

$$f\,(x) = x^3 - 7x^2 + 16x - 12$$
$$f'\,(x) = 3x^2 - 14x + 16$$

Put
$$f'\,(x) = 0$$
$$3x^2 - 14x + 16 = 0$$
$$(x-2)(3x-8) = 0$$
$$\therefore\ x = 2 \text{ or } \tfrac{8}{3}$$

Substitute $x = 2$ in $f(x)$,

$$\begin{array}{rrrr} 1 & -7 & +16 & -12 \\ & 2 & -10 & +12 \\ \hline 1 & -5 & +6 & +0 \end{array}$$

$\therefore$ 2 is a double root of $f(x) = 0$

We find $fx = (x-2)^2(x-3)$

$\therefore$ The three roots are 2, 2, and 3.

3. $x^4 - 6x^2 - 8x - 3 = 0.$

$$f(x) = x^4 - 6x^2 - 8x - 3$$
$$f'(x) = 4x^3 - 12x - 8$$

Find the H. C. F. of $f(x)$ and $f'(x)$.

$$\begin{array}{r|r|l} 4)\underline{4+0-12-8} & 1+0-6-8-3 & 1+0 \\ 1+0-\ \ 3-2 & \underline{1+0-3-2} & \\ \underline{1+2+\ \ 1} & -3)\underline{-3-6-3} & \\ -2-\ \ 4-2 & 1+2+1 & 1-2 \\ \underline{-2-\ \ 4-2} & & \end{array}$$

$\therefore$ $x^2 + 2x + 1$ is the H. C. F.

$\therefore$ $x = -1$ is a triple root.

$fx = (x+1)^3(x-3)$.

$\therefore$ The four roots are -1, -1, -1, and 3.

4. $x^4 - 7x^3 + 9x^2 + 27x - 54 = 0.$

$$f(x) = x^4 - 7x^3 + 9x^2 + 27x - 54$$
$$f'(x) = 4x^3 - 21x^2 + 18x + 27$$

Find the H. C. F. of $f(x)$ and $f'(x)$.

$$\begin{array}{r|r|l} 4-21+18+27 & 1-\ \ 7+\ \ 9+\ \ 27-\ \ 54 & \\ \underline{4-24+36} & 4-28+\ \ 36+108-216 & 1-7 \\ 3-18+27 & \underline{4-21+\ \ 18+\ \ 27} & \\ \underline{3-18+27} & -\ \ 7+\ \ 18+\ \ 81-216 & \\ & -28+\ \ 72+324-864 & \\ & \underline{-28+147-126-189} & \\ & -75)\underline{-\ \ 75+450-675} & \\ & 1-\ \ 6+\ \ 9 & 4+3 \end{array}$$

$\therefore$ $x^2 - 6x + 9$ is the H. C. F.

$\therefore$ $x = 3$ is a triple root.

$f(x) = (x-3)^3(x+2)$.

$\therefore$ The fourth roots are 3, 3, 3, and -2.

5. $x^4+6x^3+x^2-24x+16=0$.

$$f(x)=x^4+6x^3+x^2-24x+16$$
$$f'(x)=4x^3+18x^2+2x-24$$

Find the H. C. F. of $f(x)$ and $f'(x)$.

$$\begin{array}{r|r|l}
2\underline{)4+18+2-24} & 1+6+1-24+16 & \\
2+9+1-12 & 2+12+2-48+32 & 1+3 \\
\underline{2+6-8} & \underline{2+9+1-12} & \\
3+9-12 & 3+1-36+32 & \\
\underline{3+9-12} & 6+2-72+64 & \\
 & \underline{6+27+3-36} & \\
 & -25\underline{)-25-75+100} & \\
 & 1+3-4 & 2+3
\end{array}$$

$\therefore$ x^2+3x-4 is the H. C. F.

$x^2+3x-4=(x-1)(x+4)$.

$\therefore$ 1 and -4 are double roots.

$fx=(x-1)^2(x+4)^2$.

$\therefore$ The four roots are 1, 1, -4, and -4.

6. $x^5-11x^4+19x^3+115x^2-200x-500=0$.

$$f(x)=x^5-11x^4+19x^3+115x^2-200x-500$$
$$f'(x)=5x^4-44x^3+57x^2+230x-200$$

Find the H. C. F. of $f(x)$ and $f'(x)$.

$$\begin{array}{r|r|l}
5-44+57+280-200 & 1-11+19+115-200-500 & 1-11 \\
\underline{5-40+25+250} & 5-55+95+575-1000-2500 & \\
-4+32-20-200 & \underline{5-44+57+230-200} & \\
\underline{-4+32-20-200} & -11+38+345-800-2500 & \\
 & -55+190+1725-4000-12500 & \\
 & \underline{-55+484-627-2530+2200} & \\
 & -294\underline{)-294+2352-1470-14700} & \\
 & 1-8+5+50 & 5-4
\end{array}$$

$\therefore$ $x^3-8x^2+5x+50$ is the H. C. F.

$x^3-8x^2+5x+50=(x+2)(x-5)^2$

$\therefore$ -2 is a double root.

-5 is a triple root.

$f(x)=(x+2)^2(x-5)^3$

The five roots are -2, -2, 5, 5, and 5.

7. Resolve into linear factors

$$x^6-5x^5+5x^4+9x^3-14x^2-4x+8.$$

$$f(x)=x^6-5x^5+5x^4+9x^3-14x^2-4x+8$$
$$f'(x)=6x^5-25x^4+20x^3+27x^2-28x-4$$

Find the H. C. F. of $f(x)$ and $f'(x)$.

6− 25+ 20+ 27− 28− 4	1− 5+ 5+ 9− 14− 4+ 8	
390−1625+1300+1755−1820− 260	6−30+ 30+ 54− 84− 24+ 48	1 − 5
390−1572+1206+1560−1608	6−25+ 20+ 27− 28− 4	
− 53+ 94+ 195− 212− 260	− 5+ 10+ 27− 56− 20+ 48	
− 53+ 742−1749− 212+2332	−30+ 60+162−336−120+288	1
648)− 648+1944+ 0−2592	−30+125−100−135+140+ 20	
− 1+ 3+ 0− 4	− 65+262−201−260+268	− 6
	− 53+ 94+195−212−260	
	12)− 12+168−396− 48+528	1 − 11
	− 1+ 14− 33− 4+ 44	53
	− 1+ 3+ 0− 4	
	11− 33+ 0+ 44	
	11− 33+ 0+ 44	

$\therefore\ x^3 - 3x^2 + 4$ is the H. C. F.

$x^3 - 3x^2 + 4 = (x+1)(x-2)^2$

$\therefore\ -1$ is a double root.

2 is a triple root.

$f(x) = (x+1)^2(x-2)^3(x-1)$

8. Show that the equation of the form $x^n = a^n$ can have no multiple root.

$$f(x) = x^n - a^n$$
$$f'(x) = nx^{n-1}$$

If $$f'(x) = 0$$
$$x^{n-1} = 0$$
$$\therefore\ x = 0$$

But 0 is not a root of $x^n - a^n = 0$,

$\therefore f(x)$ has no multiple roots.

9. Show that the condition that the equation
$$x^3 + 3qx + r = 0$$
shall have a double root is $4q^3 + r^2 = 0$.

$$f(x) = x^3 + 3qx + r$$
$$f'(x) = 3x^2 + 3q$$

If $f(x) = 0$ has a double root,
$$f(x) = 0 \qquad (1)$$
$$f'(x) = 0 \qquad (2)$$
simultaneously.

The required condition is found by eliminating x from these two equations.

From (2), $x = \sqrt{-q}$

Substitute value of x in (1),

$$-q\sqrt{-q} + 3q\sqrt{-q} + r = 0$$
$$2q\sqrt{-q} = -r$$
$$-4q^3 = r^2$$
$$r^2 + 4q^3 = 0$$

10. Show that the condition that the equation

$$x^3 + 3px^2 + r = 0$$

shall have a double root is $r(4p^3 - r) = 0$.

$$f(x) = x^3 + 3px^2 + r$$
$$f'(x) = 3x^2 + 6px$$

If $f(x) = 0$ has a double root,

$$f(x) = 0 \qquad (1)$$
$$f'(x) = 0 \qquad (2)$$

simultaneously.

The required condition is found by eliminating x from these two equations.

From (2), $x = 0$ or $x = -2p$

Substitute these values of x in (1),

$$r = 0 \text{ or } -8p^3 + 12p^3 + r = 0$$
$$4p^3 + r = 0$$

Hence $x^3 + 3px^2 + r = 0$

has a double root if $r = 0$, or if $4p^3 + r = 0$

That is, if $r(4p^3 + r) = 0$

Exercise 76.

In the following quantics put for x the expression opposite, and reduce:

1. $x^3 - 3x^2 + 4x - 6$ $\quad x + 2.$

$$\begin{array}{rrrr|l}
1 & -3 & +4 & -6 & \underline{2} \\
 & +2 & -2 & +4 & \\ \hline
1 & -1 & +2 & -2 & \\
 & +2 & +2 & & \\ \hline
1 & +1 & +4 & & \\
 & +2 & & & \\ \hline
1 & +3 & & &
\end{array}$$

$$x^3 + 3x^2 + 4x - 2$$

2. $x^4 - 2x^2 + 6x - 3$ $\quad x + 4.$

$$\begin{array}{rrrrr|l}
1 & +0 & -2 & +6 & -3 & \underline{4} \\
 & 4 & +16 & +56 & +248 & \\ \hline
1 & +4 & +14 & +62 & +245 & \\
 & 4 & +32 & +184 & & \\ \hline
1 & +8 & +46 & +246 & & \\
 & 4 & +48 & & & \\ \hline
1 & +12 & +94 & & & \\
 & 4 & & & & \\ \hline
1 & +16 & & & &
\end{array}$$

$$x^4 + 16x^3 + 94x^2 + 246x + 245$$

3. $3x^4-2x^3+2x^2-x-4 \quad x+3.$

$$\begin{array}{rrrrrl}
3 & -2 & +2 & -1 & -4 & \underline{|3} \\
 & 9 & +21 & +69 & +204 & \\ \hline
3 & +7 & +23 & +68 & +200 & \\
 & 9 & +48 & +213 & & \\ \hline
3 & +16 & +71 & +281 & & \\
 & 9 & +75 & & & \\ \hline
3 & +25 & +146 & & & \\
 & 9 & & & & \\ \hline
3 & +34 & & & &
\end{array}$$

$3x^4+34x^3+146x^2+281x+200$

4. $2x^4-3x^3+6x^2-7x-8 \quad x-2.$

$$\begin{array}{rrrrrl}
2 & -3 & +6 & -7 & -8 & \underline{|-2} \\
 & -4 & +14 & -40 & +94 & \\ \hline
2 & -7 & +20 & -47 & +86 & \\
 & -4 & +22 & -84 & & \\ \hline
2 & -11 & +42 & -131 & & \\
 & -4 & +30 & & & \\ \hline
2 & -15 & +72 & & & \\
 & -4 & & & & \\ \hline
2 & -19 & & & &
\end{array}$$

$2x^4-19x^3+72x^2-131x+86$

5. $2x^4-2x^3+4x^2-5x-4 \quad x-3.$

$$\begin{array}{rrrrrl}
2 & -2 & +4 & -5 & -4 & \underline{|-3} \\
 & -6 & +24 & -84 & +267 & \\ \hline
2 & -8 & +28 & -89 & +263 & \\
 & -6 & +42 & -210 & & \\ \hline
2 & -14 & +70 & -299 & & \\
 & -6 & +60 & & & \\ \hline
2 & -20 & +130 & & & \\
 & -6 & & & & \\ \hline
2 & -26 & & & &
\end{array}$$

$2x^4-26x^3+130x^2-299x+263$

Exercise 77.

Multiply the roots of each of the following equations by the number placed opposite the equation.

1. $x^3-3x^2+2x-4=0 \qquad -1.$

$$x^3+3x^2+2x+4=0$$

2. $x^4+3x^2-2x-1=0 \qquad -2.$

$$x^4+3\times 2^2x^2+2\times 2^3x-2^4\equiv x^4+12x^2+16x-16=0$$

3. $2x^4-3x^3+x^2-6x-4=0 \qquad -3.$

$$2x^4+3\times 3x^3+3^2x^2+6\times 3^3x-4\times 3^4\equiv 2x^4+9x^3+9x^2+162x-324=0$$

4. $2x^4 - 3x^2 + 6x - 8 = 0 \qquad -2.$

$$2x^4 - 3 \times 2^2x^2 - 6 \times 2^3x - 8 \times 2^4 \equiv 2x^4 - 12x^2 - 48x - 128 = 0$$

or $$x^4 - 6x^2 - 24x - 64 = 0$$

5. $3x^5 - 4x^3 - 2x + 7 = 0 \qquad -2.$

$$3x^5 - 4 \times 2^2x^3 - 2 \times 2^4x - 7 \times 2^5 \equiv 3x^5 - 16x^3 - 32x - 224 = 0$$

Transform to equations with integral coefficients in the p form the equations:

6. $12x^3 - 4x^2 + 6x + 1 = 0.$

Divide by 12, $$x^3 - \tfrac{1}{3}x^2 + \tfrac{1}{2}x + \tfrac{1}{12} = 0$$

Multiply the roots by 6, $$x^3 - \frac{6}{3}x^2 + \frac{6^2}{2}x + \frac{6^3}{12} = 0$$

$$x^3 - 2x^2 + 18x + 18 = 0$$

7. $6x^3 + 10x^2 - 7x + 16 = 0.$

Divide by 6, $$x^3 + \tfrac{5}{3}x^2 - \tfrac{7}{6}x + \tfrac{8}{3} = 0$$

Multiply the roots by 6,

$$x^3 + \frac{5 \times 6}{3}x^2 - \frac{7 \times 6^2}{6}x + \frac{8 \times 6^3}{3} = 0$$

$$x^3 + 10x^2 - 42x + 576 = 0$$

8. $10x^4 + 5x^3 - 4x^2 + 25x - 30 = 0.$

Divide by 10, $$x^4 + \tfrac{1}{2}x^3 - \tfrac{2}{5}x^2 + \tfrac{5}{2}x - 3 = 0$$

Multiply the roots by 10,

$$x^4 + \frac{10}{2}x^3 - \frac{2 \times 10^2}{5}x^2 + \frac{5 \times 10^3}{2}x - 3 \times 10^4 = 0$$

$$x^4 + 5x^3 - 40 + 2500x - 30000 = 0$$

9. $6x^5 + 3x^4 + 4x^3 - 2x^2 + 6x - 18 = 0.$

Divide by 6, $$x^5 + \tfrac{1}{2}x^4 + \tfrac{2}{3}x^3 - \tfrac{1}{3}x^2 + x - 3 = 0$$

Multiply the roots by 6,

$$x^5 + \frac{6}{2}x^4 + \frac{2 \times 6^2}{3}x^3 - \frac{6^3}{3}x^2 + 6^4x - 3 \times 6^5 = 0$$

$$x^5 + 3x^4 + 24x^3 - 72x^2 + 1296x - 23328$$

Write the equations which have for their roots the reciprocals of the roots of the following equations:

10. $3x^4-2x^3+5x^2-6x+7=0.$

$$7x^4-6x^3+5x^2-2x+3=0$$

11. $2x^5-4x^3-5x^2-7x-8=0.$

$$8x^4+7x^3+5x^2+4x-2=0$$

12. $x^6-x^4+2x^2+4x-1=0.$

$$x^6-4x^5-2x^4+x^2-1=0$$

Diminish the roots of each of the following equations by the number opposite the equation:

13. $x^3-11x^2+31x-12=0$ 1.

$$\begin{array}{rrrr|l}
1 & -11 & +31 & -12 & 1 \\
 & 1 & -10 & +21 & \\
\hline
1 & -10 & +21 & +9 & \\
 & 1 & -9 & & \\
\hline
1 & -9 & +12 & & \\
 & 1 & & & \\
\hline
1 & -8 & & &
\end{array}$$

$x^3-8x^2+12x+9=0.$

14. $x^4-6x^3+4x^2+18x-5=0$ 2.

$$\begin{array}{rrrrr|l}
1 & -6 & +4 & +18 & -5 & 2 \\
 & 2 & -8 & -8 & +20 & \\
\hline
1 & -4 & -4 & +10 & +15 & \\
 & 2 & -4 & -16 & & \\
\hline
1 & -2 & -8 & -6 & & \\
 & 2 & +0 & & & \\
\hline
1 & +0 & -8 & & & \\
 & 2 & & & & \\
\hline
1 & +2 & & & &
\end{array}$$

$x^4+2x^3-8x^2-6x+15=0.$

15. $x^3+10x^2+13x-24=0$ -2.

$$\begin{array}{rrrr|l}
1 & +10 & +13 & -24 & -2 \\
 & -2 & -16 & +6 & \\
\hline
1 & +8 & -3 & -18 & \\
 & -2 & -12 & & \\
\hline
1 & +6 & -15 & & \\
 & -2 & & & \\
\hline
1 & +4 & & &
\end{array}$$

$x^3+4x^2-15x-18=0.$

16. $x^4+x^3-16x^2-4x+48=0$ 4.

$$\begin{array}{rrrrr|l}
1 & +1 & -16 & -4 & +48 & 4 \\
 & 4 & +20 & +16 & +48 & \\
\hline
1 & +5 & +4 & +12 & +96 & \\
 & 4 & +36 & +160 & & \\
\hline
1 & +9 & +40 & +172 & & \\
 & 4 & +52 & & & \\
\hline
1 & +13 & +92 & & & \\
 & 4 & & & & \\
\hline
1 & +17 & & & &
\end{array}$$

$x^4+17x^3+92x^2+172x+96=0.$

17. $x^4 + x^2 - 3x + 4 = 0$ 0.3.

$$\begin{array}{llllll}
1 & +\;0 & +\;1 & -\;3 & +\;4 & \underline{|\,0.3} \\
 & 0.3 & +0.09 & +0.327 & -0.8019 & \\
\hline
1 & +0.3 & +1.09 & -2.673 & +3.1981 & \\
 & 0.3 & +0.18 & +0.381 & & \\
\cline{1-4}
1 & +0.6 & +1.27 & -2.292 & & \\
 & 0.3 & +0.27 & & & \\
\cline{1-3}
1 & +0.9 & +1.54 & & & \\
 & 0.3 & & & & \\
\cline{1-2}
1 & +1.2 & & & &
\end{array}$$

$$x^4 + 1.2x^3 + 1.54x^2 - 2.292x + 3.1981 = 0.$$

18. $x^4 - 3x^3 - x^2 + 4x - 5 = 0$ -0.4.

$$\begin{array}{llllll}
1 & -\;3 & -\;1 & +\;4 & -\;5 & \underline{|-0.4} \\
 & -0.4 & +1.36 & -0.144 & -1.5424 & \\
\hline
1 & -3.4 & +0.36 & +3.856 & -6.5424 & \\
 & -0.4 & +1.52 & -0.752 & & \\
\cline{1-4}
1 & -3.8 & +1.88 & +3.104 & & \\
 & -0.4 & +1.68 & & & \\
\cline{1-3}
1 & -4.2 & +3.56 & & & \\
 & -0.4 & & & & \\
\cline{1-2}
1 & -4.6 & & & &
\end{array}$$

$$x^4 - 4.6x^3 + 3.56x^2 + 3.104x - 6.5424.$$

19. Form the equation which has for its roots the squares of the roots of the equation

$$x^3 - 2x^2 + 3x - 5 = 0.$$

Let y be a root of the required equation.

Then $$y = x^2$$
$$\therefore\ x = \pm\sqrt{y}$$

If we substitute this value of x in the given equation, the result is irrational in y.

But, if we write the equation in the form

$$x^3 + 3x = 2x^2 + 5$$

and square both sides

$$x^6 + 6x^4 + 9x^2 = 4x^4 + 20x^2 + 25$$

$$x^6 + 2x^4 - 11x^2 - 25 = 0$$

we have an equation which contains only powers of x^2.

Putting in this equation $y = x^2$ we have the result

$$y^3 + 2y^2 - 11y - 25 = 0.$$

20. Form the equation which has for its roots the squares of the differences of the roots of

$$x^3 - 4x^2 - 2x - 3 = 0. \qquad (1)$$

Let

$$\begin{aligned} y &= (\alpha - \beta)^2 \\ &= \alpha^2 + \beta^2 + \gamma^2 - \gamma^2 - 2\alpha\beta \\ &= 12 - \gamma^2 - \frac{2\alpha\beta\gamma}{\gamma} \\ &= 12 - \gamma^2 - \frac{6}{\gamma} \end{aligned}$$

Put

$$\gamma = x$$

Then

$$y = 12 - x^2 - \frac{6}{x}$$

or

$$x^3 + (y - 12)x + 6 = 0 \qquad (2)$$

But

$$x^3 - 4x^2 + 2x - 3 = 0 \qquad (1)$$

Subtract

$$4x^2 + (y - 14)x + 9 = 0 \qquad (3)$$

Multiply (3) by x, (1) by 4 and subtract

$$(y + 2)x^2 + x + 12 = 0 \qquad (4)$$

Eliminate x between (3) and (4)

$$\begin{vmatrix} 4 & y - 14 & 9 & 0 \\ 0 & 4 & y - 14 & 9 \\ y + 2 & 1 & 12 & 0 \\ 0 & y + 2 & 1 & 12 \end{vmatrix} = 0$$

From 4 times second row subtract 3 times fourth.

$$\begin{vmatrix} 4 & y - 14 & 9 \\ 0 & -3y + 10 & 4y - 59 \\ y + 2 & 1 & 12 \end{vmatrix} = 0$$

which reduces to $y^3 - 20y^2 + 100y - 547 = 0$, the required equation.

21. Given the equation

$$x^3 - 2x^2 + 4x - 4 = 0,$$

find the equation in y where $y = 2x^2 - 3$.

Write the given equation in the form

$$x^3 + 4x = 2x^2 + 4$$

Square both sides,

$$x^6 + 8x^4 + 16x^2 = 4x^4 + 16x^2 + 16$$

$$x^6 + 4x^4 - 16 = 0$$

But $$y = 2x^2 - 3$$

$$\therefore\ x^2 = \frac{y+3}{2}$$

Substitute the value of x^2 in the equation just obtained.

$$\left(\frac{y+3}{2}\right)^3 + 8\left(\frac{y+3}{2}\right)^2 - 16 = 0$$

$$\frac{y^3 + 9y^2 + 27y + 27}{8} + 2(y^2 + 6y + 9) - 16 = 0$$

$$y^3 + 25y^2 + 123y + 43 = 0$$

which is the required equation.

Exercise 78.

All the roots of the equations given below are real; determine their signs.

1. $x^4 + 4x^3 - 43x^2 - 58x + 240 = 0.$

The order of signs is $+ \quad + \quad - \quad - \quad +$

There are two variations and two permanences.

$\therefore$ There are two positive roots and two negative roots.

2. $x^3 - 22x^2 + 155x - 350 = 0.$

The order of signs is $+ \quad - \quad + \quad -$

There are 3 variations. $\therefore$ There are 3 positive roots.

3. $x^4 + 4x^3 - 35x^2 - 78x + 360 = 0.$

The order of signs is $+ \quad + \quad - \quad - \quad +$

There are two variations and two permanences.

$\therefore$ There are two positive roots and two negative roots.

4. $x^3 - 12x^2 - 43x - 30 = 0.$

The order of signs is $+ \quad - \quad - \quad -$

There is one variation and two permanences.

$\therefore$ There is one positive root and two negative roots.

5. $x^5 - 3x^4 - 5x^3 + 15x^2 + 4x - 12 = 0.$

The order of signs is $+ \quad - \quad - \quad + \quad + \quad -$

There are three variations and two permanences.

$\therefore$ There are three positive roots and two negative roots.

6. $x^3 - 12x^2 + 47x - 60 = 0.$

The order of signs is $+ \quad - \quad + \quad -$

There are three variations. $\therefore$ There are three positive roots.

7. $x^4 - 2x^3 - 13x^2 + 38x - 24 = 0.$

The order of signs is $+ \quad - \quad - \quad + \quad -$

There are three variations and one permanence.

$\therefore$ There are three positive roots and one negative root.

8. $x^5 - x^4 - 187x^3 - 359x^2 + 186x + 360 = 0.$

The order of signs is $+ \quad - \quad - \quad - \quad + \quad +$

There are two variations and three permanences.

$\therefore$ There are two positive roots and three negative roots.

9. $x^6 - 10x^5 + 19x^4 + 110x^3 - 536x^2 + 800x - 384 = 0.$

The order of signs is $+ \quad - \quad + \quad + \quad - \quad + \quad -$

There are five variations and one permanence.

$\therefore$ There are five positive roots and one negative root.

10. If an equation involves only even powers of x, and the signs are all positive, the equation has no real root.

For, if there were a real root, its even powers would all be positive, and when these are multiplied by the positive coefficients and the results added, the sum will be positive, and therefore cannot be equal to 0, unless the root is itself 0. With this exception there can therefore be no real root.

11. If an equation involves only odd powers of x, and the signs are all positive, the equation has the root 0, and no other real root.

For the equation is divisible by x, so that $x = 0$ gives one root. But in the quotient all the powers of x are even, and since the coefficients are all positive, there is no other real root.

12. Show that the equation

$$x^6 - 3x^2 - x + 1$$

has at least two imaginary roots.

The order of signs is $+ \quad 0 \quad 0 \quad 0 \quad - \quad - \quad +$

If we take the 0 terms as all positive, we have

$$+ \quad + \quad + \quad + \quad - \quad - \quad + \qquad \text{2 variations,}$$

so that there are not more than two positive roots; and if we take the first two negative and the third positive, we have

$$+ \quad - \quad - \quad + \quad - \quad - \quad + \qquad \text{2 permanences,}$$

so that there are not more than two negative roots.

$\therefore$ There are then not more than two positive roots, and not more than two negative roots.

$\therefore$ There are at least two imaginary roots.

13. Show that the equation

$$x^4 + 15x^2 + 7x - 11 = 0$$

has two imaginary roots, and determine the signs of the real roots.

The order of signs is

$$+ \quad 0 \quad + \quad + \quad -$$

We may write the signs

either $\qquad + \quad + \quad + \quad + \quad -$ one variation,

or $\qquad + \quad - \quad + \quad + \quad -$ one permanence.

$\therefore$ There is not more than one positive root, and not more than one negative root.

$\therefore$ There are at least two imaginary roots.

There cannot be four imaginary roots. For if $\alpha + \beta i$, $\alpha - \beta i$, $\gamma + \delta i$, and $\gamma - \delta i$ are roots, their product is $(\alpha^2 + \beta^2)(\gamma^2 + \delta^2)$, which is positive, whereas -11 is negative.

$\therefore$ There is one positive root, one negative root, and two imaginary roots.

14. Show that the equation $x^3 + qx + r = 0$ has one negative and two imaginary roots when q and r are both positive; and determine the character of the roots when q is negative and r positive.

If p and q are both positive the equation has no positive root.

It must therefore have either one real negative root and two imaginary roots, or three real negative roots.

But in the latter case the sum of the three negative roots would also be negative, whereas in the equation the coefficient of x^2 is 0, that is the sum of the roots is 0.

There must therefore be one negative and two imaginary roots in this case.

If q is negative and r positive, the order of the signs is

$$+ \quad 0 \quad - \quad +$$

which we may take

either $\quad + \ + \ - \ +$ one permanence, two variations,

or $\quad + \ - \ - \ +$ one permanence, two variations.

∴ There is not more than one negative root, and not more than two positive roots.

There may therefore be one negative and two positive roots,
or one negative and two imaginary roots,
or one positive and two imaginary roots.

The last case cannot occur. For, if the two imaginary roots are $\alpha + \beta i$ and $\alpha - \beta i$ their product is $\alpha^2 + \beta^2$, which is positive. So that the product of the three roots would be positive. But r is positive, hence the product of the three roots must be negative.

∴ There are either

one negative and two positive roots,
or one negative and two imaginary roots.

15. Show that the equation $x^n - 1 = 0$ has but two real roots, $+1$ and -1, when n is even; and but one real root, $+1$, when n is odd.

The order of signs is

$$+ \ 0 \ 0 \ \cdots\cdots \ -$$

If we take the zeros all positive we have

$$+ \ + \ + \ \cdots\cdots \ + \ -$$

This gives only one variation and consequently only one positive root, namely $+1$.

If n is even, the equation is unchanged when we put $-x$ for x.

There will therefore be only one negative root, namely -1.

But if n is odd, the sign of x^n is changed when we put $-x$ for x, and the order of signs becomes

$$- \ 0 \ 0 \ 0 \ - \ - \ - \ -$$

which we may take

$$- \ - \ - \ \cdots\cdots \ - \ -$$

Here there is no variation and consequently the equation, $x^n - 1 = 0$, has no negative root in this case.

16. Show that the equation $x^n + 1 = 0$ has no real root when n is even; and but one real root, -1, when n is odd.

The order of signs is $\quad + \ 0 \ 0 \ \cdots\cdots \ +$

This may be taken as $\quad + \ + \ + \ \cdots\cdots \ + \ +$

There is then no variation, and therefore no positive root.

Or, if n is even, we may take

$$+ \quad - \quad + \quad - \quad + \quad - \quad \cdots\cdots \quad - \quad +$$

There is then no permanence, and consequently no negative root.

But, if n is odd, we have

$$+ \quad - \quad + \quad - \quad + \quad - \quad \cdots\cdots \quad - \quad + \quad +$$

There is only one permanence, and therefore only the one negative root, -1.

Exercise 79.

Find superior limits to the positive roots of the following equations:

1. $x^3-2x^2+4x+3=0$.

$$x^2(x-2)+4x+3=0$$

The left member is positive if $x>2$.

$\therefore$ 2 is a superior limit.

2. $2x^4-x^3-x+1=0$.

$$x^2(x^2-x)+x(x^3-1)+1=0$$

The left member is positive if $x>1$.

$\therefore$ 1 is a superior limit.

3. $3x^4+5x^3-12x^2+10x-18=0$.

$$3x^2(x^2-1)+5x^2(x-2)+x^2+10(x-2)+2=0$$

The left member is positive if $x>2$.

$\therefore$ 2 is a superior limit.

4. $4x^4-3x^3-x^2+7x+5=0$.

$$3x^3(x-1)+x^2(x^2-1)+7x+5=0$$

The left member is positive if $x>1$.

$\therefore$ 1 is a superior limit.

5. $x^4-x^3-2x^2-4x-24=0$.

$$4x^4-4x^3-8x^2-16x-96=0$$

$$x^3(x-4)+x^2(x^2-8)+x(x^3-16)+x^4-96=0$$

The left member is positive if $x>4$.

$\therefore$ 4 is a superior limit.

6. $4x^5-8x^4+22x^3+90x^2-60x+1=0$.

$$4x^4(x-2)+22x^3+90x(x-1)+30x+1=0$$

The left member is positive if $x>2$.

$\therefore$ 2 is a superior limit.

Exercise 80.

Find the commensurable roots, and if possible all the roots, of each of the following equations:

1. $x^4 - 4x^3 - 8x + 32 = 0$.

Try 2,

$$\begin{array}{rrrrr|l} 32 & -8 & +0 & -4 & +1 & 2 \\ & 16 & +4 & +2 & -1 & \\ \hline & 8 & +4 & -2 & +0 & \end{array}$$

Try 4,

$$\begin{array}{rrrr|l} 16 & +4 & +2 & -1 & 4 \\ & 4 & +2 & +1 & \\ \hline & 8 & +4 & +0 & \end{array}$$

The depressed equation is

$$x^2 + 2x + 4 = 0$$

$$x = -1 \pm \sqrt{-3}$$

∴ The four roots are 2, 4, and $-1 \pm \sqrt{-3}$.

2. $x^3 - 6x^2 + 10x - 8 = 0$.

Try 4,

$$\begin{array}{rrrr|l} -8 & +10 & -6 & +1 & 4 \\ & -2 & +2 & -1 & \\ \hline & 8 & -4 & +0 & \end{array}$$

The depressed equation is

$$x^2 - 2x + 2 = 0$$

$$x = 1 \pm \sqrt{-1}$$

∴ The three roots are 4 and $1 \pm \sqrt{-1}$.

3. $x^4 + 2x^3 - 7x^2 - 8x + 12 = 0$.

Try 1,

$$\begin{array}{rrrrr|l} 12 & -8 & -7 & +2 & +1 & 1 \\ & 12 & +4 & -3 & -1 & \\ \hline & 4 & -3 & -1 & +0 & \end{array}$$

Try 2,

$$\begin{array}{rrrr|l} 12 & +4 & -3 & -1 & 2 \\ & 6 & +5 & +1 & \\ \hline & 10 & +2 & +0 & \end{array}$$

The depressed equation is

$$x^2 + 5x + 6 = 0$$

$$x = -2, \text{ or } -3$$

∴ The four roots are 1, 2, -2, and -3.

4. $x^3+3x^2-30x+36=0$.

Try 3,
$$\begin{array}{rrrrl} 36 & -30 & +3 & +1 & \underline{|3} \\ & 12 & -6 & -1 & \\ \hline & -18 & -3 & +0 & \end{array}$$

The depressed equation is

$$x^2+6x-12=0$$
$$x=-3\pm\sqrt{21}$$

$\therefore$ The three roots are 3 and $-3\pm\sqrt{21}$.

5. $x^4-12x^3+32x^2+27x-18=0$.

Try -1,
$$\begin{array}{rrrrrl} -18 & +27 & +32 & -12 & +1 & \underline{|-1} \\ & 18 & -45 & +13 & -1 & \\ \hline & 45 & -13 & +1 & +0 & \end{array}$$

Try 6,
$$\begin{array}{rrrrl} 18 & -45 & +13 & -1 & \underline{|6} \\ & 3 & -7 & +1 & \\ \hline & -42 & +6 & +0 & \end{array}$$

The depressed equation is

$$x^2-7x+3=0$$
$$x=\frac{7\pm\sqrt{37}}{2}$$

$\therefore$ The four roots are -1, 6, and $\frac{7\pm\sqrt{37}}{2}$.

6. $x^4-9x^3+17x^2+27x-60=0$.

Try 4,
$$\begin{array}{rrrrrl} -60 & +27 & +17 & -9 & +1 & \underline{|4} \\ & -15 & +3 & +5 & -1 & \\ \hline & 12 & +20 & -4 & +0 & \end{array}$$

Try 5,
$$\begin{array}{rrrrl} -15 & +3 & +5 & -1 & \underline{|5} \\ & -3 & +0 & +1 & \\ \hline & 0 & +5 & +0 & \end{array}$$

The depressed equation is

$$x^2-3=0$$
$$x=\pm\sqrt{3}$$

$\therefore$ The four roots are 4, 5, and $\pm\sqrt{3}$.

7. $x^5-5x^4+3x^3+17x^2-28x+12=0$.

Try 1,
$$\begin{array}{rrrrrrl} 12 & -28 & +17 & +3 & -5 & +1 & \underline{|1} \\ & 12 & -16 & +1 & +4 & -1 & \\ \hline & -16 & +1 & +4 & -1 & +0 & \end{array}$$

Try 1,

$$\begin{array}{rrrrrl} 12 & -16 & +1 & +4 & -1 & \lfloor\underline{1} \\ & 12 & -4 & -3 & +1 & \\ \hline & -4 & -3 & +1 & +0 & \end{array}$$

Try 2,

$$\begin{array}{rrrrl} 12 & -4 & -3 & +1 & \lfloor\underline{2} \\ & 6 & +1 & -1 & \\ \hline & 2 & -2 & +0 & \end{array}$$

The depressed equation is

$$x^2 - x - 6 = 0$$

$$x = 3, \text{ or } -2$$

∴ The five roots are 1, 1, 2, 3, and −2.

8. $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0.$

Try 1,

$$\begin{array}{rrrrrl} 24 & -50 & +35 & -10 & +1 & \lfloor\underline{1} \\ & 24 & -26 & +9 & -1 & \\ \hline & -26 & +9 & -1 & +0 & \end{array}$$

Try 2,

$$\begin{array}{rrrrl} 24 & -26 & +9 & -1 & \lfloor\underline{2} \\ & 12 & -7 & +1 & \\ \hline & -14 & +2 & +0 & \end{array}$$

The depressed equation is

$$x^2 - 7x + 12 = 0$$

$$x = 3, \text{ or } 4$$

∴ The four roots are 1, 2, 3, and 4.

9. $x^5 - 8x^4 + 11x^3 + 29x^2 - 36x - 45 = 0.$

Try 3,

$$\begin{array}{rrrrrrl} -45 & -36 & +29 & +11 & -8 & +1 & \lfloor\underline{3} \\ & -15 & -17 & +4 & +5 & -1 & \\ \hline & -51 & +12 & +15 & -3 & +0 & \end{array}$$

Try 5,

$$\begin{array}{rrrrrl} -15 & -17 & +4 & +5 & -1 & \lfloor\underline{5} \\ & -3 & -4 & +0 & +1 & \\ \hline & -20 & +0 & +5 & +0 & \end{array}$$

Try −1,

$$\begin{array}{rrrrl} -3 & -4 & +0 & +1 & \lfloor\underline{-1} \\ & 3 & +1 & -1 & \\ \hline & -1 & +1 & +0 & \end{array}$$

The depressed equation is

$$x^2 - x - 3 = 0$$

$$x = \frac{1 \pm \sqrt{13}}{2}$$

∴ The five roots are 3, 5, −1, and $\frac{1 \pm \sqrt{13}}{2}$.

10. $x^5 - x^4 - 6x^3 + 9x^2 + x - 4 = 0.$

Try 1,

$$\begin{array}{rrrrrrl} -4 & +1 & +9 & -6 & -1 & +1 & \underline{\lfloor 1} \\ & -4 & -3 & +6 & +0 & -1 & \\ \hline & -3 & +6 & +0 & -1 & +0 & \end{array}$$

∴ 1 is a root.

There are no other commensurable roots.

11. $2x^4 - 3x^3 - 20x^2 + 27x + 18 = 0.$

Try 2,

$$\begin{array}{rrrrrl} 18 & +27 & -20 & -3 & +2 & \underline{\lfloor 2} \\ & 9 & +18 & -1 & -2 & \\ \hline & 36 & -2 & -4 & +0 & \end{array}$$

Try 3,

$$\begin{array}{rrrrl} 9 & +18 & -1 & -2 & \underline{\lfloor 3} \\ & 3 & +7 & +2 & \\ \hline & 21 & +6 & +0 & \end{array}$$

The depressed equation is

$$2x^2 + 7x + 3 = 0$$

$$x = -\tfrac{1}{2}, \text{ or } -3$$

∴ The four roots are 2, 3, $-\frac{1}{2}$, and -3.

12. $2x^4 - 9x^3 - 27x^2 + 134x - 120 = 0$

Try 2,

$$\begin{array}{rrrrrl} -120 & +134 & -27 & -9 & +2 & \underline{\lfloor 2} \\ & -60 & +37 & +5 & -2 & \\ \hline & 74 & +10 & -4 & +0 & \end{array}$$

Try 5,

$$\begin{array}{rrrrl} -60 & +37 & +5 & -2 & \underline{\lfloor 5} \\ & -12 & +5 & +2 & \\ \hline & 25 & +10 & +0 & \end{array}$$

The depressed equation is

$$2x^2 + 5x - 12 = 0$$

$$x = \tfrac{3}{2}, \text{ or } -4$$

∴ The four roots are 2, 5, $\frac{3}{2}$, and -4.

13. $x^6 + 3x^5 - 2x^4 - 15x^3 - 15x^2 + 8x + 20 = 0.$

Try 1,

$$\begin{array}{rrrrrrrl} 20 & +8 & -15 & -15 & -2 & +3 & +1 & \underline{\lfloor 1} \\ & 20 & +28 & +13 & -2 & -4 & -1 & \\ \hline & 28 & +13 & -2 & -4 & -1 & +0 & \end{array}$$

Try 2,

$$\begin{array}{rrrrrrl} 20 & +28 & +13 & -2 & -4 & -1 & \underline{\lfloor -2} \\ & -10 & -9 & -2 & +2 & +1 & \\ \hline & 18 & +4 & -4 & -2 & +0 & \end{array}$$

∴ 1 and -2 are roots.

There are no other commensurable roots.

14. $18x^3 + 3x^2 - 7x - 2 = 0.$

Multiply the roots by 6,

$$18x^3 + 18x^2 - 252x - 432 = 0$$
$$x^3 + x^2 - 14x - 24 = 0$$

Try 4,
$$\begin{array}{rrrr|l} -24 & -14 & +1 & +1 & \underline{4} \\ & -6 & -5 & -1 & \\ \hline & -20 & -4 & +0 & \end{array}$$

The depressed equation is

$$x^2 + 5x + 6 = 0$$
$$x = -2, \text{ or } -3$$

∴ The three roots of the given equation are $\frac{4}{6}$, $-\frac{2}{6}$, and $-\frac{3}{6}$; or $\frac{2}{3}$, $-\frac{1}{3}$, and $-\frac{1}{2}$.

15. $24x^3 - 34x^2 - 5x + 3 = 0.$

Multiply the roots by 12,

$$24x^3 - 12 \times 34x^2 - 5 \times 144x + 3 \times 1728 = 0$$
$$x^3 - 17x^2 - 30x + 216 = 0$$

Try 3,
$$\begin{array}{rrrr|l} 216 & -30 & -17 & +1 & \underline{3} \\ & 72 & +14 & -1 & \\ \hline & 42 & -3 & +0 & \end{array}$$

The depressed equation is

$$x^2 - 14x - 72 = 0$$
$$x = 18, \text{ or } -4$$

∴ The roots of the given equation are $\frac{3}{12}$, $\frac{18}{12}$, and $-\frac{4}{12}$; or $\frac{1}{4}$, $\frac{3}{2}$, and $-\frac{1}{3}$.

16. $27x^3 - 18x^2 - 3x + 2 = 0.$

Multiply the roots by 3,

$$27x^3 - 3 \times 18x^2 - 3 \times 9x + 2 \times 27 = 0$$
$$x^3 - 2x^2 - x + 2 = 0$$

Try 1,
$$\begin{array}{rrrr|l} 2 & -1 & -2 & +1 & \underline{1} \\ & 2 & +1 & -2 & \\ \hline & 1 & -1 & +0 & \end{array}$$

The depressed equation is

$$x^2 - x - 2 = 0$$
$$x = 2, \text{ or } -1$$

∴ The three roots of the given equation are $\frac{1}{3}$, $\frac{2}{3}$, or $-\frac{1}{3}$.

17. $18x^4 + 9x^3 + 10x^2 - 8x + 1 = 0$.

Multiply the roots by 6,

$$\begin{aligned} 18x^4 + 9 \times 6x^3 + 10 \times 36x^2 - 8 \times 216x + 1296 &= 0 \\ x^4 + 3x^3 + 20x^2 - 96x + 72 &= 0 \end{aligned}$$

Try 1,

$$\begin{array}{rrrrr|l} 72 & -96 & +20 & +3 & +1 & \underline{1} \\ & 72 & -24 & -4 & -1 & \\ \hline & -24 & -4 & -1 & & \end{array}$$

Try 2,

$$\begin{array}{rrrr|l} 72 & -24 & -4 & -1 & \underline{2} \\ & 36 & +6 & +1 & \\ \hline & 12 & +2 & & \end{array}$$

The depressed equation is

$$x^2 + 6x + 36 = 0$$
$$x = -3 \pm \sqrt{-27}$$

∴ The roots of the given equation are $\frac{1}{6}$, $\frac{2}{6}$, and $\frac{-3 \pm \sqrt{-27}}{6}$, or $\frac{1}{6}$, $\frac{1}{3}$, and $\frac{-1 \pm \sqrt{-3}}{2}$.

18. $36x^4 + 48x^3 - 23x^2 - 17x + 6 = 0$.

Multiply the roots by 6,

$$\begin{aligned} 36x^4 + 48 \times 6x^3 - 23 \times 36x^2 - 17 \times 216x + 6 \times 1296 &= 0 \\ x^4 + 8x^3 - 23x^2 - 102x + 216 &= 0 \end{aligned}$$

Try 2,

$$\begin{array}{rrrrr|l} 216 & -102 & -23 & +8 & +1 & \underline{2} \\ & 108 & +3 & -10 & -1 & \\ \hline & 6 & -20 & -2 & +0 & \end{array}$$

Try 3,

$$\begin{array}{rrrr|l} 108 & +3 & -10 & -1 & \underline{3} \\ & 36 & +13 & +1 & \\ \hline & 39 & +3 & +0 & \end{array}$$

The depressed equation is

$$x^2 + 13x + 36 = 0$$
$$x = -4, \text{ or } -9$$

∴ The four roots of the given equations are $\frac{2}{6}$, $\frac{3}{6}$, $-\frac{4}{6}$, and $-\frac{9}{6}$; or $\frac{1}{3}$, $\frac{1}{2}$, $-\frac{2}{5}$, and $-\frac{3}{2}$.

Exercise 81.

Determine the first significant figure of each real root of the following equations:

1. $x^3 - x^2 - 2x + 1 = 0.$

There are not more than two positive roots and not more than one negative root.

$f(0) = 1$
$f(0.4) = 0.104$ $\quad\therefore$ one root is $+0.4+$
$f(0.5) = 0.125$
$f(1) = -1$
$f(2) = +1$ $\quad\therefore$ one root is $+1.+$
$f(-1) = 1$
$f(-2) = -7$ $\quad\therefore$ one root is $-1.+$

2. $x^3 - 5x - 3 = 0.$

There are not more than two negative roots and not more than one positive root.

$f(0) = -3$
$f(1) = -7$
$f(2) = -5$
$f(3) = 9$ $\quad\therefore$ one root is $+2.+$
$f(-0.6) = -0.216$
$f(-0.7) = 0.157$ $\quad\therefore$ one root is $-0.6+$
$f(-1) = 1$
$f(-2) = -1$ $\quad\therefore$ one root is $-1.+$

3. $x^3 - 5x^2 + 7 = 0.$

There are not more than two positive roots and not more than one negative root.

$f(0) = 7$
$f(1) = 3$
$f(2) = -5$ $\quad\therefore$ one root is $+1.+$
$f(3) = -11$
$f(4) = -9$
$f(5) = 7$ $\quad\therefore$ one root is $+4.+$
$f(-1) = 1$
$f(-2) = -21$ $\quad\therefore$ one root is $-1.+$

4. $x^3 + 2x^2 - 30x + 39 = 0.$

There are not more than two positive roots and not more than one negative root.

$f(0) = 39$
$f(1) = 12$
$f(2) = -5$ $\therefore$ one root is $+1.+$

$f(3) = -6$
$f(4) = 15$ $\therefore$ one root is $+3.+$

$f(-1) = 70$
$f(-2) = 99$
$f(-3) = 120$
$f(-4) = 127$
$f(-5) = 114$
$f(-6) = 75$
$f(-7) = 4$
$f(-8) = -105$ $\therefore$ one root is $-7.+$

5. $x^3 - 6x^2 + 3x + 5 = 0.$

There are not more than two positive roots and not more than one negative root.

$f(0) = 5$
$f(1) = 3$
$f(2) = -5$ $\therefore$ one root is $+1.+$

$f(3) = -13$
$f(4) = -15$
$f(5) = -5$
$f(6) = 23$ $\therefore$ one root is $+5.+$

$f(-1) = -5$
$f(-0.7) = -0.383$
$f(-0.6) = +0.824$ $\therefore$ one root is $-0.6+$

6. $x^3 + 9x^2 + 24x + 17 = 0.$

There is no positive root.

$f(0) = 17$
$f(-1) = 1$
$f(-2) = -3$ $\therefore$ one root is $-1.+$

$f(-3) = -1$
$f(-4) = 1$ ∴ one root is $-3.+$

$f(-5) = -3$ ∴ one root is $-4.+$

7. $x^3 - 15x^2 + 63x - 50 = 0.$

There is no negative root.

$f(0) = -50$
$f(1) = -1$
$f(2) = 24$ ∴ one root is $+1.+$

$f(3) = 31$
$f(4) = 26$
$f(5) = 15$
$f(6) = 4$
$f(7) = -1$ ∴ one root is $+6.+$

$f(8) = 6$ ∴ one root is $+7.+$

8. $x^4 - 8x^3 + 14x^2 + 4x - 8 = 0.$

There are not more than three positive roots and not more than one negative root.

$f(0) = -8$
$f(0.7) = -0.8439$
$f(0.8) = +0.4736$ ∴ one root is $+0.7+$

$f(1) = 3$
$f(2) = 8$
$f(3) = -5$ ∴ one root is $+2.+$

$f(4) = -24$
$f(5) = -13$
$f(6) = 88$ ∴ one root is $+5.+$

$f(-0.7) = -.9559$
$f(-0.8) = +2.2656$ ∴ one root is $-0.7+$

Exercise 82.

Compute for each of the following equations the root of which the first figure is the number in parenthesis opposite the equation. Carry out the work to three places of decimals:

1. $x^3 + 3x - 5 = 0$ (1).

1	+ 0	+ 3	− 5	1.154 +
	+ 1	+ 1	+ 4	
	+ 1	+ 4	− 1000	
	+ 1	+ 2	+ 631	
	+ 2	+ 600	− 369000	
	+ 1	+ 31	+ 339875	
	+ 30	+ 631	− 29125000	
	+ 1	+ 32	+ 27925264	
	+ 31	+ 66300	− 1199736	
	+ 1	+ 1675		
	+ 32	+ 67975		
	+ 1	+ 1700		
	+ 330	+ 6967500		
	+ 5	+ 13816		
	+ 335	+ 6981316		
	+ 5	+ 13832		
	+ 340	6995148		
	+ 5			
	+ 3450			
	+ 4			
	+ 3454			
	+ 4			
	+ 3458			
	+ 4			
	+ 3462			

2. $x^3 - 6x - 12 = 0$ (3).

1	+0	− 6	− 12	3.134 +
	+3	+ 9	+ 9	
	+3	+ 3	− 3000	
	+3	+ 18	+ 2191	
	+6	+ 2100	− 809000	
	+3	+ 91	+ 693297	
	+90	+ 2191	− 115703000	
	+ 1	+ 92	+ 93713104	
	+91	+ 228300	− 21989896	
	+ 1	+ 2799		
	+92	+ 231099		
	+ 1	+ 2808		
	+930	+ 23390700		
	+ 3	+ 37576		
	+933	+ 23428276		
	+ 3	+ 37592		
	+936	+ 23465868		
	+ 3			
	+9390			
	+ 4			
	+9394			
	+ 4			
	+9398			
	+ 4			
	+9402			

3. $x^3 + x^2 + x - 100 = 0$ (4).

1	$+1$	$+1$	-100	$4.264+$
	$+4$	$+20$	$+84$	
	$+5$	$+21$	-16000	
	$+4$	$+36$	$+11928$	
	$+9$	$+5700$	-4072000	
	$+4$	$+264$	$+3788376$	
	$+130$	$+5964$	-283624000	
	$+2$	$+268$	$+256071744$	
	$+132$	$+623200$	-27552256	
	$+2$	$+8196$		
	$+134$	$+631396$		
	$+2$	$+8232$		
	$+1360$	$+63962800$		
	$+6$	$+55136$		
	$+1366$	$+64017936$		
	$+6$	$+55152$		
	$+1372$	$+64073088$		
	$+6$			
	$+13780$			
	$+4$			
	$+13784$			
	$+4$			
	$+13788$			
	$+4$			
	$+13792$			

4. $x^3 + 10x^2 + 6x - 120 = 0$ (2).

1	+ 10	+ 6	− 120	2.833+
	+ 2	+ 24	+ 60	
	+ 12	+ 30	− 60000	
	+ 2	+ 28	+ 57152	
	+ 14	+ 5800	− 2848000	
	+ 2	+ 1344	+ 2582187	
	+ 160	+ 7144	− 265813000	
	+ 8	+ 1408	+ 260046537	
	+ 168	+ 855200	− 5766463	
	+ 8	+ 5529		
	+ 176	+ 860729		
	+ 8	+ 5538		
	+ 1840	+ 86626700		
	+ 3	+ 55479		
	+ 1843	+ 86682179		
	+ 3	+ 55488		
	+ 1846	+ 86737667		
	+ 3			
	+ 18490			
	+ 3			
	+ 18493			
	+ 3			
	+ 18496			
	+ 3			
	+ 18499			

5. $x^3 + 9x^2 + 24x + 17 = 0$ (-4).

Change the sign of the roots.

The resulting equation is $x^3 - 9x^2 + 24x - 17 = 0$.

1	-9	$+24$	-17	$4.532+$
	$+4$	-20	$+16$	
	-5	$+4$	-1000	
	$+4$	-4	$+875$	
	-1	000	-125000	
	$+4$	$+175$	$+116577$	
	$+30$	$+175$	-8423000	
	$+5$	$+200$	$+8063768$	
	$+35$	$+37500$	-359232	
	$+5$	$+1359$		
	$+40$	$+38859$		
	$+5$	$+1368$		
	$+450$	$+4022700$		
	$+3$	$+9184$		
	$+453$	$+4031884$		
	$+3$	$+9188$		
	$+456$	$+4041072$		
	$+3$			
	$+4590$			
	$+2$			
	$+4592$			
	$+2$			
	$+4594$			
	$+2$			
	$+4596$			

Root is $-4.532+$

6. $x^4 - 12x^3 + 12x - 3 = 0 \qquad (-1).$

Change the sign of the roots.

The resulting equation is $x^4 + 12x^3 - 12x - 3 = 0.$

1	+ 12	+ 0	− 12	− 3	1.064+
	+ 1	+ 13	+ 13	+ 1	
	+ 13	+ 13	+ 1	− 200000000	
	+ 1	+ 14	+ 27	+ 183466896	
	+ 14	+ 27	+ 28000000	− 165331040000	
	+ 1	+ 15	+ 2577816	+ 133572525216	
	+ 15	+ 420000	+ 30577816	− 31758514784	
	+ 1	+ 9636	+ 2635848		
	+ 1600	+ 429636	+ 33213664000		
	+ 6	+ 9672	+ 179467304		
	+ 1606	+ 439308	+ 33393131304		
	+ 6	+ 9708	+ 179726272		
	+ 1612	+ 44801600	+ 33572857576		
	+ 6	+ 64976			
	+ 1618	+ 44866576			
	+ 6	+ 64992			
	+ 16240	+ 44931568			
	+ 4	+ 65008			
	+ 16244	+ 44996576			
	+ 4				
	+ 16248				
	+ 4				
	+ 16252				
	+ 4				
	+ 16256				

7. $x^4 - 8x^3 + 14x^2 + 4x - 8 = 0$ (−0).

Change the sign of the roots.

The resulting equation is

$$x^4 + 8x^3 + 14x^2 - 4x - 8 = 0.$$

1	+ 80	+ 1400	− 4000	− 80000	0.732 +
	+ 7	+ 609	+ 14063	+ 70441	
	+ 87	+ 2009	+ 10063	− 95590000	
	+ 7	+ 658	+ 18669	+ 89261841	
	+ 94	+ 2667	+ 28732000	− 63281590000	
	+ 7	+ 707	+ 1021947	+ 61710292976	
	+ 101	+ 337400	+ 29753947	− 1571297024	
	+ 7	+ 3249	+ 1031721		
	+ 1080	+ 340649	+ 30785668000		
	+ 3	+ 3258	+ 69478488		
	+ 1083	+ 343907	+ 30855146488		
	+ 3	+ 3267	+ 69522184		
	+ 1086	+ 34717400	+ 30924668672		
	+ 3	+ 21844			
	+ 1089	+ 34739244			
	+ 3	+ 21848			
	+ 10920	+ 34761092			
	+ 2	+ 21852			
	+ 10922	+ 34782944			
	+ 2				
	+ 10924				
	+ 2				
	+ 10926				
	+ 2				
	+ 10928				

Root is − 0.732 −

Exercise 83.

Calculate to six places of decimals the positive roots of the following equations:

1. $x^3 - 3x - 1 = 0$.

There is not more than one positive root.

The positive root lies between 1 and 2.

1	+ 0	− 3	− 1	1.879385 +
	+ 1	+ 1	− 2	
	+ 1	− 2	− 3000	
	+ 1	+ 2	+ 2432	
	+ 2	000	− 568000	
	+ 1	+ 304	+ 497203	
	+ 30	+ 304	− 70797000	
	+ 8	+ 368	+ 67871439	
	+ 38	+ 67200	− 2925561	
	+ 8	+ 3829	+ 2278080	
	+ 46	+ 71029	− 647481	
	+ 8	+ 3878	+ 607688	
	+ 540	+ 7490700	− 39793	
	+ 7	+ 50571	+ 37985	
	+ 547	+ 7541271	− 1808	
	+ 7	+ 50652		
	+ 554	+ 7591923		
	+ 7	+ 759192		
	+ 5610	+ 168		
	+ 9	+ 759360		
	+ 5619	+ 168		
	+ 9	+ 759528		
	+ 5628	+ 75953		
	+ 9	+ 8		
	+ 5637	+ 75961		
	+ 56	+ 8		
	+ 1	+ 75969		
		+ 7597		

2. $x^3 + 2x^2 - 4x - 43 = 0$.

There is only one positive root. It lies between 3 and 4.

1	+ 2	− 4	− 43	3.263389 +
	+ 3	+ 15	+ 33	
	+ 5	+ 11	− 10000	
	+ 3	+ 24	+ 7448	
	+ 8	+ 3500	− 2552000	
	+ 3	+ 224	+ 2413176	
	+ 110	+ 3724	− 138824	
	+ 2	+ 228	+ 122877	
	+ 112	+ 395200	− 15947	
	+ 2	+ 6996	+ 12297	
	+ 114	+ 402196	− 3650	
	+ 2	+ 7032	+ 3280	
	+ 1160	+ 409228	− 370	
	+ 6	+ 40923	+ 369	
	+ 1166	+ 36	− 1	
	+ 6	+ 40959		
	+ 1172	+ 36		
	+ 6	+ 40995		
	+ 1178	+ 4099		
	+ 12	+ 410		
		+ 41		

3. $3x^3 + 3x^2 + 8x - 32 = 0.$

There is only one positive root. It lies between 1 and 2.

3	+ 3	+ 8	− 32	1.580947 +
	+ 3	+ 6	+ 14	
	+ 6	+ 14	− 18000	
	+ 3	+ 9	+ 14875	
	+ 9	+ 2300	− 3125000	
	+ 3	+ 675	+ 3087136	
	+ 120	+ 2975	− 37864	
	+ 15	+ 750	+ 35955	
	+ 135	+ 372500	− 1909	
	+ 15	+ 13392	+ 1596	
	+ 150	+ 385892	− 313	
	+ 15	+ 13584	+ 280	
	+ 1650	+ 399476	− 33	
	+ 24	+ 39948		
	+ 1674	+ 3995		
	+ 24	+ 399		
	+ 1698	+ 40		
	+ 24			
	+ 1722			
	+ 17			

4. $2x^3 - 26x^2 + 131x - 202 = 0.$

There is only one positive root. It lies between 2 and 3.

2	− 26	+ 131	− 202	2.656116 +
	+ 4	− 44	+ 174	
	− 22	+ 87	− 28000	
	+ 4	− 36	+ 25992	
	− 18	+ 5100	− 2008000	
	+ 4	− 768	+ 1792250	
	− 140	+ 4332	− 215750	
	+ 12	− 696	+ 211650	
	− 128	+ 363600	− 4100	
	+ 12	− 5150	+ 3521	
	− 116	+ 358450	− 579	
	+ 12	− 5100	+ 352	
	− 1040	+ 353350	− 227	
	+ 10	+ 35335	+ 210	
	− 1030	− 60	− 17	
	+ 10	+ 35275		
	− 1020	− 60		
	+ 10	+ 35215		
	− 1010	+ 3521		
	− 10	+ 352		
		+ 35		

5. $x^4 - 12x + 7 = 0$.

There are two positive roots, one between 0 and 1, and one between 2 and 3.

1	+ 00	+ 000	− 12000	+ 70000	0.593685+
	+ 5	+ 25	+ 125	− 59375	
	+ 5	+ 25	− 11875	+ 106250000	
	+ 5	+ 50	+ 375	− 102132639	
	+ 10	+ 75	− 11500000	+ 4117361	
	+ 5	+ 75	+ 151929	− 3351663	
	+ 15	+ 15000	− 11348071	+ 765698	
	+ 5	+ 1881	+ 169587	− 669882	
	+ 200	+ 16881	− 11178484	− 95816	
	+ 9	+ 1962	627	+ 89304	
	+ 209	+ 18843	− 1117221	− 6512	
	+ 9	+ 2043	627	+ 5580	
	+ 218	+ 20886	− 1116594	− 932	
	+ 9	+ 209	12		
	+ 227	+ 2	− 111647		
	+ 9		12		
	+ 236		− 111635		

1	+ 0	+ 0	− 12	+ 7	2.047275
	+ 2	+ 4	+ 8	− 8	
	+ 2	+ 4	− 4	− 100000000	
	+ 2	+ 8	+ 24	+ 83891456	
	+ 4	+ 12	+ 20000000	− 16108544	
	+ 2	+ 12	+ 972864	+ 15493758	
	+ 6	+ 240000	+ 20972864	− 614786	
	+ 2	+ 3216	+ 985792	+ 446296	
	+ 800	+ 243216	+ 21958656	− 168490	
	+ 4	+ 3232	+ 2195866	+ 156205	
	+ 804	+ 246448	+ 17528	− 12285	
	+ 4	+ 3248	+ 2213394	+ 11155	
	+ 808	+ 249696	+ 17577	− 1130	
	+ 4	+ 2497	+ 2230971		
	+ 812	+ 7	+ 223098		
	+ 1	+ 2504	+ 50		
		+ 7	+ 223148		
		+ 2511	+ 22315		
		+ 7	+ 2231		
		+ 2518			
		+ 25			

6. $x^4 - 5x^3 + 2x^2 - 13x + 55 = 0$.

There are two positive roots, one between 2 and 3, and one between 4 and 5.

1	−5	+2	−13	+55	2.381966+
	+2	−6	−8	−42	
	−3	−4	−21	+130000	
	+2	−2	−12	−101709	
	−1	−6	−33000	+282910000	
	+2	+2	−903	−276123136	
	+1	−400	−33903	+6786864	
	+2	+99	−579	−3452059	
	+30	−301	−34482000	+3334805	
	+3	+108	−33392	−3106827	
	+33	−193	−34515392	+227978	
	+3	+117	−5488	−207120	
	+36	−7600	−34520880	+20858	
	+3	+3424	−3452088	−20712	
	+39	−4174	+29	+146	
	+3	+3488	−3452059		
	+420	−686	+29		
	+8	+3552	−3452030		
	+428	+2866	−345203		
	+8	+29	−34520		
	+436		−3452		
	+8				
	+444				
	+8				
	+452				

1	−5	+ 2	−13	+ 55	4.618035+
	+4	− 4	− 8	−84	
	−1	− 2	−21	−290000	
	+4	+ 12	+ 40	+ 275856	
	+3	+ 10	+ 19000	−141440000	
	+4	+ 28	+ 26976	+ 77944941	
	+7	+ 3800	+ 45976	− 63495059	
	+4	+ 696	+ 31368	+ 63216592	
	+ 110	+ 4496	+ 77344000	−278467	
	+ 6	+ 732	+ 600941	+ 238524	
	+ 116	+ 5228	+ 77944941	− 39943	
	+ 6	+ 768	+ 602283	+ 39755	
	+ 122	+ 599600	+ 78547224	− 188	
	+ 6	+ 1341	+ 47352		
	+ 128	+ 600941	+ 7902074		
	+ 6	+ 1342	+ 48416		
	+ 1340	+ 602283	+ 7950490		
	+ 1	+ 1343	+ 79505		
	+ 1341	+ 603626	+ 3		
	+ 1	+ 8	+ 79508		
	+ 1342	+ 6044	+ 3		
	+ 1	+ 8	+ 79511		
	+ 1343	+ 6052	+ 7951		
	+ 1	+ 8			
	+ 1344	+ 6060			
	1	+ 61			
		+ 1			

7. $x^3 = 35,499$.

There is only one real root. It lies between 30 and 40.

1 + 0	+ 0	− 35499	32.865383 +
+ 30	+ 900	+ 27000	
+ 30	+ 900	− 8499	
+ 30	+ 1800	+ 5768	
+ 60	+ 2700	− 2731000	
+ 30	+ 184	+ 2519552	
+ 90	+ 2884	− 211448000	
+ 2	+ 188	+ 194005656	
+ 92	+ 307200	− 17442344	
+ 2	+ 7744	+ 16199170	
+ 94	+ 314944	− 1243174	
+ 2	+ 7808	+ 972108	
+ 960	+ 32275200	− 271066	
+ 8	+ 59076	+ 259232	
+ 968	+ 32334276	− 11834	
+ 8	+ 59112	+ 9720	
+ 976	+ 32393388	− 2114	
+ 8	+ 3239339		
+ 9840	+ 495		
+ 6	+ 3239834		
+ 9846	+ 495		
+ 6	+ 3240329		
+ 9852	+ 324033		
+ 6	+ 3		
+ 9858	+ 324036		
+ 99	+ 3		
+ 1	+ 324039		
	+ 32404		

8. $x^3 = 242{,}970{,}624$.

There is only one real root. It lies between 600 and 700.

1	+ 0	+ 0	− 242970624	624
	+ 600	+ 360000	+ 216000000	
	+ 600	+ 360000	− 26970624	
	+ 600	+ 720000	+ 22328000	
	+ 1200	+ 1080000	− 4642624	
	+ 600	+ 36400	+ 4642624	
	+ 1800	+ 1116400	0	
	+ 20	+ 36800		
	+ 1820	+ 1153200		
	+ 20	+ 7456		
	+ 1840	+ 1160656		
	+ 20			
	+ 1860			
	+ 4			
	+ 1864			

9. $x^4 = 707{,}281$.

There is only one positive root. It lies between 20 and 30.

There is only one negative root. It lies between − 20 and − 30.

1	+ 0	+ 0	+ 0	− 707281	29
	+ 20	+ 400	+ 8000	+ 160000	
	+ 20	+ 400	+ 8000	− 547281	
	+ 20	+ 800	+ 24000	+ 547281	
	+ 40	+ 1200	+ 32000	0	
	+ 20	+ 1200	+ 28809		
	+ 60	+ 2400	+ 60809		
	+ 20	+ 801			
	+ 80	+ 3201			
	+ 9				
	+ 89				

± 29 are the only real roots

10. $x^5 = 147,008,443$.

There is only one real root. It lies between 40 and 50.

1	+ 0	+ 0	+ 0	+ 0	− 147008443	43
	+ 40	+ 1600	+ 64000	+ 2560000	+ 102400000	
	+ 40	+ 1600	+ 64000	+ 2560000	− 44608443	
	+ 40	+ 3200	+ 192000	+ 10240000	+ 44608443	
	+ 80	+ 4800	+ 256000	+ 12800000	0	
	+ 40	+ 4800	+ 384000	+ 2069481		
	+ 120	+ 9600	+ 640000	+ 14869481		
	+ 40	+ 6400	+ 49827			
	+ 160	+ 16000	+ 689827			
	+ 40	+ 609				
	+ 200	+ 16609				
	+ 3					
	+ 203					

11. $x^3 + 2x + 20 = 0$.

There is only one real root. It lies between − 2 and − 3.

1	+ 0	+ 2	− 20	2.469547+
	+ 2	+ 4	+ 12	
	+ 2	+ 6	− 8000	
	+ 2	+ 8	+ 6624	
	+ 4	+ 1400	− 1376000	
	+ 2	+ 256	+ 1182936	
	+ 60	+ 1656	− 193064	
	+ 4	+ 272	+ 181962	
	+ 64	+ 192800	− 11102	
	+ 4	+ 4356	+ 10140	
	+ 68	+ 197156	− 962	
	+ 4	+ 4392	+ 812	
	+ 720	+ 201548	− 150	
	+ 6	+ 20155	+ 140	
	+ 726	+ 63	− 10	
	+ 6	+ 20218		
	+ 732	+ 63		
	+ 6	+ 20281		
	+ 738	+ 2028		
	+ 7	+ 203		
		+ 20		

The real root is − 2.469547+

12. $x^3 - 10x^2 + 8x + 120 = 0.$

There is only one real root. It lies between -2 and -3.

1	+ 10	+ 8	− 120	2.768345+
	+ 2	+ 24	+ 64	
	+ 12	+ 32	− 56000	
	+ 2	+ 28	+ 50183	
	+ 14	+ 6000	− 5817000	
	+ 2	+ 1169	+ 5097576	
	+ 160	+ 7169	− 719424	
	+ 7	+ 1218	+ 689576	
	+ 167	+ 838700	− 29848	
	+ 7	+ 10896	+ 25902	
	+ 174	+ 849596	− 3946	
	+ 7	+ 10932	+ 3452	
	+ 1810	+ 860528	− 494	
	+ 6	+ 86053	+ 430	
	+ 1816	+ 144	− 64	
	+ 6	+ 86197		
	+ 1822	+ 144		
	+ 6	+ 86341		
	+ 1828	+ 8634		
	+ 18	+ 863		
		+ 86		

The real root is −2.768345+

13. $x^3-3x^2-4x+13=0.$

This equation has two roots between 2 and 3.

1	−3	−4	+13	2.3568958
	+2	−2	−12	
	−1	−6	+ **1000**	
	+2	+2	− 903	
	+1	**−400**	**+ 97000**	
	+2	+ 99	−86625	
	30	−301	**+ 10375000**	
	3	+108	− 9048984	
	33	**−19300**	**+ 1326016**	
	3	+ 1975	− 1184408	
	36	−17325	+ 141608	
	3	+ 2000	− 132948	
	390	**−1532500**	+ 8660	
	5	+ 24336	− 7385	
	395	−1508164	+ 1275	
	5	+ 24372	− 1184	
	400	**−1483792**		
	5	−148379		
	4050	+ 328		
	6	−148051		
	4056	+ 328		
	6	−147723		
	4062	−14772		
	6	−1477		
	4068	−148		
	41			

To six places 2.356896

The roots are separated after the first transformation, for: $\frac{2 \times 1000}{400}$ gives 5, and $\frac{400}{2 \times 30}$ gives 6 for the next figure of the root. By trial 4 is too large. We therefore try 3 as above.

The greater root after the first transformation as follows:

1	30	− 400	+ 1000	2.692021
	6	+ 216	− 1104	
	36	− 184	− 104000	
	6	+ 252	+ 100809	
	42	6800	− 3191000	
	6	4401	+ 3156888	
	480	11201	− 34112	
	9	4482	+ 31772	
	489	1568300	− 2340	
	9	10144	+ 1589	
	498	1578444	− 751	
	9	10148		
	5070	1588592		
	2	15886		
	5072	1589		
	2			
	5074			

The change of sign in the last term shows that we have passed over a root, viz.: the less root calculated above.

14. $2x^4 + 8x^3 - 35x^2 - 36x + 117 = 0.$

The two nearly equal roots lie between 2 and 3.

2	+ 8	− 35	− 36	+ 117	2.121320+
	+ 4	+ 24	− 22	− 116	
	+ 12	− 11	− 58	+ 10000	
	+ 4	+ 32	+ 42	− 9658	
	+ 16	+ 21	− 16000	+ 3420000	
	+ 4	+ 40	+ 6342	− 3391328	
	+ 20	+ 5100	− 9658	+ 28672	
	+ 4	+ 242	+ 6586	− 23955	
	+ 240	+ 6342	− 3072000	+ 4717	
	+ 2	+ 244	+ 1376336	− 4461	
	+ 242	+ 6586	− 1695664	+ 256	
	+ 2	+ 246	+ 1386288	− 250	
	+ 244	+ 683200	− 309376	+ 6	
	+ 2	+ 4968	− 30938		
	+ 246	+ 688168	+ 6983		
	+ 2	+ 4976	− 23955		
	+ 2480	+ 693144	+ 6985		
	+ 4	+ 4984	− 16970		
	+ 2484	+ 698128	− 1697		
	+ 4	+ 6981	+ 210		
	+ 2488	+ 2	+ 1487		
	+ 4	+ 6983	− 210		
	+ 2492	+ 2	− 1267		
	+ 4	+ 6985	− 127		
	+ 2496	+ 2	+ 2		
	+ 2	+ 6987	− 125		
		+ 70	+ 2		
		+ 1	− 123		

2	+ 2	+ 6981	− 30938	+ 28672	2.123113+
		+ 6	+ 20961	− 39931	
		+ 6987	− 9977	− 1259	
		+ 6	+ 20979	+ 1107	
		+ 6993	+ 11002	− 152	
		+ 7	+ 1100	+ 111	
			+ 7	− 41	
			+ 1107	+ 33	
			+ 7	− 8	
			+ 1114		
			+ 111		
			+ 11		

15. $x^3 + 11x^2 - 102x + 181 = 0.$

The two nearly equal roots lie between 3 and 4.

1	+ 11	− 102	+ 181	3.213127 +
	+ 3	+ 42	− 180	
	+ 14	− 60	+ 1000	
	+ 3	+ 51	− 992	
	+ 17	− 900	+ 8000	
	+ 3	+ 404	− 6739	
	+ 200	− 496	+ 1261000	
	+ 2	+ 408	− 1217403	
	+ 202	− 8800	+ 43597	
	+ 2	+ 2061	− 34183	
	+ 204	− 6739	+ 9414	
	+ 2	+ 2062	− 6788	
	+ 2060	− 467700	+ 2626	
	+ 1	+ 61899	− 2373	
	+ 2061	− 405801	− 253	
	+ 1	+ 61908		
	+ 2062	− 343893		
	+ 1	− 34389		
	+ 20630	+ 206		
	+ 3	− 34183		
	+ 20633	+ 206		
	+ 3	− 33977		
	+ 20636	− 3398		
	+ 3	+ 4		
	+ 20639	− 3394		
	+ 206	+ 4		
	+ 2	− 3390		
		− 339		

1	+ 2060	− 8800	+ 8000	3.229521 +
	+ 2	+ 4124	− 9352	
	+ 2062	− 4676	− 1352000	
	+ 2	+ 4128	+ 1180989	
	+ 2064	− 54800	− 171011	
	+ 2	+ 186021	+ 163835	
	+ 20660	+ 131221	− 7176	
	+ 9	+ 186102	+ 6768	
	+ 20669	+ 317323	− 408	
	+ 9	+ 31732	+ 339	
	+ 20678	+ 1035	− 69	
	+ 9	+ 32767		
	+ 20687	+ 1035		
	+ 207	+ 33802		
	+ 2	+ 3380		
		+ 4		
		+ 3384		
		+ 4		
		+ 3388		
		+ 339		

Exercise 84.

Determine by Sturm's Theorem the number and situation of the real roots of the following equations:

1. $x^3 - 4x^2 - 11x + 43 = 0.$

$$f(x) = x^3 - 4x^2 - 11x + 43$$
$$f'(x) = 3x^2 - 8x - 11$$

3 − 8 − 11	1 − 4 − 11 + 43	
6 − 16 − 22	3 − 12 − 33 + 129	1 − 4
6 − 21	3 − 8 − 11	
5 − 22	− 4 − 22 + 129	
10 − 44	− 12 − 66 + 387	
10 − 35	− 12 + 32 + 44	
− 9	− 98 + 343	
+ 9	2 − 7	3

$$\therefore f(x) = x^3 - 4x^2 - 11x + 43$$
$$f'(x) = 3x^2 - 8x - 11$$
$$f_2(x) = 2x - 7$$
$$f_3(x) = 9$$

	$f(x)$	$f'(x)$	$f_2(x)$	$f_3(x)$	
$x = -\infty$	−	+	−	+	**3 variations.**
$x = 0$	+	−	−	+	**2 variations.**
$x = +\infty$	+	+	+	+	**0 variations.**

$\therefore$ There is one negative root and two positive roots.

Again $\qquad f(-4) = -41, \qquad f(-3) = +13$

$\therefore$ The negative root lies between -3 and -4.

$$f(3) = +1, \qquad f(4) = -1, \qquad f(5) = +13$$

$\therefore$ One positive root lies between 3 and 4, the other between 4 and 5.

2. $x^3 - 6x^2 + 7x - 3 = 0.$

$$f(x) = x^3 - 6x^2 + 7x - 3$$
$$f'(x) = 3x^2 - 12x + 7$$

$$\begin{array}{r|r|l}
3 - 12 + 7 & 1 - 6 + 7 - 3 & \\
6 - 24 + 14 & 3 - 18 + 21 - 9 & 1 - 2 \\
6 - 3 & 3 - 12 + 7 & \\
\cline{1-2}
-21 + 14 & -6 + 14 - 9 & \\
-6 + 4 & -6 + 24 - 14 & \\
\cline{2-2}
-6 + 3 & -10 + 5 & \\
\cline{1-1}
+1 & 2 - 1 & 3 - 3 \\
-1 & &
\end{array}$$

$$\therefore f(x) = x^3 - 6x^2 + 7x - 3$$
$$f'(x) = 3x^2 - 12x + 7$$
$$f_2(x) = 2x - 1$$
$$f_3(x) = -1$$

	$f(x)$	$f'(x)$	$f_2(x)$	$f_3(x)$	
$x = -\infty$	$-$	$+$	$-$	$-$	2 variations.
$x = 0$	$-$	$+$	$-$	$-$	2 variations.
$x = +\infty$	$+$	$+$	$+$	$-$	1 variation.

$\therefore$ There is one positive root and no negative roots.

Again $\qquad f(4) = -7, \qquad f(5) = +7$

$\therefore$ The only real root lies between 4 and 5.

3. $x^4 - 4x^3 + x^2 + 6x + 2 = 0.$

$$f(x) = x^4 - 4x^3 + x^2 + 6x + 2$$
$$f'(x) = 4x^3 - 12x^2 + 2x + 6$$

$$\begin{array}{r|r|l}
4 - 12 + 2 + 6 & 1 - 4 + 1 + 6 + 2 & \\
2 - 6 + 1 + 3 & 2 - 8 + 2 + 12 + 4 & 1 - 1 \\
10 - 30 + 5 + 15 & 2 - 6 + 1 + 3 & \\
\cline{2-2}
10 - 20 - 14 & -2 + 1 + 9 + 4 & \\
\cline{1-1}
-10 + 19 + 15 & -2 + 6 - 1 - 3 & \\
\cline{2-2}
-10 + 20 + 14 & -5 + 10 + 7 & \\
\cline{1-1}
-1 + 1 & 5 - 10 - 7 & 2 - 2 \\
1 - 1 & 5 - 5 & 5 - 5 \\
\cline{2-2}
 & -5 - 7 & \\
 & -5 + 5 & \\
\cline{2-2}
 & -12 & \\
 & +12 &
\end{array}$$

$$\begin{aligned}
\therefore f(x) &= x^4 - 4x^3 + x^2 + 6x + 2\\
f'(x) &= 4x^3 - 12x^2 + 2x + 6\\
f_2(x) &= 5x^2 - 10x - 7\\
f_3(x) &= x - 1\\
f_4(x) &= +12
\end{aligned}$$

	$f(x)$	$f'(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	
$x = -\infty$	+	−	+	−	+	4 variations.
$x = 0$	+	+	−	−	+	2 variations.
$x = +\infty$	+	+	+	+	+	0 variations.

∴ There are two positive roots and two negative roots.

Again, $f(0) = 2,\ f(-\frac{1}{2}) = -\frac{3}{16},\ f(-1) = 2$

∴ One negative root lies between 0 and $-\frac{1}{2}$, and one between $-\frac{1}{2}$ and -1.

Also, $f(2) = 2,\ f(2\frac{1}{2}) = -0.1875,\ f(3) = 2$

∴ One positive root lies between 2 and $2\frac{1}{2}$, and one between $2\frac{1}{2}$ and 3.

4. $x^4 - 5x^3 + 10x^2 - 6x - 21 = 0.$

$$\begin{aligned}
f(x) &= x^4 - 5x^3 + 10x^2 - 6x - 21\\
f'(x) &= 4x^3 - 15x^2 + 20x - 6
\end{aligned}$$

$$\begin{array}{r|r|l}
4 - 15 + 20 - 6 & 1 - 5 + 10 - 6 - 21 & \\
20 - 75 + 100 - 30 & 4 - 20 + 40 - 24 - 84 & 1 - 5\\
20 + 112 - 1464 & 4 - 15 + 20 - 6 & \\
\cline{1-2}
-187 + 1564 - 30 & -5 + 20 - 18 - 84 & \\
-935 + 7820 - 150 & -20 + 80 - 72 - 336 & \\
-935 - 5236 + 68442 & -20 + 75 - 100 + 30 & \\
\cline{1-2}
13056 - 68592 & 5 + 28 - 366 & \\
-272 + 1429 & -5 - 28 + 366 & -4 + 187\\
 & -1360 - 7616 + 99552 & +5\\
 & -1360 + 7145 & \\
\cline{2-2}
 & -14761 + 99552 & \\
 & & \\
\cline{2-2}
 & + & \\
 & - &
\end{array}$$

$$\begin{aligned}
\therefore f(x) &= x^4 - 5x^3 + 10x^2 - 6x - 21\\
f(x) &= 4x^3 - 15x^2 + 20x - 6\\
f_2(x) &= -5x^2 - 28x + 366\\
f_3(x) &= -272x + 1429\\
f_4(x) &= -
\end{aligned}$$

	$f(x)$	$f'(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	
$x=-\infty$	+	−	−	+	−	3 variations.
$x=0$	−	−	+	+	−	2 variations.
$x=+\infty$	+	+	−	−	−	1 variation.

∴ There is one positive root, one negative root, and two imaginary roots.

Again, $f(0)=-21, \qquad f(-1)=+5$

∴ The negative root lies between 0 and -1.

Also, $f(3)=-3, \qquad f(4)=+51$

∴ The positive root lies between 3 and 4.

5. $x^4-x^3-x^2+6=0.$

$$f(x)=x^4-x^3-x^2+6$$
$$f'(x)=4x^3-3x^2-2x$$

$$\begin{array}{r|r|l}
4-3-2+0 & 1-1-1+0+6 & \\
44-33-22+0 & 4-4-4+0+24 & 1-1 \\
44+8-384 & 4-3-2+0 & \\
\hline
-41+362+0 & -1-2+0+24 & \\
-451+3982+0 & -4-8+0+96 & \\
-451-82+3936 & -4+3+2+0 & \\
\hline
4064-3936 & -11-2+96 & \\
-127+123 & 11+2-96 & 4-41 \\
 & 1397+254-12192 & -11 \\
 & 1397-1353 & \\
\cline{2-2}
 & 1607-12192 & \\
\cline{2-2}
 & - & \\
 & + &
\end{array}$$

$$\therefore f(x)=x^4-x^3-x^2+6$$
$$f'(x)=4x^3-3x^2-2x$$
$$f_2(x)=11x^2+2x-96$$
$$f_3(x)=-127x+123$$
$$f_4(x)=+$$

	$f(x)$	$f'(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	
$x=-\infty$	+	−	+	+	+	2 variations.
$x=0$	+	0	−	+	+	2 variations.
$x=+\infty$	+	+	+	−	+	2 variations.

∴ There are four imaginary roots.

6. $x^4 - 2x^3 - 3x^2 + 10x - 4 = 0.$

$$f(x) = x^4 - 2x^3 - 3x^2 + 10x - 4$$
$$f(x) = 4x^3 - 6x^2 - 6x + 10$$

$$\begin{array}{r|r|l}
4 - 6 - 6 + 10 & 1 - 2 - 3 + 10 - 4 & \\
2 - 3 - 3 + 5 & 2 - 4 - 6 + 20 - 8 & 1 - 1 \\
18 - 27 - 27 + 45 & 2 - 3 - 3 + 5 & \\
\cline{2-2}
\underline{18 - 54 + 22} & -1 - 3 + 15 - 8 & \\
27 - 49 + 45 & -2 - 6 + 30 - 16 & \\
\underline{27 - 81 + 33} & -2 + 3 + 3 - 5 & \\
\cline{2-2}
32 + 12 & -9 + 27 - 11 & \\
-8 - 3 & 9 - 27 + 11 & 2 + 3 \\
 & 72 - 216 + 88 & -9 + 243 \\
 & 72 + 27 & \\
\cline{2-2}
 & -243 + 88 & \\
 & -1944 + 704 & \\
 & -1944 - 729 & \\
\cline{2-2}
 & +1433 & \\
 & -1433 &
\end{array}$$

$$\therefore f(x) = x^4 - 2x^3 - 3x^2 + 10x - 4$$
$$f'(x) = 4x^3 - 6x^2 - 6x + 10$$
$$f_2(x) = 9x^2 - 27x + 11$$
$$f_3(x) = -8x - 3$$
$$f_4(x) = -1433$$

	$f(x)$	$f'(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	
$x = -\infty$	+	−	+	+	−	3 variations.
$x = 0$	−	+	+	−	−	2 variations.
$x = +\infty$	+	+	+	−	−	1 variation.

∴ There is one positive root, one negative root, and two imaginary roots.

Again, $f(-2) = -4,$ $f(-3) = +74$

∴ The negative root lies between -2 and -3.

Also, $f(0) = -4,$ $f(1) = +2$

∴ The positive root lies between 0 and 1.

7. $x^5 + 2x^4 + 3x^3 + 3x^2 - 1 = 0.$

$$f(x) = x^5 + 2x^4 + 3x^3 + 3x^2 - 1$$
$$f'(x) = 5x^4 + 8x^3 + 9x^2 + 6x$$

Both $f(x)$ and $f'(x)$ vanish when $x = -1$.

∴ $x = -1$ is therefore a double root of $f(x) = 0$.

Divide $f(x)$ by $(x + 1)^2$.

The depressed equation is

$$x^3 + 2x - 1 = 0$$

Now let $f(x) = x^3 + 2x - 1$

Then $f'(x) = 3x^2 + 2$

$$\begin{array}{r|r|l}
3 + 0 + 2 & 1 + 0 + 2 - 1 & \\
12 + 0 + 8 & 3 + 0 + 6 - 3 & 1 \\
12 - 9 & 3 + 0 + 2 & \\
\cline{1-2}
9 + 8 & 4 - 3 & -3 - 9 \\
36 + 32 & -4 + 3 & \\
36 - 27 & & \\
\cline{1-1}
+59 & & \\
-59 & &
\end{array}$$

$$\begin{aligned}
\therefore f(x) &= x^3 + 2x - 1 \\
f'(x) &= 3x^2 + 2 \\
f_2(x) &= -4x + 3 \\
f_3(x) &= -59
\end{aligned}$$

	$f(x)$	$f'(x)$	$f_2(x)$	$f_3(x)$	
$x = -\infty$	−	+	+	−	2 variations.
$x = 0$	−	+	+	−	2 variations.
$x = +\infty$	+	+	−	−	1 variation.

∴ There is one positive root and two imaginary roots.

Again, $f(0) = -1$, $f(1) = +2$

∴ The incommensurable real root lies between 0 and 1.

8. $x^5 + x^3 - 2x^2 + 3x - 2 = 0.$

$$\begin{aligned}
f(x) &= x^5 + x^3 - 2x^2 + 3x - 2 \\
f'(x) &= 5x^4 + 3x^2 - 4x + 3
\end{aligned}$$

$$\begin{array}{r|r|l}
5 + 0 + 3 - 4 + 3 & 1 + 0 + 1 - 2 + 3 - 2 & \\
5 - 15 + 30 - 25 & 5 + 0 + 5 - 10 + 15 - 10 & 1 \\
\cline{1-1}
15 - 27 + 21 + 3 & 5 + 0 + 3 - 4 + 3 & \\
\cline{2-2}
15 - 45 + 90 - 75 & 2 - 6 + 12 - 10 & \\
\cline{1-1}
18 - 69 + 78 & -1 + 3 - 6 + 5 & -5 - 15 \\
-6 + 23 - 26 & -6 + 18 - 36 + 30 & 1 + 1 \\
-210 + 805 - 910 & -6 + 23 - 26 & \\
\cline{2-2}
-210 + 372 & -5 - 10 + 30 & \\
\cline{1-1}
433 - 910 & -6 - 12 + 36 & \\
15155 - 31850 & -6 + 23 - 26 & \\
\cline{2-2}
15155 - 26896 & -35 + 62 & \\
\cline{1-1}
- & 35 - 62 & -6 + 433 \\
+ & &
\end{array}$$

$$f\ (x) = x^5 + x^3 - 2x^2 + 3x - 2$$
$$f'(x) = 5x^4 + 3x^2 - 4x + 3$$
$$f_2(x) = -x^3 + 3x^2 - 6x + 5$$
$$f_3(x) = -6x^2 + 23x - 26$$
$$f_4(x) = 35x - 62$$
$$f_5(x) = +$$

	$f(x)$	$f'(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	$f_5(x)$	
$x = -\infty$	−	+	+	−	−	+	3 variations.
$x = 0$	−	+	+	−	−	+	3 variations.
$x = +\infty$	+	+	−	−	+	+	2 variations.

∴ There is one positive root and four imaginary roots.

Again $f(0) = -2$, $f(1) = +1$

∴ The real root lies between 0 and 1.

Exercise 85.

Solve the equations:

1. $x^4 + 7x^3 - 7x - 1 = 0.$

$$x^4 - 1 + 7x(x^2 - 1) = 0$$
$$(x^2 - 1)(x^2 + 7x + 1) = 0$$
$$\therefore\ x = \pm 1,\ \text{or}\ \frac{-7 \pm 3\sqrt{5}}{2}$$

2. $x^4 + 2x^3 + x^2 + 2x + 1 = 0.$

Divide by x^2,

$$x^2 + \frac{1}{x^2} + 2\left(x + \frac{1}{x}\right) + 1 = 0$$

Put $x + \frac{1}{x} = z$

Then $z^2 - 2 + 2z + 1 = 0$

$$z^2 + 2z - 1 = 0$$
$$z = -1 \pm \sqrt{2}$$

$$x + \frac{1}{x} = -1 + \sqrt{2} \qquad x + \frac{1}{x} = -1 - \sqrt{2}$$
$$x^2 + (1 - \sqrt{2})x + 1 = 0 \qquad x^2 + (1 + \sqrt{2})x + 1 = 0$$
$$\therefore x = \frac{-1 + \sqrt{2} \pm \sqrt{-2\sqrt{2} - 1}}{2} \qquad \therefore x = \frac{-1 - \sqrt{2} \pm \sqrt{2\sqrt{2} - 1}}{2}$$

3. $x^6 - 3x^5 + 5x^4 - 5x^2 + 3x - 1 = 0.$

$$x^6 - 1 - 3x(x^4 - 1) + 5x^2(x^2 - 1) = 0$$

Divide by $x^2 - 1$,

$$x^4 + x^2 + 1 - 3x(x^2 + 1) + 5x^2 = 0$$
$$x^4 - 3x^3 + 6x^2 - 3x + 1 = 0$$

Divide by x^2, $$x^2+\frac{1}{x^2}-3\left(x+\frac{1}{x}\right)+6=0$$

Put $$x+\frac{1}{x}=z$$

$$z^2-2-3z+6=0$$

$$z^2-3z+4=0$$

$$z=\frac{3\pm\sqrt{-7}}{2}$$

$$x+\frac{1}{x}=\frac{3+\sqrt{-7}}{2}\qquad\qquad x+\frac{1}{x}=\frac{3-\sqrt{-7}}{2}$$

$$x^2-\left(\frac{3+\sqrt{-7}}{2}\right)x+1=0\qquad\qquad x^2-\left(\frac{3-\sqrt{-7}}{2}\right)x+1=0$$

$$\therefore x=\frac{3+\sqrt{-7}\pm\sqrt{-14+6\sqrt{-7}}}{4}\qquad\qquad \therefore x=\frac{3-\sqrt{-7}\pm\sqrt{-14-6\sqrt{-7}}}{4}$$

4. $x^4-5x^3+6x^2-5x+1=0.$

Divide by x^2, $$x^2+\frac{1}{x^2}-5\left(x+\frac{1}{x}\right)+6=0$$

Put $$x+\frac{1}{x}=z$$

$$z^2-2-5z+6=0$$

$$z^2-5z+4=0$$

$$z=1,\text{ or }4$$

$$x+\frac{1}{x}=1\qquad\qquad x+\frac{1}{x}=4$$

$$x^2-x+1=0\qquad\qquad x^2-4x+1=0$$

$$x=\frac{1\pm\sqrt{-3}}{2}\qquad\qquad x=2\pm\sqrt{3}$$

5. $2x^4-5x^3+6x^2-5x+2=0.$

Divide by x^2, $$2\left(x^2+\frac{1}{x^2}\right)-5\left(x+\frac{1}{x}\right)+6=0$$

Put $$x+\frac{1}{x}=z$$

$$2(z^2-2)-5z+6=0$$

$$2z^2-5z+2=0$$

$$\therefore z=2,\text{ or }\tfrac{1}{2}$$

$$x+\frac{1}{x}=2\qquad\qquad x+\frac{1}{x}=\frac{1}{2}$$

$$x^2-2x+1=0\qquad\qquad x^2-\tfrac{1}{2}x+1=0$$

$$x=1,1\qquad\qquad x=\frac{1\pm\sqrt{-15}}{4}$$

6. $x^5 - 4x^4 + x^3 + x^2 - 4x + 1 = 0.$

$$x^5 + 1 - 4x(x^3 + 1) + x^2(x + 1) = 0$$

Divide by $x + 1$,

$$x^4 - x^3 + x^2 - x + 1 - 4x(x^2 - x + 1) + x^2 = 0$$

$$x^4 - 5x^3 + 6x^2 - 5x + 1 = 0$$

Divide by x^2, $\left(x^2 + \frac{1}{x^2}\right) - 5\left(x + \frac{1}{x}\right) + 6 = 0$

Put $x + \frac{1}{x} = z$

$$z^2 - 2 - 5z + 6 = 0$$

$$z^2 - 5z + 4 = 0$$

$$z = 1, \text{ or } 4$$

$x + \frac{1}{x} = 1$ $\qquad$ $x + \frac{1}{x} = 4$

$x^2 - x + 1 = 0$ $\qquad$ $x^2 - 4x + 1 = 0$

$x = \frac{1 \pm \sqrt{-3}}{2}$ $\qquad$ $x = 2 \pm \sqrt{3}$

7. $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0.$

Divide by x^2, $\left(x^2 + \frac{1}{x^2}\right) - 10\left(x + \frac{1}{x}\right) + 26 = 0$

Put $x + \frac{1}{x} = z,$

$$z^2 - 2 - 10z + 26 = 0$$

$$z^2 - 10z + 24 = 0$$

$$z = 4, \text{ or } 6$$

$x + \frac{1}{x} = 4$ $\qquad$ $x + \frac{1}{x} = 6$

$x^2 - 4x + 1 = 0$ $\qquad$ $x^2 - 6x + 1 = 0$

$x = 2 \pm \sqrt{3}$ $\qquad$ $x = 3 \pm 2\sqrt{2}$

8. $x^3 + mx^2 + mx + 1 = 0.$

$$x^3 + 1 + mx(x + 1) = 0$$

Divide by $x + 1$, $x^2 - x + 1 + mx = 0$

$$x^2 + (m - 1)x + 1 = 0$$

$$\therefore\ x = \frac{1 - m \pm \sqrt{m^2 - 2m - 3}}{2}$$

$$\therefore\ x = -1,\ \frac{1 - m \pm \sqrt{m^2 - 2m - 3}}{2}$$

9. $x^5 + x^4 - x^3 - x^2 + x + 1 = 0.$

$$x^5 + 1 + x(x^3 + 1) - x^2(x + 1) = 0$$

Divide by $x + 1$,

$$x^4 - x^3 + x^2 - x + 1 + x(x^2 - x + 1) - x^2 = 0$$

$$x^4 - x^2 + 1 = 0$$

Divide by x^2, $$x^2 + \frac{1}{x^2} - 1 = 0$$

Put $$x + \frac{1}{x} = z$$

$$z^2 - 2 - 1 = 0$$

$$z = \pm\sqrt{3}$$

$$x + \frac{1}{x} = \sqrt{3} \qquad\qquad x + \frac{1}{x} = -\sqrt{3}$$

$$x^2 - \sqrt{3}x + 1 = 0 \qquad\qquad x^2 + \sqrt{3}x + 1 = 0$$

$$x = \frac{\sqrt{3} \pm \sqrt{-1}}{2} \qquad\qquad \therefore x = \frac{-\sqrt{3} \pm \sqrt{-1}}{2}$$

$$\therefore x = -1, \quad \frac{\sqrt{3} \pm \sqrt{-1}}{2}, \text{ or } \frac{-\sqrt{3} \pm \sqrt{-1}}{2}$$

10. $3x^5 - 2x^4 + 5x^3 - 5x^2 + 2x - 3 = 0.$

$$3(x^5 - 1) - 2x(x^3 - 1) + 5x^2(x - 1) = 0$$

Divide by $x - 1$,

$$3(x^4 + x^3 + x^2 + x + 1) - 2x(x^2 + x + 1) + 5x^2 = 0$$

$$3x^4 + x^3 + 6x^2 + x + 3 = 0$$

Divide by x^2 $$3\left(x^2 + \frac{1}{x^2}\right) + x + \frac{1}{x} + 6 = 0$$

Put $$x + \frac{1}{x} = z$$

$$3(z^2 - 2) + z + 6 = 0$$

$$3z^2 + z = 0$$

$$\therefore z = 0 \text{ or } -\tfrac{1}{3}$$

$$x = \frac{1}{x} = 0 \qquad\qquad x + \frac{1}{x} = -\frac{1}{3}$$

$$x^2 + 1 = 0 \qquad\qquad x^2 + \tfrac{1}{3}x + 1 = 0$$

$$x = \pm\sqrt{-1} \qquad\qquad x = \frac{-1 \pm \sqrt{-35}}{6}$$

Exercise 86.

Solve the binomial equations:

1. $x^6+1=0.$

Put $x^2=y$

$$\therefore\ y^3+1=0$$
$$\therefore\ y=-1,\ -\omega,\ -\omega^2$$
$$\therefore\ x=\pm\sqrt{-1},\ \pm\sqrt{-\omega},\ \pm\sqrt{-\omega^2}$$

But $\omega=\omega^4$

$$\therefore\ \sqrt{-\omega}=\sqrt{-\omega^4}$$
$$=\omega^2\sqrt{-1}$$
$$\therefore\ x=\pm\sqrt{-1},\ \pm\omega^2\sqrt{-1},\ \pm\omega\sqrt{-1}$$

2. $x^8-1=0.$

$$(x^4-1)(x^4+1)=0$$

Let $x^4-1=0$

$$(x^2-1)(x^2+1)=0$$
$$\therefore\ x=\pm 1,\ \pm\sqrt{-1}$$

Let $x^4+1=0$

$$x^2+\frac{1}{x^2}=0$$
$$\left(x+\frac{1}{x}\right)^2-2=0$$
$$\therefore\ x+\frac{1}{x}=\pm\sqrt{2}$$

$$x+\frac{1}{x}=\sqrt{2}$$
$$x^2-\sqrt{2}\,x+1=0$$
$$\therefore\ x=\frac{\sqrt{2}\pm\sqrt{-2}}{2}$$

$$x+\frac{1}{x}=-\sqrt{2}$$
$$x^2+\sqrt{2}\,x+1=0$$
$$\therefore\ x=\frac{-\sqrt{2}\pm\sqrt{-2}}{2}$$

3. $x^9-1=0.$

Let $x^3=y$

$$\therefore\ y^3-1=0$$
$$\therefore\ y=1,\ \omega,\ \omega^2$$

Let $\therefore\ x^3=1$

$$\therefore\ x=1,\ \omega,\ \omega^2$$

Let $$x^3 = \omega$$

$$\therefore\ x = \sqrt[3]{\omega},\ \omega\sqrt[3]{\omega},\ \omega^2\sqrt[3]{\omega}$$

Let $$x^3 = \omega^2$$

$$\therefore\ x = \sqrt[3]{\omega^2},\ \omega\sqrt[3]{\omega^2},\ \omega^2\sqrt[3]{\omega^2}$$

$$\therefore\ x = 1,\ \omega,\ \omega^2,\ \sqrt[3]{\omega},\ \omega\sqrt[3]{\omega},\ \omega^2\sqrt[3]{\omega},\ \sqrt[3]{\omega^2},\ \omega\sqrt[3]{\omega^2},\ \omega^2\sqrt[3]{\omega^2}.$$

4. $x^5 - 243 = 0.$

$$(x-3)(x^4 + 3x^3 + 9x^2 + 27x + 81) = 0$$

$$\therefore\ x = 3$$

or $$x^4 + 3x^3 + 9x^2 + 27x + 81 = 0$$

Divide by $9x^2$, $$\frac{x^2}{9} + \frac{9}{x^2} + \frac{x}{3} + \frac{3}{x} + 1 = 0$$

Put $$\frac{x}{3} + \frac{3}{x} = z$$

$$z^2 - 2 + z + 1 = 0$$

$$z^2 + z - 1 = 0$$

$$\therefore\ z = \frac{-1 \pm \sqrt{5}}{2}$$

$$\frac{x}{3} + \frac{3}{x} = \frac{-1+\sqrt{5}}{2}$$

$$x^2 + \frac{3-3\sqrt{5}}{2}x + 9 = 0$$

$$\therefore\ x = \frac{-3+3\sqrt{5} \pm \sqrt{-90-18\sqrt{5}}}{4}$$

$$= \tfrac{3}{4}(-1+\sqrt{5}+\sqrt{10+2\sqrt{5}}\sqrt{-1})$$

$$\frac{x}{3} + \frac{3}{x} = \frac{-1-\sqrt{5}}{2}$$

$$x^2 + \frac{3+3\sqrt{5}}{2}x + 9 = 0$$

$$\therefore\ x = \frac{-3-3\sqrt{5} \pm \sqrt{-90-18\sqrt{5}}}{4}$$

$$= \tfrac{3}{4}(-1-\sqrt{5}+\sqrt{10+2\sqrt{5}}\sqrt{-1})$$

5. Find the quintic on which the solution of the equation $x^{11} = 1$ depends.

$$x^{11} - 1 = 0$$

$$(x-1)(x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1) = 0$$

$$\therefore\ x = 1$$

or $$x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$$

Divide by x^5,

$$x^5 + \frac{1}{x^5} + x^4 + \frac{1}{x^4} + x^3 + \frac{1}{x^3} + x^2 + \frac{1}{x^2} + x + \frac{1}{x} + 1 = 0$$

Add $$5x^3 + 10x + \frac{10}{x} + \frac{5}{x^3}$$

and $$4x^2 + 6 + \frac{4}{x^2}$$

to both sides of the equation.

Then

$$x^5 + 5x^3 + 10x + \frac{10}{x} + \frac{5}{x^3} + \frac{1}{x^5} + x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4}$$
$$+ x^3 + \frac{1}{x^3} + x^2 + \frac{1}{x^2} + x + \frac{1}{x} + 1$$
$$= 5x^3 + 10x + \frac{10}{x} + \frac{5}{x^3} + 4x^2 + 6 + \frac{4}{x^2}$$

$$\therefore \left(x + \frac{1}{x}\right)^5 + \left(x + \frac{1}{x}\right)^4 + x^3 + \frac{1}{x^3} + x^2 + \frac{1}{x^2} + x + \frac{1}{x} + 1$$
$$= 5x^3 + 10x + \frac{10}{x} + \frac{5}{x^3} + 4x^2 + 6 + \frac{4}{x^2}$$

$$\therefore \left(x + \frac{1}{x}\right)^5 + \left(x + \frac{1}{x}\right)^4 - 4\left(x^3 + \frac{1}{x^3}\right) - 3\left(x^2 + \frac{1}{x^2}\right) - 9\left(x + \frac{1}{x}\right) - 5 = 0$$

Again, subtract $12x + \frac{12}{x}$ and 6 from both sides of the equation.

Then

$$\left(x+\frac{1}{x}\right)^5 + \left(x+\frac{1}{x}\right)^4 - 4\left(x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}\right) - 3\left(x^2 + 2 + \frac{1}{x^2}\right) - 9\left(x + \frac{1}{x}\right) - 5$$
$$= -12\left(x + \frac{1}{x}\right) - 6$$

$$\therefore \left(x + \frac{1}{x}\right)^5 + \left(x + \frac{1}{x}\right)^4 - 4\left(x + \frac{1}{x}\right)^3 - 3\left(x + \frac{1}{x}\right)^2 + 3\left(x + \frac{1}{x}\right) + 1 = 0$$

Put $\quad x + \frac{1}{x} = z$

Then $\quad z^5 + z^4 - 4z^3 - 3z^2 + 2z + 1 = 0$

and this is the required quintic.

6. Show that $x^3 \pm y^3 \equiv (x \pm y)(x \pm \omega y)(x + \omega^2 y)$.

$$x^3 \pm y^3 = y^3\left(\frac{x^3}{y^3} \pm 1\right)$$

Let $\quad \frac{x}{y} = z$

Then

$$x^3 \pm y^3 = y^3(z^3 \pm 1)$$
$$= x^3(z + 1)(z \pm \omega)(z \pm \omega^2)$$
$$= y^3\left(\frac{x}{y} \pm 1\right)\left(\frac{x}{y} \pm \omega\right)\left(\frac{x}{y} \pm \omega^2\right)$$
$$= (x \pm y)(x \pm \omega y)(x \pm \omega^2 y)$$

7. Show that

$$x^2 + y^2 + z^2 - yz - zx - xy \equiv (x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z).$$

$$\begin{array}{l}
x + \omega y \quad + \omega^2 z \\
\underline{x + \omega^2 y \quad + \omega z} \\
x^2 + \omega xy \quad + \omega^2 xz \\
\quad + \omega^2 xy \qquad\qquad + \omega^3 y^2 + \omega^4 yz \\
\qquad\qquad + \omega xz \qquad\qquad + \omega^2 yz + \omega^3 z^2 \\
\hline
x^2 + (\omega + \omega^2)xy + (\omega + \omega^2)\,xz + \omega^3 y^2 + (\omega^4 + \omega^2)yz + \omega^3 z^2
\end{array}$$

But $\omega^3 = 1$

$\therefore\ \omega^4 = \omega$

and $\omega + \omega^2 = -1$

$\therefore$ The product is

$$x^2 + y^2 + z^2 - xy - yz - xz.$$

8. If α is an imaginary root of $x^5 - 1 = 0$, show that

$$(1 - \alpha)(1 - \alpha^2)(1 - \alpha^3)(1 - \alpha^4) = 5.$$

If $x^5 - 1 = 0$

$$(x - 1)(x^4 + x^3 + x^2 + x + 1) = 0$$

$$\therefore\ x = 1$$

or $x^4 + x^3 + x^2 + x + 1 = 0$

The four imaginary roots are the roots of this latter equation.

Since α is one root, α^2, α^3, and α^4 are the other three roots, denote these three roots by β, γ, and δ.

Now $(1 - \alpha)(1 - \beta)(1 - \gamma)(1 - \delta)$

$$= 1 - (\alpha + \beta + \gamma + \delta) + (\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$$
$$- (\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta) + \alpha\beta\gamma\delta$$

But $\alpha + \beta + \gamma + \delta = -1$

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = 1$$
$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -1$$
$$\alpha\beta\gamma\delta = 1$$
$$\therefore\ (1 - \alpha)(1 - \beta)(1 - \gamma)(1 - \delta) = 1 + 1 + 1 + 1 + 1$$
$$= 5$$

Exercise 87.

Find the three roots of:

1. $x^3 + 6x^2 = 36$.

$a = 1, \quad b = 2, \quad c = 0, \quad d = -36$

$\therefore\ H = -4, \quad G = -20$

$$\therefore\ u = \frac{20 \pm \sqrt{400 - 256}}{2} = 16, \text{ or } 4$$

$$v = -\frac{H^3}{u} = 4, \text{ or } 16$$

$\therefore$ The values of z are

$$2\sqrt[3]{2} + \sqrt[3]{4}, \quad 2\omega\sqrt[3]{2} + \omega^2\sqrt[3]{4}, \quad 2\omega^2\sqrt[3]{2} + \omega\sqrt[3]{4}$$

And the values of x are

$$-2 + 2\sqrt[3]{2} + \sqrt[3]{4}, \quad -2 + 2\omega\sqrt[3]{2} + \omega^2\sqrt[3]{4}, \quad -2 + 2\omega^2\sqrt[3]{2} + \omega\sqrt[3]{4}$$

2. $3x^3 - 6x^2 - 2 = 0.$

$$a = 3, \quad b = -2, \quad c = 0, \quad d = -2,$$

$$\therefore\ H = -4, \quad G = -34$$

$$\therefore\ u = \frac{34 \pm \sqrt{1156 - 256}}{2} = 2 \text{ or } 32$$

$$v = -\frac{H^3}{u} = 32 \text{ or } 2$$

The values of z are

$$\sqrt[3]{2} + 2\sqrt[3]{4}, \quad \omega\sqrt[3]{2} + 2\omega^2\sqrt[3]{4}, \quad \omega^2\sqrt[3]{2} + 2\omega\sqrt[3]{4}$$

And the values of x are

$$\tfrac{1}{3}(2 + \sqrt[3]{2} + 2\sqrt[3]{4}), \quad \tfrac{1}{3}(2 + \omega\sqrt[3]{2} + 2\omega^2\sqrt[3]{4}), \quad \tfrac{1}{3}(2 + \omega^2\sqrt[3]{2} + 2\omega\sqrt[3]{4})$$

3. $x^3 - 3x^2 - 6x - 4 = 0.$

$$a = 1, \quad b = -1, \quad c = -2, \quad d = -4$$

$$\therefore\ H = -3, \quad G = -12$$

$$u = \frac{12 \pm \sqrt{144 - 108}}{2} = 3, \text{ or } 9$$

$$v = -\frac{H^3}{u} = 9, \text{ or } 3$$

$\therefore$ The values of z are

$$\sqrt[3]{3} + \sqrt[3]{9}, \quad \omega\sqrt[3]{3} + \omega^2\sqrt[3]{9}, \quad \omega^2\sqrt[3]{3} + \omega\sqrt[3]{9}$$

And the values of x are

$$1 + \sqrt[3]{3} + \sqrt[3]{9}, \quad 1 + \omega\sqrt[3]{3} + \omega^2\sqrt[3]{9}, \quad 1 + \omega^2\sqrt[3]{3} + \omega\sqrt[3]{9}$$

4. $9x^2 - 54x^2 + 90x - 50 = 0.$

$$a = 9, \quad b = -18, \quad c = 30, \quad d = -50$$

$$\therefore\ H = -54, \quad G = -1134$$

$$u = \frac{1134 \pm \sqrt{9^4 \times 14^2 - 9^4 \times 96}}{2} = 162, \text{ or } 972$$

$$v = -\frac{H^3}{u} = 972, \text{ or } 162$$

∴ The values of z are

$$\sqrt[3]{162}+\sqrt[3]{972},\quad \omega\sqrt[3]{162}+\omega^2\sqrt[3]{972},\quad \omega^2\sqrt[3]{162}+\omega\sqrt[3]{972}$$

or

$$3\sqrt[3]{6}+3\sqrt[3]{36},\quad 3\omega\sqrt[3]{6}+3\omega^2\sqrt[3]{36},\quad 3\omega^2\sqrt[3]{6}+3\omega\sqrt[3]{36}$$

And the values of x are

$$2+\tfrac{1}{3}(\sqrt[3]{6}+\sqrt[3]{36}),\quad 2+\tfrac{1}{3}(\omega\sqrt[3]{6}+\omega^2\sqrt[3]{36}),\quad 2+\tfrac{1}{3}(\omega^2\sqrt[3]{6}+\omega\sqrt[3]{36})$$

5. $z^3+3mz^2=m^2(m+1)^2.$

$$a=1,\ b=m,\ c=0,\ d=-m^2(m+1)^2$$

$$\therefore\ H=-m^2,\ G=-m^2(m+1)^2+2m^3=-m^4-m^2$$

$$\begin{aligned}u&=\frac{m^4+m^2\pm\sqrt{m^8+2m^6+m^4-4m^6}}{2}\\&=\frac{m^4+m^2\pm(m^4-m^2)}{2}\\&=m^4,\text{ or }m^2\\v&=-\frac{H^3}{u}\\&=m^2,\text{ or }m^4\end{aligned}$$

∴ The values of z are

$$m\sqrt[3]{m}+\sqrt[3]{m^2},\quad \omega m\sqrt[3]{m}+\omega^2\sqrt[3]{m^2},\quad \omega^2 m\sqrt[3]{m}+\omega\sqrt[3]{m^2}.$$

And the values of x are

$$-m+m\sqrt[3]{m}+\sqrt[3]{m^2},\quad -m+\omega m\sqrt[3]{m}+\omega^2\sqrt[3]{m^2},\quad -m+\omega^2\sqrt[3]{m}+\omega\sqrt[3]{m^2}.$$

6. In the case of the cubic, putting

$$L\equiv\alpha+\omega\beta+\omega^2\gamma,\quad M\equiv\alpha+\omega^2\beta+\omega\gamma,$$

show that:

$$\begin{aligned}L^3+M^3&=2\Sigma\alpha^3-3\Sigma\alpha^2\beta+12\alpha\beta\gamma\\&=-27\left(\frac{d}{a}-\frac{3bc}{a^2}+\frac{2b^3}{a^3}\right)\\&=-\frac{27G}{a^3}\\LM&=\alpha^2+\beta^2+\gamma^2-\beta\gamma-\gamma\alpha-\alpha\beta\\&=-\frac{9H}{a^2}\\L^3-M^3&=-3\sqrt{-3}(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)\end{aligned}$$

$$\begin{aligned}(\alpha+\omega\beta+\omega^2\gamma)^3&=\alpha^3+\omega^3\beta^3+\omega^6\gamma^3+3\omega\alpha^2\beta+3\omega^2\alpha\beta^2+3\omega^2\alpha^2\gamma+3\omega^4\alpha\gamma^2\\&\quad+3\omega^4\beta^2\gamma+3\omega^5\beta\gamma^2+6\omega^3\alpha\beta\gamma\\&=\alpha^3+\beta^3+\gamma^3+3\omega\alpha^2\beta+3\omega^2\alpha\beta^2+3\omega^2\alpha^2\gamma+3\omega\alpha\gamma^2\\&\quad+3\omega\beta^2\gamma+3\omega^2\beta\gamma^2+6\alpha\beta\gamma\end{aligned}$$

$$\begin{aligned}(\alpha + \omega^2\beta + \omega\gamma)^3 &= \alpha^3 + \omega^6\beta^3 + \omega^3\gamma^3 + 3\,\omega^2\alpha^2\beta + 3\,\omega^4\alpha\beta^2 + 3\,\omega\alpha^2\gamma + 3\,\omega^2\alpha\gamma^2\\ &\quad + 3\,\omega^5\beta^2\gamma + 3\,\omega^4\beta\gamma^2 + 6\,\omega^3\alpha\beta\gamma\\ &= \alpha^3 + \beta^3 + \gamma^3 + 3\,\omega^2\alpha^2\beta + 3\,\omega\alpha\beta^2 + 3\,\omega\alpha^2\gamma + 3\,\omega^2\alpha\gamma^2\\ &\quad + 3\,\omega^2\beta^2\gamma + 3\,\omega\beta\gamma^2 + 6\,\alpha\beta\gamma\end{aligned}$$

$$\begin{aligned}\therefore\ L^3 + M^3 &= 2(\alpha^3 + \beta^3 + \gamma^3) + 3(\omega + \omega^2)(\alpha^2\beta + \alpha\beta^2 + \alpha^2\gamma + \alpha\gamma^2\\ &\quad + \beta^2\gamma + \beta\gamma^2) + 12\,\alpha\beta\gamma\\ &= 2\,\Sigma\alpha^3 - 3\,\Sigma\alpha^2\beta + 12\,\alpha\beta\gamma\end{aligned}$$

We have
$$\alpha + \beta + \gamma = -\frac{3\,b}{a}$$
$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{3\,c}{a}$$
$$\alpha\beta\gamma = -\frac{d}{a}$$

$$\begin{aligned}\therefore\ \alpha^2\beta + \beta^2\alpha + \alpha^2\gamma + \alpha\gamma^2 + \beta^2\gamma + \beta\gamma^2 &= (\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha + \beta + \gamma) - 3\,\alpha\beta\gamma\\ &= -\frac{9\,bc}{a^2} + 3\frac{d}{a}\\ &= \frac{3\,ad - 9\,bc}{a^2}\end{aligned}$$

$$\begin{aligned}\alpha^3 + \beta^3 + \gamma^3 &= (\alpha + \beta + \gamma)^3 - 3(\alpha^2\beta + \alpha\beta^2 + \alpha^2\gamma + \alpha\gamma^2 + \beta^2\gamma + \beta\gamma^2) - 6\,\alpha\beta\gamma\\ &= -\frac{27\,b^3}{a^3} + \frac{27\,bc}{a^2} - 9\frac{d}{a} + 6\frac{d}{a}\\ &= \frac{27\,abc - 3\,a^2d - 27\,b^3}{a^3}\end{aligned}$$

$$\begin{aligned}\therefore\ L^3 + M^3 &= \frac{54\,abc - 6\,a^2d - 54\,b^3 - 9\,a^2d + 27\,abc - 12\,a^2d}{a^3}\\ &= \frac{81\,abc - 27\,a^2d - 54\,b^3}{a^3}\\ &= \frac{-27(a^2d - 3\,abc + 2\,b^3)}{a^3}\\ &= -\frac{27\,G}{a^3}\end{aligned}$$

$$\begin{aligned}LM &= (\alpha + \omega\beta + \omega^2\gamma)(\alpha + \omega^2\beta + \omega\gamma)\\ &= \alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha\\ &= (\alpha + \beta + \gamma)^2 - 3(\alpha\beta + \beta\gamma + \gamma\alpha)\\ &= \frac{9\,b^2}{a^2} - \frac{9\,c}{a}\\ &= \frac{9(b^2 - ca^2)}{a^2}\\ &= -\frac{9\,H}{a^2}\end{aligned}$$

$$\begin{aligned}L^3 - M^3 &= 3(\omega - \omega^2)(\alpha^2\beta - \alpha\beta^2 - \alpha^2\gamma + \alpha\gamma^2 + \beta^2\gamma - \beta\gamma^2)\\ &= 3\sqrt{-3}\,[\alpha\beta(\alpha-\beta) - (\alpha+\beta)(\alpha-\beta)\gamma + (\alpha-\beta)\gamma^2]\\ &= 3\sqrt{-3}(\alpha-\beta)(\alpha\beta - \alpha\gamma - \beta\gamma + \gamma^2)\\ &= 3\sqrt{-3}(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)\\ &= -3\sqrt{-3}(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)\end{aligned}$$

7. From Ex. 6, and the relation

$$(L^3 - M^3)^2 \equiv (L^3 + M^3)^2 - 4L^3M^3,$$

show that

$$a^6(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2 = -27\,(G^2 + 4H^3),$$

and thence deduce the conditions of § 513.

We have

$$\begin{aligned}(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta) &= \frac{L^3 - M^3}{-3\sqrt{-3}}\\ \therefore\ (\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2 &= \frac{(L^3 - M^3)^2}{-27}\\ a^6(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2 &= \frac{a^6(L^3 - M^3)^2}{-27}\\ &= \frac{a^6[(L^3 + M^3)^2 - 4\,ML)^3]}{-27}\\ &= \frac{a^6\left[\left(-\frac{27\,G}{a^3}\right)^2 - 4\left(-\frac{9\,H}{a^2}\right)^3\right]}{-27}\\ &= \frac{27^2\,G^2 + 4 \times 27^2\,H^3}{-27}\\ &= -27\,(G^2 + 4\,H^2)\end{aligned}$$

Now, if α, β, and γ are all real and unequal, $(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2$ will be real and positive.

$G^2 + 4\,H^3$ is therefore in this case negative.

If two roots are equal, then one of the quantities $\alpha-\beta$, $\beta-\gamma$, $\gamma-\alpha$ is 0.

In this case $G^2 + 4\,H^3$ must be 0.

If two of the roots are imaginary, say β and γ, denote them by $a + bi$ and $a - bi$. Then $\alpha - \beta = \alpha - a - bi$, $\alpha - \gamma = \alpha - a + bi$, and $\beta - \gamma = 2bi$. $\alpha - \beta$ and $\alpha - \gamma$ are therefore conjugate imaginaries, and their product is therefore positive, and its square is positive. But the square of $\beta - \gamma$ is negative. Hence in this case $(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2$ is negative, and therefore $G^2 + 4\,H^3$ is positive.

Exercise 88.

Find the four roots of:

1. $x^4 - 12x^3 + 50x^2 - 84x + 49 = 0.$

Put $$z = x - 3$$

Diminish the roots of the given equation by 3.

$$\begin{array}{rrrrrl}
1 & -12 & +50 & -84 & +49 & \underline{3} \\
 & +3 & -27 & +69 & -45 & \\
\cline{2-5}
 & -9 & +23 & -15 & +4 & \\
 & +3 & -18 & +15 & & \\
\cline{2-4}
 & -6 & +5 & +0 & & \\
 & +3 & -9 & & & \\
\cline{2-3}
 & -3 & -4 & & & \\
 & +3 & & & & \\
\cline{2-2}
 & 0 & & & &
\end{array}$$

∴ The equation in z is

$$z^4 - 4z^2 + 4 = 0$$
$$(z^2 - 2)^2 = 0$$
$$\therefore z^2 = 2, 2$$
$$z = \pm\sqrt{2}, \pm\sqrt{2}$$
$$\therefore x = 3 \pm \sqrt{2}, 3 \pm \sqrt{2}.$$

2. $x^4 - 17x^2 - 20x - 6 = 0.$

Compare the equation with

$$(x^2 + px + q)(x^2 - px + q') = 0$$

We have $$q + q' - p^2 = -17 \qquad (1)$$
$$pq' - pq = -20 \qquad (2)$$
$$qq' = -6 \qquad (3)$$

From (1), $$q + q' = p^2 - 17$$

From (2), $$q' - q = -\frac{20}{p}$$

$$\therefore 2q = p^2 - 17 + \frac{20}{p}$$
$$2q' = p^2 - 17 - \frac{20}{p}$$

$$\therefore \left(p^2 - 17 + \frac{20}{p}\right)\left(p^2 - 17 - \frac{20}{p}\right) = -24$$
$$p^6 - 34p^4 + 289p^2 - 400 = -24p^2$$

$$p^6 - 34p^4 + 313p^2 - 400 = 0$$
$$(p^2 - 16)(p^4 - 18p^2 + 25) = 0$$
$$\therefore\ p = \pm 4 \text{ are two roots.}$$

Hence we have

$$2q = 4$$
$$q = 2$$
$$2q' = -6$$
$$q' = -3$$
$$\therefore\ x^4 - 17x^2 - 20x - 6 \equiv (x^2 + 4x + 2)(x^2 - 4x - 3)$$

But $x^2 + 4x + 2 = 0$

gives $x = -2 \pm \sqrt{2}$

And $x^2 - 4x - 3 = 0$

gives $x = 2 \pm \sqrt{7}$

$\therefore$ The four roots are $-2 \pm \sqrt{2}$, $2 \pm \sqrt{7}$.

3. $x^4 - 8x^3 + 16x - 21 = 0$.

Let $z = x - 2$

Diminish the roots of the given equation by 2.

$$\begin{array}{rrrrr|l}
1 & -8 & +20 & -16 & -21 & \underline{2} \\
 & +2 & -12 & +16 & +\ 0 & \\
\hline
 & -6 & +\ 8 & +\ 0 & -21 & \\
 & +2 & -\ 8 & +\ 0 & & \\
\hline
 & -4 & +\ 0 & +\ 0 & & \\
 & +2 & -\ 4 & & & \\
\hline
 & -2 & -\ 4 & & & \\
 & +2 & & & & \\
\hline
 & 0 & & & &
\end{array}$$

$\therefore$ The equation in z is

$$z^4 - 4z^2 - 21 = 0$$
$$\therefore\ z^2 = -3, \text{ or } 7$$
$$z = \pm\sqrt{-3}, \text{ or } \pm\sqrt{7}$$
$$x = z + 2$$
$$= 2 \pm \sqrt{-3}, \text{ or } 2 \pm \sqrt{7}$$

4. $x^4 - 11x^3 + 46x^2 - 117x + 45 = 0$.

$$a = 1,\quad b = -\tfrac{11}{4},\quad c = \tfrac{23}{3},\quad d = -\tfrac{117}{4},\quad e = 45$$
$$\therefore\ H = \tfrac{23}{3} - \tfrac{121}{16} = \tfrac{5}{48}$$
$$G = -\tfrac{117}{4} + \tfrac{253}{4} - \tfrac{1331}{32} = -\tfrac{243}{32}$$
$$I = 45 - \tfrac{1287}{4} + \tfrac{529}{3} = -\tfrac{1205}{12}$$

Euler's cubic is

$$t^3+\frac{5}{16}t^2+\frac{6435}{16^2}t-\frac{3^{10}}{16^3}=0$$

Multiply the roots of this equation by 16.

The resulting equation is

$$s^3+5s^2+6435s-3^{10}=0$$

or $$(s-9)(s^2+14s+3^8)=0$$

$$\therefore\ s=9$$

or $$s^2+14s+6561=0$$

$$\therefore\ s=-7\pm4\sqrt{-407}$$

$$t=\frac{9}{16},\quad\frac{-7\pm4\sqrt{-407}}{16},\quad\frac{-7-4\sqrt{-407}}{16}$$

$$u=\sqrt{\tfrac{9}{16}}=\pm\tfrac{3}{4}$$

$$v=\pm\sqrt{\frac{-7+4\sqrt{-407}}{16}}=\pm\frac{2\sqrt{-11}+\sqrt{37}}{4}$$

$$w=\pm\sqrt{\frac{-7-4\sqrt{-407}}{16}}=\pm\frac{2\sqrt{-11}-\sqrt{37}}{4}$$

$$\therefore\ ax+b=\frac{3}{4}+\frac{2\sqrt{-11}+\sqrt{37}}{4}-\frac{2\sqrt{-11}-\sqrt{37}}{4}=\frac{3}{4}+\frac{\sqrt{37}}{2}$$

or $$ax+b=\frac{3}{4}-\frac{2\sqrt{-11}+\sqrt{37}}{4}+\frac{2\sqrt{-11}-\sqrt{37}}{4}=\frac{3}{4}-\frac{\sqrt{37}}{2}$$

or $$ax+b=-\frac{3}{4}+\frac{2\sqrt{-11}+\sqrt{37}}{4}+\frac{2\sqrt{-11}-\sqrt{37}}{4}=-\frac{3}{4}+\sqrt{-11}$$

or $$ax+b=-\frac{3}{4}-\frac{2\sqrt{-11}+\sqrt{37}}{4}-\frac{2\sqrt{-11}-\sqrt{37}}{2}=-\frac{3}{4}-\sqrt{-11}$$

These are the only allowable combinations, since $\sqrt{u}\,\sqrt{v}\,\sqrt{w}=-\frac{G}{2}$.

From these we have

$$x=2+\sqrt{-11},\quad 2-\sqrt{-11},\quad \tfrac{7}{2}+\tfrac{1}{2}\sqrt{37},\quad \tfrac{7}{2}-\tfrac{1}{2}\sqrt{37}$$

5. $x^4-7x^3-60x^2+221x-169=0$.

$$a=1,\quad b=-\tfrac{7}{4},\quad c=-10,\quad d=\tfrac{221}{4},\quad e=-169$$

$$\therefore\ H=-10-\tfrac{49}{16}=-\tfrac{209}{16}$$

$$G=\tfrac{221}{4}-\tfrac{210}{4}-\tfrac{343}{32}=-\tfrac{255}{32}$$

$$I=-169+\frac{7\times221}{4}+300=\frac{2071}{4}$$

$\therefore$ Euler's cubic is

$$t^3-\frac{627}{16}t^2+\frac{97907}{16^2}t-\frac{65025}{16^3}=0$$

Multiply the roots by 16.

The resulting equation is

$$s^3 - 627\,s^2 + 97907\,s - 65025 = 0$$

Or
$$(s - 289)(s^2 - 338\,s + 225) = 0$$

$$\therefore\ s = 289, \text{ or } 169 \pm 4\sqrt{1771}$$

$$\therefore\ u = \frac{289}{16}, \qquad v = \frac{169 + 4\sqrt{1771}}{16}, \qquad w = \frac{169 - 4\sqrt{1771}}{16}$$

$$\sqrt{u} = \frac{17}{4}, \qquad \sqrt{v} = \frac{2\sqrt{23} + \sqrt{77}}{4}, \qquad \sqrt{w} = \frac{2\sqrt{23} - \sqrt{77}}{4}$$

$$\therefore\ z = \pm\frac{17}{4} \pm \left(\frac{2\sqrt{23} + \sqrt{77}}{4}\right) \pm \left(\frac{2\sqrt{23} - \sqrt{77}}{4}\right)$$

But from the equation

$$\sqrt{u}\,\sqrt{v}\,\sqrt{w} = -\frac{G}{2} = +\frac{255}{64}$$

it appears that the order of the signs must be one of the following form:

$$\begin{array}{ccc} + & + & + \\ + & - & - \\ - & - & + \\ - & + & - \end{array}$$

$$\therefore\ z = \frac{17}{4} + \frac{2\sqrt{23} + \sqrt{77}}{4} + \frac{2\sqrt{23} - \sqrt{77}}{4} = \frac{17}{4} + \sqrt{23}$$

or
$$z = \frac{17}{4} - \frac{2\sqrt{23} + \sqrt{77}}{4} - \frac{2\sqrt{23} - \sqrt{77}}{4} = \frac{17}{4} - \sqrt{23}$$

or
$$z = -\frac{17}{4} - \frac{2\sqrt{23} + \sqrt{77}}{4} + \frac{2\sqrt{23} - \sqrt{77}}{4} = -\frac{17}{4} - \frac{\sqrt{77}}{2}$$

or
$$z = -\frac{17}{4} + \frac{2\sqrt{23} + \sqrt{77}}{4} - \frac{2\sqrt{23} - \sqrt{77}}{4} = -\frac{17}{4} + \frac{\sqrt{77}}{2}$$

$$x = z + \tfrac{7}{4}$$

$$\therefore\ x = 6 + \sqrt{23},\ \ 6 - \sqrt{23},\ \ \frac{-5 + \sqrt{77}}{2},\ \ \frac{-5 - \sqrt{77}}{2}$$

6. Show that the biquadratic can be solved by quadratics if $G = 0$.

If $G = 0$, the Eulerian cubic

$$t^3 + 3\,Ht^2 + \left(3\,H^2 - \frac{a^2I}{4}\right)t - \frac{G^2}{4} = 0$$

reduces to

$$t^3 + 3\,Ht^2 + \left(3\,H^2 - \frac{a^2I}{4}\right)t = 0$$

One root of this equation is 0, and the other two roots satisfy the quadratic equation

$$t^2 + 3Ht + \left(3H^2 - \frac{a^2I}{4}\right) = 0$$

Of the three quantities u, v, and w, u is therefore 0, and v and w are the roots of a quadratic equation. x is then obtained by extracting the square root of v and w, and combining the results.

Hence the process of obtaining x involves only the solution of quadratic equations.

7. Show that the two biquadratic equations

$$ax^4 + 6cx^2 \pm 4dx + e = 0$$

have the same reducing cubic.

Since $b = 0$, we have

$$H = ac, \qquad G = \pm a^2d, \qquad I = ae + 3c^2$$

But since the Eulerian cubic involves G only in the second power, it will make no difference whether

$$G = + a^2d \text{ or } - a^2d$$

8. Solve the biquadratic for the two particular cases in which $I = 0$ and $J = 0$.

(1) $$I = 0$$

In this case the reducing cubic reduces to

$$4a^3\theta^3 + J = 0$$

$\therefore$ the three values of $a\theta$ are the three cube roots of $-\frac{J}{4}$.

By § 515, $$u = a\sqrt[3]{-\frac{J}{4}} - H, \text{ etc.};$$

whence the four values of x are easily found.

If $J = 0$, the reducing cubic becomes

$$(t + H)^3 - \frac{a^2I}{4}(t + H) = 0$$

$$\therefore t + H = 0$$

or $$(t + H)^2 - \frac{a^2I}{4} = 0$$

$$\therefore t + H = \pm \frac{a}{2}\sqrt{I}$$

$$\therefore u = -H, \quad v = \frac{a}{2}\sqrt{I}, \quad w = -\frac{a}{2}\sqrt{I}$$

$$\therefore\ z = \pm\sqrt{-H} \pm \sqrt{\frac{a}{2}\sqrt{I}} \pm \sqrt{-\frac{a}{2}\sqrt{I}}$$

$$x = -\frac{b}{a} + \frac{1}{a}\left(\pm\sqrt{-H} \pm \sqrt{\frac{a}{2}\sqrt{I}} \pm \sqrt{-\frac{a}{2}\sqrt{I}}\right)$$

9. Show that if H is positive, the biquadratic has either two or four imaginary roots.

In the Eulerian cubic

$$t^3 + 3Ht^2 + \left(3H^2 - \frac{a^2I}{4}\right)t - \frac{G^2}{4} = 0$$

the last term is always negative, since G^2 is always positive.

If H is positive, the order of signs is therefore

$$\begin{array}{cccc} + & + & + & - \\ + & + & - & - \end{array}$$

according as $3H^2 - \frac{a^2I}{4}$ is positive or negative.

In either case the cubic has, by Descartes' Rule of Signs, not more than one positive root, and not more than two negative roots.

It may therefore have

one positive root and two negative roots,

or one positive root and two imaginary roots,

or one negative root and two imaginary roots.

The last case is, however, excluded, because the product of the roots must be positive, since $-\frac{G^2}{4}$ is negative.

But the first two cases are Case II. and Case III. of Art. 515. Hence the biquadratic has either two or four imaginary roots.

10. Find the reducing cubic of

$$x^4 - 6ax^2 + 8x\sqrt{a^3 + b^3 + c^3 - 3abc} + (12bc - 3a^2) = 0.$$

Let $6c' = -6a$, $4d' = 8\sqrt{a^3 + b^3 + c^3 - 3abc}$, and $e' = 12bc - 3a^2$.

Then for the biquadratic

$$x^4 + 16c'x^2 + 4d'x + e' = 0$$

we have

$$I = e' + 3c'^2 = 12bc$$

$$J = c'e' - d'^2 - c'^3 = -4b^3 - 4c^3$$

$\therefore$ The reducing cubic is

$$4t^3 - 12bct - 4(b^3 + c^3) = 0$$

or

$$t^3 - 3bct - (b^3 + c^3) = 0$$

11. Prove that J vanishes for the biquadratic

$$3a(x-2a)^4 = 2a(x-3a)^4.$$

Expanding $3(x^4 - 8ax^3 + 24a^2x^2 - 32a^3x + 16a^4)$

$$= 2(x^4 - 12ax^3 + 54a^2x^3 - 108a^3x + 81a^4)$$

$$x^4 - 36a^2x^2 + 120a^3x - 114x = 0$$

$$\therefore H = -6a^2, \quad G = 30a^3, \quad I = 3H^2 - 114a^4 = -6a^4$$

$$J = 36a^6 - 900a^6 + 864a^6 = 0$$

12. If the roots of a biquadratic are all real, and are in harmonical progression, prove that the roots of Euler's cubic are in arithmetical progression.

In the case of harmonic division $\dfrac{(\alpha-\beta)(\gamma-\delta)}{(\alpha-\delta)(\beta-\gamma)} = 1$

$$(\alpha-\beta)(\gamma-\delta) = (\alpha-\delta)(\beta-\gamma)$$

By (7), § 515,

$$a(\alpha-\beta) = 2(\sqrt{u} - \sqrt{v})$$

$$a(\gamma-\delta) = -2(\sqrt{u} + \sqrt{v})$$

$$a^2(\alpha-\beta)(\gamma-\delta) = -4(u-v)$$

$$a(\alpha-\delta) = -2(\sqrt{v} + \sqrt{w})$$

$$a(\beta-\gamma) = 2(\sqrt{v} - \sqrt{w})$$

$$a^2(\alpha-\delta)(\beta-\gamma) = -4(v-w)$$

$$\therefore v - w = u - v$$

$\therefore$ u, v, w are in arithmetical progression.

Exercise 89.

1. $(a+bi)^4 + (a-bi)^4$.

$$(a+bi)^4 = a^4 + 4a^3bi - 6a^2b^2 - 4ab^3i + b^4$$

$$(a-bi)^4 = a^4 - 4a^3bi - 6a^2b^2 + 4ab^3i + b^4$$

Add, $(a+bi)^4 + (a-bi)^4 = 2a^4 - 12a^2b^2 + 2b^4$

2. $\dfrac{1+i}{1+2i} + \dfrac{1-i}{1-2i}$.

$$\frac{1+i}{1+2i} = \frac{(1+i)(1-2i)}{(1+2i)(1-2i)} = \frac{3-i}{5}$$

$$\frac{1-i}{1-2i} = \frac{(1-i)(1+2i)}{(1-2i)(1+2i)} = \frac{3+i}{5}$$

$$\therefore \frac{1+i}{1+2i} + \frac{1-i}{1-2i} = \frac{6}{5}$$

3. $\dfrac{2+36i}{6+8i}+\dfrac{7-26i}{3-4i}$.

$$\frac{2+36i}{6+8i}=\frac{(2+36i)(6-8i)}{(6+8i)(6-8i)}=\frac{300+200i}{100}=3+2i$$

$$\frac{7-26i}{3-4i}=\frac{(7-26i)(3+4i)}{(3-4i)(3+4i)}=\frac{125-50i}{25}=5-2i$$

$$\therefore\ \frac{2+36i}{6+8i}+\frac{7-26i}{3-4i}=8$$

4. Show that $[(\sqrt{3}+1)+(\sqrt{3}-1)i]^3=16+16i$.

$$\begin{aligned}&[(\sqrt{3}+1)+(\sqrt{3}-1)i]^3\\ &=(\sqrt{3}+1)^3+3(\sqrt{3}+1)^2(\sqrt{3}-1)i-3(\sqrt{3}+1)(\sqrt{3}-1)^2-(\sqrt{3}-1)^3i\\ &=3\sqrt{3}+9+3\sqrt{3}+1+(6\sqrt{3}+6)i-(6\sqrt{3}-6)-(3\sqrt{3}-9+3\sqrt{3}-1)i\\ &=16+16i\end{aligned}$$

5. If $\sqrt[3]{x+yi}=a+bi$, show that

$$4(a^2-b^2)=\frac{x}{a}+\frac{y}{b}$$

Cube both sides of the equation.

$$x+yi=(a+bi)^3=a^3+3a^2bi-3ab^2-b^3i$$

Equate the real parts and the imaginary parts.

$$x=a^3-3ab^2 \qquad y=3a^2b-b^3$$

$$\frac{x}{a}=a^2-3b^2 \qquad \frac{y}{b}=3a^2-b^2$$

$$\therefore\ \frac{x}{a}+\frac{y}{b}=4(a^2-b^2)$$

6. Find the modulus of $\dfrac{(3-4i)(2+3i)}{(6-4i)(15+8i)}$.

$$\begin{aligned}\text{Modulus }(3-4i)&=5\\ \text{Modulus }(2+3i)&=\sqrt{13}\\ \text{Modulus }(6-4i)&=2\sqrt{13}\\ \text{Modulus }(15+8i)&=17\end{aligned}$$

$$\therefore\ \text{Modulus }\frac{(3-4i)(2+3i)}{(6-4i)(15+8i)}=\frac{5\times\sqrt{13}}{2\sqrt{13}\times 17}=\frac{5}{34}$$

7. Find the three cube roots of $1 + i$.

$$\tan\phi = 1$$
$$\therefore\ \phi = 45^\circ$$
$$\text{Modulus } (1 + i) = \sqrt{2}$$

$\therefore$ The three cube roots are

$$\sqrt[6]{2}(\cos 15^\circ + i\sin 15^\circ), \qquad \sqrt[6]{2}(\cos 135^\circ + i\sin 135^\circ),$$
$$\sqrt[6]{2}(\cos 265^\circ + i\sin 265^\circ),$$

or

$$\sqrt[6]{2}\left(\frac{1+\sqrt{3}}{2\sqrt{2}} + i\frac{\sqrt{3}-1}{2\sqrt{2}}\right), \qquad \sqrt[6]{2}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right),$$
$$\sqrt[6]{2}\left(\frac{1-\sqrt{3}}{2\sqrt{2}} - i\frac{1+\sqrt{3}}{2\sqrt{2}}\right),$$

or

$$\frac{1}{2\sqrt[3]{2}}[1 + \sqrt{3} + i(\sqrt{3} - 1)], \qquad \frac{1}{\sqrt[3]{2}}(-1 + i),$$
$$\frac{1}{2\sqrt[3]{2}}[1 - \sqrt{3} - i(1 + \sqrt{3})].$$

8. Find the five fifth roots of 1.

The five fifth roots are

$$\begin{aligned} 1 &= 1 \\ \cos 72^\circ + i\sin 72^\circ &= 0.3090 + i\,0.9551 \\ \cos 144^\circ + i\sin 144^\circ &= -0.5878 + i\,0.8090 \\ \cos 216^\circ + i\sin 216^\circ &= -0.5878 - i\,0.8090 \\ \cos 288^\circ + i\sin 288^\circ &= 0.3090 - i\,0.9551 \end{aligned}$$

Or by the rule for inscribing a regular decagon,

$$\cos 72^\circ = \frac{\sqrt{5} - 1}{4}, \qquad \sin 72^\circ = \frac{\sqrt{10 + 2\sqrt{5}}}{4}$$

from which the roots may be expressed by radicals.

9. Find the four fourth roots of $3 + 4i$.

$$\tan\phi = \tfrac{4}{3}$$
$$\therefore\ \phi = 53^\circ\,8'$$
$$\text{Modulus } 3 + 4i = 5$$

$\therefore$ The four fourth roots are

$$\begin{aligned} \sqrt[4]{5}(\cos 13^\circ 17' + i\sin 13^\circ 17') &= \sqrt[4]{5}(\cos 13^\circ 17' + i\sin 13^\circ 17') \\ \sqrt[4]{5}(\cos 103^\circ 17' + i\sin 103^\circ 17') &= \sqrt[4]{5}(-\sin 13^\circ 17' + i\cos 13^\circ 17') \\ \sqrt[4]{5}(\cos 193^\circ 17' + i\sin 163^\circ 17') &= \sqrt[4]{5}(-\cos 13^\circ 17' - i\sin 13^\circ 17') \\ \sqrt[4]{5}(\cos 283^\circ 17' + i\sin 283^\circ 17') &= \sqrt[4]{5}(\sin 13^\circ 17' - i\cos 13^\circ 17') \end{aligned}$$

$$\log 5 \quad = 0.6990 \qquad \log\cos 13^\circ 17' = 9.9882$$
$$\therefore \log \sqrt[4]{5} = 0.1747 \qquad \log\sin 13^\circ 17' = 9.3614$$
$$\therefore \log(\sqrt[4]{5}\cos 13^\circ 17') = 0.1629$$
$$\sqrt[4]{5}\cos 13^\circ 17' = 1.455$$
$$\log(\sqrt[4]{5}\sin 13^\circ 17') = 9.5361$$
$$\sqrt[4]{5}\sin 13^\circ 17' = 0.3436$$

∴ The four roots are

$$1.455 + i\,0.3436$$
$$-0.3436 + i\,1.455$$
$$-1.455 - i\,0.3436$$
$$0.3436 - i\,1.455$$

By the formulas for $\sin\frac{\phi}{2}$ and $\cos\frac{\phi}{2}$ we find,

$$\sin\frac{\phi}{4} = \sqrt{\frac{1}{2} - \frac{3}{2\sqrt{10}}}, \qquad \cos\frac{\phi}{4} = \sqrt{\frac{1}{2} + \frac{3}{2\sqrt{10}}}$$

from which the roots may be expressed by radicals.

10. Solve the equation $x^3 - 12x + 3 = 0$.

$$a = 1, \qquad b = 0, \qquad c = -4, \qquad d = 3$$
$$\therefore H = -4, \qquad G = 3$$
$$\tan\phi = \frac{\sqrt{-(9-256)}}{3} = \frac{7}{3}\sqrt{5}$$
$$\log 7 = 0.8451$$
$$\log\sqrt{5} = 0.3455$$
$$\operatorname{colog} 3 = 9.5229$$
$$\therefore \log\tan\phi = 0.7125$$
$$\phi = 79^\circ 2'$$
$$\frac{\phi}{3} = 26^\circ 21', \qquad 146^\circ 21', \qquad 266^\circ 21'$$
$$\cos\frac{\phi}{3} = 0.8961, \qquad -0.8325, \qquad -0.0636$$

The three values of x are

$$2\sqrt{4}\cos 26^\circ 21' = 3.5844$$
$$2\sqrt{4}\cos 146^\circ 21' = -3.3300$$
$$2\sqrt{4}\cos 266^\circ 21' = -0.2544$$

Check:

$$\begin{array}{r} 3.5844 \\ -3.3300 \\ \underline{-0.2544} \\ 0.0000 \end{array}$$

11. Solve the equation $2x^3 + 3x^2 - 3x - 1 = 0$.

$$a = 2, \qquad b = 1, \qquad c = -1, \qquad d = -1$$

$$\therefore\ H = -3, \qquad G = 4$$

$$\tan\phi = \frac{\sqrt{-(16 - 108)}}{4} = \frac{\sqrt{23}}{2}$$

$$\log\sqrt{23} = 0.6806$$

$$\text{colog}\,2 = 9.6990$$

$$\therefore\ \log\tan\phi = 0.3796$$

$$\phi = 24^\circ\,40'$$

$$\frac{\phi}{3} = 8^\circ\,13', \qquad 128^\circ\,13', \qquad 248^\circ\,13'$$

$$\log\cos\frac{\phi}{3} = 9.9955, \qquad 9.7915, \qquad 9.5694$$

The three values of x are

$$-\tfrac{1}{2} + \sqrt{3}\cos 8.13', \quad -\tfrac{1}{2} + \sqrt{3}\cos 128^\circ\,13', \quad -\tfrac{1}{2} + \sqrt{3}\cos 248^\circ\,13'$$

or $\quad 1.212, \quad -1.571, \quad -1.1427$

Check:

$$\begin{array}{r} 1.212 \\ -1.571 \\ -1.143 \\ \hline -1.502 \end{array}$$

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